Fixed Functionals in Quantum Einstein Gravity

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setup

- tool: \( k \frac{d \Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} k \frac{d \mathcal{R}_k}{dk} \right] \)

- truncation: \( \Gamma_k^{\text{grav}} \left[ g_{\mu \nu} \right] = \int d^d x \sqrt{g} f_k(R) \)
  
  [0705.1769, 0712.0445, 1204.3541, 1211.0955, ...]

- background field method \( g_{\mu \nu} = \bar{g}_{\mu \nu} + h_{\mu \nu} \)

- conformal reduction: \( h_{\mu \nu} = \frac{1}{d} \bar{g}_{\mu \nu} \phi \)
  
  [0801.3287, ...]

- maximally symmetric background: \( S^d \ (R > 0) \)
type II regulator

operator: \( \Box := -\bar{D}^2 + \mathbf{E} \) (potential term \( \mathbf{E} \) containing \( \bar{R} \))

define regulator \( \mathcal{R}_k(\Box) \)

\[
\Gamma_k^{(2)}(\Box) + \mathcal{R}_k(\Box) \overset{\text{def.}}{=} \Gamma^{(2)}_k(\Box + R_k(\Box)),
\]

where \( R_k \) is the profile function (Litim’s cutoff).

flow equation

\[
\int d^d x \sqrt{g} \partial_t f_k(\bar{R}) = \frac{1}{2} \text{Tr} W(\Box)
\]
operator trace

spectral sum:

$$\text{Tr } W(\Box) = \sum_i D_i W(\lambda_i)$$

eigenvalues $\lambda_i$ and multiplicities $D_i$ of $\Box$

integrating out eigenmodes:

$$\text{optimised cutoff} \implies W(\lambda_i) \propto \theta(k^2 - \lambda_i)$$
$$\implies W(\lambda_i) \neq 0 \iff k^2 \geq \lambda_i$$

■ every time $k$ crosses $\lambda_i$ the eigenmode is integrated out

■ spectral sum is a finite sum (ACHTUNG: $\lambda_i \propto R$)
definition: fixed function

\[ R = k^2 r, \quad E = k^2 e, \quad f_k(R) = k^d \varphi_k(R/k^2) \]

- flow equation: partial differential equation for \( \varphi_k(r) \)

- fixed functions: (global) \( k \) stationary solutions \( \iff \partial_t \varphi_k(r) = 0 \)

- third order equation: three parameter family of solutions
pole structure of flow equation

the coefficient of $\varphi'''(r)$

- three poles $r_0$, $r_1$ and $r_2$ (in $d = 3, 4$)
- global solutions should cross the poles
regular singular points and pole crossing

generic ODE: \( y^{(n)}(x) = f(y^{(n-1)}, \ldots, y', y, x) \)

r.h.s. \( f \) can have singular points

\[
f(y^{(n-1)}, \ldots, y', y, x) = \frac{e(y^{(n-1)}(x_0), \ldots, y'(x_0), y(x_0), x_0)}{x - x_0} + \mathcal{O}((x - x_0)^0)
\]

- pole crossing (global) solution \( \iff e \big|_{x=x_0} = 0 \) (regularity condition)

- additional boundary condition reduces number of free parameters
analytic example

initial value problem:

\[ y''(x) = -\frac{y(x)}{x - 1}, \quad y(0) = y_0, \quad y'(0) = y_1 \]

solutions are modified Bessel functions \( I_n \). For most \( y_0, y_1 \): no global solution.

add BC \( y(1)=0 \): pole crossing solution

\[ y(x; y_0) = y_0 \frac{\sqrt{1-x} I_1(2\sqrt{1-x})}{I_1(2)} \]

self similarity in initial values \( y(x; y_0) = \lambda^{-1} y(x; \lambda y_0) \)
analytic example

initial value problem:

\[ y''(x) = -\frac{y(x)}{x - 1}, \quad y(0) = y_0, \quad y'(0) = y_1 \]

polynomial expansion:

\[ y(x) = \sum_{n=0}^{k} a_n x^n \]

- fixing all coefficients at \( x = 0 \) \( \implies \) \( a_i = 0 \)

- improved strategy: fix \( a_n \) at \( x = 0 \) for \( n \geq 2 \) and fix \( a_1 \) at \( x = 1 \)
  \( \implies \) series expansion of analytic solution recovered

- self similarity \( y(x) = y_0 \sum_{n=0}^{k} \tilde{a}_n x^n \)
numerical shooting I

expand around $r = 0$

$$\varphi(r; a_0, a_1) = a_0 + a_1 r + \sum_{n=2}^{k} a_n(a_0, a_1) r^n$$

1. fix $a_n, n \geq 2$ by using regularity condition at $r = 0$

2. initial conditions for numerical integration $(\varepsilon > 0)$

$$\varphi_{\text{init}}(\varepsilon) = \varphi^{(n)}(\varepsilon; a_0, a_1), \quad n = 0, 1, 2$$

3. regularity condition at $r_1$

$$e : (a_0, a_1) \mapsto \mathbb{R}$$
pole crossing solutions

- green: \( e(a_0, a_1) > 0 \)
- red: \( e(a_0, a_1) < 0 \)
- black line: \( e(a_0, a_1) \approx 0 \)
polynomial expansion

- polynomial expansion
  \[ \varphi(r) = \sum_{k=0}^{n} a_k r^k \]

- boundary condition at \( r = 0 \) \((a_2, a_3, \ldots)\)

- boundary condition at \( r = r_1 \) \((a_1)\)

- \( a_1 \) as function of \( a_0 \) (dashed blue line)
numerical shooting II

algorithm for second shooting

1. parametrize regular line by $a_0$

2. initial conditions for second numerical integration

$$\varphi^{(n)}_{\text{init}}(r_1 + \varepsilon) = \varphi^{(n)}_{\text{num}}(r_1 - \varepsilon; a_0), \quad n = 0, 1, 2$$

3. numerically integrate up to $r_2$ and check regularity condition

$$e_3 : a_0 \mapsto \mathbb{R}$$
regularity at $r_2$

- There are two distinct zeros $\implies$ two fixed functions
- improved stability: at most three relevant directions
preliminary results in $d = 4$

- numerical shooting and polynomial expansion yield regular line
- one (nontrivial) fixed function identified
fixed function in $d = 4$
summary

- flow of conformally reduced $f(R)$-gravity
- interpretation of integrating out eigenmodes
- numerical and analytical techniques for pole crossing
- two fixed functions in $d = 3$
- one fixed function in $d = 4$
Thank You!