Critical behavior in spherical and hyperbolic spaces

Dario Benedetti

Albert Einstein Institute, Potsdam, Germany

September 23, 2014

based on [arXiv:1403.6712]
Motivations

- Quantum field theory on curved background
- Recent surge of interest on IR effects in dS spacetime [see Serreau’s talk]
- Background dependence in FRG approach to asymptotically safe gravity:
  - $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $\Delta_k S[\bar{g}; h] \Rightarrow \Gamma_k = \Gamma_k[\bar{g}; h]$, mWI
  - Nonperturbative (in $\bar{R}$) background effects in $f(R)$ approximation
- Usually we study asymptotic safety in Euclidean signature
  $\Rightarrow$ Euclidean QFT (≡ statistical field theory) on curved background
- In condensed matter the effect of curvature can be of interest for a number of reasons (e.g. for theoretical modeling of 3d frustration in simplified 2d models), and it has been studied in the context of liquids, percolation, Ising model, XY model, self-avoiding walks and more

This talk: How does curvature affect critical behavior in a simple model and how do we see that with the FRGE
Outline

- Two simple backgrounds: $d$-dimensional spheres and hyperboloids
- Effective dimension and general expectations
- FRGE in the presence of background curvature
The $d$-dimensional sphere

- Homogeneous space: $S^d \simeq SO(d + 1)/SO(d)$

$$\sum_{A=1}^{d+1} (X^A)^2 = a^2$$

$$ds^2_{(S^d)} = a^2 d\Omega_d = a^2 d\theta_d^2 + a^2 \sin^2(\theta_d) d\Omega_{d-1}$$

- Maximally symmetric:

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

with positive curvature:

$$R = \frac{d(d-1)}{a^2}$$

- Compact space $\Rightarrow$ discrete spectrum, including a zero mode

$$-\nabla^2 \psi_{n,j} = \frac{n(n+d-1)}{a^2} \psi_{n,j}$$

with multiplicity $D_n = \frac{(n+d-2)! (2n+d-1)}{n!(d-1)!}$, $j = 1, 2, \ldots D_n$, and $n = 0, 1, 2, \ldots + \infty$
The $d$-dimensional hyperboloid

- Homogeneous space: $H^d \simeq SO(d, 1)/SO(d)$

$$
\sum_{A=1}^{d} (X^A)^2 - (X^{d+1})^2 = -a^2
$$

$$
\text{d} s_{(H^d)}^2 = \text{d}\tau^2 + a^2 \sinh^2 (\tau/a) \text{d}\Omega_{d-1}
$$

- Also maximally symmetric, but with negative curvature:

$$
R = - \frac{d(d - 1)}{a^2}
$$

- Non-compact space $\Rightarrow$ continuous spectrum

$$
-\nabla^2 \phi_{\lambda,l} = \frac{1}{a^2} (\lambda^2 + \rho^2) \phi_{\lambda,l}
$$

where $\rho = (d - 1)/2$, $\lambda \in [0, +\infty)$, and $l = 0, 1, 2, ... + \infty$

note: no zero mode (not normalizable)
Effective dimension

Hausdorff dimension:

\[ L^{d_H} = \int_L d^d x \sqrt{g} \]

where the integral extends over the set of points for which \( \sigma(x,0) \leq L \).

- Sphere: \( d_H \to 0 \) for \( L \to \infty \) (\( \sim \) it looks like a point from far)
- Hyperboloid: \( d_H \to \infty \) for \( L \to \infty \) (due to exponential growth \( e^L \))

Spectral dimension \( \Rightarrow \) same result
Effective dimension

Hausdorff dimension:

\[ L^{d_H} = \int_{\mathcal{L}} d^d x \sqrt{g} \]

where the integral extends over the set of points for which \( \sigma(x,0) \leq L \).

- Sphere: \( d_H \to 0 \) for \( L \to \infty \) (\( \sim \) it looks like a point from far)
- Hyperboloid: \( d_H \to \infty \) for \( L \to \infty \) (due to exponential growth \( e^L \))

Spectral dimension \( \Rightarrow \) same result

\( \Rightarrow \) We expect mean field behavior on hyperboloid, and no phase transition on sphere

(We reach same expectations by using Ginzburg criterion for scalar field)
FRGE on curved background – (I)

- LPA
  \[
  \Gamma_k[\phi] = \int d^d x \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V_k(\phi) \right]
  \]

- FRGE
  \[
  k \partial_k V_k(\phi) = \frac{1}{2} \text{Tr}(\mathcal{M}) \left[ \frac{k \partial_k R_k(-\nabla^2/k^2)}{-Z_k \nabla^2 + V_k''(\phi) + R_k(-\nabla^2/k^2)} \right] \bigg|_{\phi = \text{const.}}
  \]

- Using the optimized cutoff and dimensionless variables
  \[
  k \partial_k \tilde{V}_k(\tilde{\phi}) + d \tilde{V}_k(\tilde{\phi}) - \frac{d - 2}{2} \tilde{\phi} \tilde{V}_k'(\tilde{\phi}) = \frac{1}{1 + \tilde{V}_k''(\tilde{\phi})} F(\mathcal{M})(\tilde{a})
  \]

  where
  \[
  F(\mathcal{M})(\tilde{a}) = \text{Tr}(\mathcal{M})[\theta(1 - \tilde{\Delta})], \quad \text{and} \quad \tilde{a} = ak
  \]

  ⇒ All the background dependence is in the spectral counting function \( F(\mathcal{M})(\tilde{a}) \).
FRGE on curved background – (II)

- In flat space, by Fourier transform:

\[ F_{(E^3)}(\infty) = \frac{\Omega_{d-1}}{d(2\pi)^d} \xrightarrow{d=3} \frac{1}{6\pi^2} \]

- Hyperboloid \((d = 3)\):

\[ F_{(H^3)}(ak) = \frac{1}{6\pi^2} \left(1 - \frac{1}{a^2k^2}\right)^{\frac{3}{2}} \theta \left(1 - \frac{1}{a^2k^2}\right) \]

- Sphere \((d = 3)\):

\[ F_{(S^3)}(ak) = \frac{1}{2\pi^2 a^3k^3} \mathcal{P}(\lfloor N_3 \rfloor) \]

where \([x]\) is the floor function,

\[ \mathcal{P}(N) = \sum_{n=0}^{N} D_n = \frac{1}{6}(1 + N)(2 + N)(3 + 2N) \]

\[ N_3 = -1 + \sqrt{1 + a^2k^2} \]

The spherical case gives rise to a staircase function, as a combined effect of the discrete spectrum and the use of a step function in the cutoff.
Non-autonomous system

\[ F_{(\mathcal{M})}(\tilde{a}) = \tilde{\text{Tr}}_{(\mathcal{M})}[\theta(1 - \tilde{\Delta})] \]

\[ \downarrow \]

- Non-autonomous equation: explicit dependence on \( k \) via \( \tilde{a} = ak \)

No rescaling of variables can turn the equation into an autonomous one

\[ \Rightarrow \] Non-trivial fixed points are unlikely (\( k \)-dependence should factorize in \( \beta \)'s)

- Non-autonomous equations found also in:

  - quantum field theory at finite temperature [Tetradis, Wetterich - '93],
  - non-commutative spacetime [Gurau, Rosten - '09],
  - RG for matrix/tensor models [DB, BenGeloun, Oriti - to appear],
  - gravity, if we eliminate \( G \) (treating it as inessential parameter) [Percacci, Perini - '04]
Scaling dimensions in the deep IR

Deep IR, $k \rightarrow 0$:

- **Hyperboloid:** $F_{(H^d)}(\tilde{a}) \rightarrow 0$ (due to mass gap)

  For $ak < (d - 1)/2$ (due to optimized cutoff)

  $$k \partial_k \tilde{V}_k(\tilde{\phi}) + d \tilde{V}_k(\tilde{\phi}) - \frac{d - 2}{2} \tilde{\phi} \tilde{V}_k'(\tilde{\phi}) = 0$$

  $\equiv$ classical scaling equation
Scaling dimensions in the deep IR

Deep IR, $k \to 0$:

- **Hyperboloid**: $F_{(H^d)}(\tilde{a}) \to 0$ (due to mass gap)

  For $ak < (d-1)/2$ (due to optimized cutoff)

  $$k \partial_k \tilde{V}_k(\tilde{\phi}) + d \tilde{V}_k(\tilde{\phi}) - \frac{d-2}{2} \tilde{\phi} \tilde{V}_k'(\tilde{\phi}) = 0$$

  $\equiv$ classical scaling equation

- **Sphere**: $F_{(S^d)}(\tilde{a}) \to \infty$ (due to zero mode, and compactness)

  In order to absorb divergence of FRGE:

  $$\tilde{\phi} = a^{d/2} k \phi, \quad \tilde{V}(\tilde{\phi}) = a^d V(a^{-d/2} k^{-1} \phi)$$

  The resulting equation for $k^2 < d/a^2$ is

  $$k \partial_k \tilde{V}_k(\tilde{\phi}) + \tilde{\phi} \tilde{V}_k'(\tilde{\phi}) = \frac{1}{\Omega_d} \frac{1}{1 + \tilde{V}_k''(\tilde{\phi})}$$

  $\equiv$ flat FRG equation for $d = 0$
Solve numerically the flow equation, and integrating towards $k = 0$ observe different behavior as function of initial condition.

Blue curve: initial condition $V_Λ(\phi) = \lambda_Λ(\phi^2 - ρ_Λ)^2$
Despite the large value of the initial symmetry breaking parameter (here $\rho_\Lambda = 25$), symmetry restoration still takes place.

No true phase transition!
Numerical integration – Hyperboloid

Phase transition is there:

Note: no zero mode ⇒ convexity of $\Gamma$ does not imply convexity of the potential

Convexity of the effective action: all the eigenvalues of $\Gamma^{(2)}[\bar{\phi}]$ are non-negative

If $p^2 = 0$ is in the spectrum ⇒ $V''(\bar{\phi}) \geq 0$ (because $\Gamma^{(2)}[\bar{\phi}] = V''(\bar{\phi})$ at $p^2 = 0$)

In hyperbolic space the smallest eigenvalue of the Laplacian is $\nu_0 = \rho^2/a^2 > 0$ (with eigenfunction $\varphi_{0,l}$) ⇒ $\Gamma^{(2)}[\bar{\phi}] \cdot \varphi_{0,l} \neq V''(\bar{\phi}) \cdot \varphi_{0,l}$

In agreement with the mean field approximation, in which the potential in the broken phase is not convex
No nontrivial fixed points

A simple truncation

$$\tilde{V}_k(\tilde{\phi}) = v_0(k) + v_2(k) \tilde{\phi}^2 + v_4(k) \tilde{\phi}^4$$

$$\downarrow$$

$$k \partial_k v_2 = -2 v_2 - 12 v_4 \frac{F(\mathcal{M})(\tilde{a})}{(1 + 2 v_2)^2}$$

$$k \partial_k v_4 = (d - 4) v_4 + 144 v_4^2 \frac{F(\mathcal{M})(\tilde{a})}{(1 + 2 v_2)^3}$$

$$\downarrow$$

$$k \partial_k v_2^* = k \partial_k v_4^* = 0 \quad \Rightarrow \quad v_2^* = \frac{4 - d}{2d - 32}, \quad v_4^* = \frac{12(d - 4)}{(d - 16)^3 F(\mathcal{M})(\tilde{a})}$$

Hyperboloid: $v_4^* \to \infty$ for $k \to 0$

$\Rightarrow$ only Gaussian fixed point $\Rightarrow$ mean field exponents
Conclusions

- Strong IR effects produce very different physics on spheres and hyperboloids.
- We have in these cases some general arguments (effective dimensionality, Ginzburg criterion) to give us some indications on what to expect.
- FRGE can be used to nicely derive such properties.
- Many possible calculations and extensions possible ($\eta$, large-$N$, other spaces...).
- Open question: can critical behavior be modified in a less trivial way by the background? (i.e. not explainable in terms of effective dimension).