Derivative expansion:
numerical results at order six,
convergence and optimization

with L. Canet, H. Chaté and I. Balog
(to be published....)

Questions: - Does the DE converge?
- Can we optimize it?

Many papers on this subject (D. Klein, J. Pawlowski, L. Canet, H. Chaté, D. Machanna, H. Tissier, J. Vidal)

but... not a well-posed question!

⇒ e.g.: perturbation theory: not convergent!

⇒ at best asymptotic and Borel-summable (but QED works well!)

⇒ in this case, convergence means:
convergence of the resummed Pade’-Borel (for instance) series!

⇒ optimization w.r.t. which criterion?
Optimization before convergence

two kinds of optimization \{ "global", "local"

**global**: optimize generic properties of the RG flow through the choice of cut-off function \( R_{\kappa} (q) \), independently of the physical quantity studied.

**local**: choose \( R_{\kappa} (q) \) to optimize a given physical quantity at a given order of the DE.

Example: \( \psi^4, N=1, d=3, \) LPA

* Litim's claim on global optimization:
  There exists regulators that are "optimal"

  e.g. \( R_{\kappa} (q) = \frac{3.32}{e^{q^2} - 1}; R_{\kappa} (q) = (k^2 - q^2) \Theta (k^2 - q^2) \)

  \( R_{\kappa} (q) = \frac{k^2}{q^2} \); .....
* "local" optimization of the critical exponent $\nu$:

1. Choose a family of cut-off functions
   
   \[ R_\infty(q) = \frac{q^2}{e^{q^2/b^2} - 1} \]

2. Compute (numerically) $\nu(b)$

   ![Graph showing $\nu$ vs. $b$ with $\nu_{\text{opt}} \approx 6$ and $b_{\text{opt}} \approx 6$]

   Principle of
   Minimal Sensitivity

   (PHS)

3. Repeat "ad nauseam" steps 1 and 2 with different families of $R_\infty$ and compare the different $\nu_{\text{opt}}$ (for instance $R_\infty(q) = 6 (k^2 - q^2) \delta(k^2 - q^2)$ with $b_{\text{opt}} = 1$)

**Results**

* at LPA, $\nu_{\text{opt}}$ is always a minimum and above $\nu_{\text{exact}}$

\[
\begin{align*}
\nu_{\text{opt}} &= 0.6503 \\
\nu_{\text{opt}} &= 0.6494 \\
\end{align*}
\]

(Canet 2004) very close!
Questions:
- Does the PMS apply at each order of the DE and for any physical quantity? (one and only one extremum)
- Does the extremum point to the right direction?
- Does the series of numbers $\nu_{0+1}^{opt}$, $\nu_{0(2)}^{opt}$, $\nu_{0(4)}^{opt}$, $\nu_{0(6)}^{opt}$ seem to converge to the exact result?
- Does the series of numbers $\nu_{-1}^{exp}$, $\nu_{0(2)}^{exp}$, $\nu_{0(4)}^{exp}$ and idem $\nu_{0(4)}^{opt}$, $\nu_{0(6)}^{opt}$ converge to the same number?

A global criterion

\[ \bar{\Gamma}_{\phi}^{(12)} = \bullet \quad + \quad \circ \]

in general a graph $\int \Phi_{\phi(q)} (\text{Propag})^m (\text{vertex})^k$

DE $\Rightarrow$ 1 Propag $= \frac{1}{\Gamma^{(12)}_{\phi} + \Phi_{\phi(q)}} = \frac{1}{\text{Polyn}(q) + \Phi_{\phi(q)}}$

\( \Rightarrow \) finite radius of convergence of these expansions in $\mathbb{R}_\phi$

\( \Rightarrow \) very bad beyond this radius of conv.

\( \Rightarrow \) need to cut-off efficiently for $\Phi_{\phi(q)}$. 

\[ DE_{\phi} \]
Best choice: \( R_k (q > k) \equiv 0 ? \)

e.g.: \( R_k (q) = (k^2 - q^2)^\theta (k^2 - q^2) ? \)

No because this cut-off does not regularize the DE at "large" orders.

\[
\implies R_k (q) \approx \frac{k^2}{4} \left( 1 - \frac{q^2}{k^2} \right)^m \Theta (k^2 - q^2)
\]

Remark: At order of DE, \( \Im n / \text{DE is regularized} \)

BUT...

\( \forall m \), \( \Im n \) order of DE / DE is not regular.

Remark: At order \( 0(q^6) \) of DE, we need:

\[
\partial_c \Gamma^{(6)}_{q_4} = \ldots
\]

\[ \implies \text{a product of 6 } \Gamma^{(3)}_{q_4} \text{ functions} \]

\[ \implies m_1 m_2 \ldots m_6 q^{30} \]

\[ \implies \text{difficult to make accurate and controlled calculations with } q^{30}! \]

\[ \implies \text{keep only terms up to (momentum)}^6 \]
\[ \Gamma_{\phi} = \int d^4 x \left\{ V_{\phi} + \frac{1}{2} Z_{\phi} (\nabla \phi)^2 ight. \\
+ \frac{1}{2} W_{\phi} (\nabla \phi)^2 + \frac{1}{2} W_{\phi} (\psi \phi\phi) + \frac{1}{2} X_{\phi} (\nabla \phi)^4 \\
+ \frac{1}{2} X_{\phi} (\nabla_\mu \nabla_\nu \phi)^2 + \cdots + \frac{1}{36} X_{\phi} (\nabla \phi)^6 \right\} \]

2. \[ R_{\phi}(q) = b Z_{\phi} \frac{q^2}{\exp(q^2) - 1}, \quad R_{\phi}(q) = b Z_{\phi} \frac{q^2}{\exp(q^2) - 1} \]

(we have also studied three other families of \( R_{\phi} \) functions).

3. Compute the running \( \beta_{\phi} = -\frac{d}{d \ln \mu} \log Z_{\phi} \) by imposing a renormalization prescription on \( Z_{\phi}(\rho) \) at a given \( \rho_0 : \quad Z_{\phi}(\rho) = Z_{\phi}(\rho_0) \Rightarrow 3(\rho) = 1. \)

\[ \Rightarrow \text{two kinds of arbitrariness} \]

- choice of \( R_{\phi} \)
- choice of \( \rho_0 \)

One can prove that changing \( b \) or \( \rho_0 \)
is equivalent (equivalence theorem? reparametrization invariance?)
Izing: exact results in $d=3$

$\nu = 0.62929(5)$  
$\eta = 0.03631(3)$  
$\omega = 0.8303(18)$

**Our results**

<table>
<thead>
<tr>
<th></th>
<th>Exponential cut-off</th>
<th>$0(\phi^4)$</th>
<th>$0(\phi^6)$</th>
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</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>0.6503</td>
<td>0.6281</td>
<td>0.63051</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0</td>
<td>0.04427</td>
<td>0.03454</td>
</tr>
<tr>
<td>$\omega$</td>
<td></td>
<td>0.872</td>
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**$\Theta$-cut-off**

<table>
<thead>
<tr>
<th></th>
<th>m=3</th>
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<th>m=5</th>
<th>m=6</th>
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<tbody>
<tr>
<td>$\nu$</td>
<td>0.6302</td>
<td>0.6302</td>
<td>0.6303</td>
<td>0.6303</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.03507</td>
<td>0.03481</td>
<td>0.03589</td>
<td>0.03583</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.05445</td>
<td>0.05445</td>
<td>0.05445</td>
<td>0.05445</td>
</tr>
</tbody>
</table>

**Other results**

6-loop: $\nu = 0.63004(15)$, $\eta = 0.03635(25)$, $\omega = 0.793(11)$

HT: $\nu = 0.63012(16)$, $\eta = 0.03633(15)$, $\omega = 0.825(50)$

MC: $\nu = 0.63002(10)$, $\eta = 0.03680(20)$, $\omega = 0.824(15)$

$\approx 0.37$