# Classical and Quantum Aspects of Higher-Derivative Scalar Theories

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- In certain higher-derivative field theories scattering can take place at a length scale $r_*$ much larger than the typical scale $L_*$ of the nonrenormalizable terms in the Lagrangian.
  (Dvali, GIudice, Gomez, Kehagias, Pirtskhalava, Grojean...)
- The center-of-mass energy can be used to define the analogue of the Schwarzschild radius: classicalization radius $r_*$.
- If all scattering takes place at $r_* \gg L_*$, the fundamental scale $L_*$ is irrelevant and no UV completion of the theory is needed.
- The DBI theory is a typical theory that can support classicalons.
- I shall present a numerical study of the scenario.
- Classical solutions that describe shock fronts may also be relevant for scattering. These solutions also describe throats connecting two branes.

- The cubic Galileon theory describes the dynamics of the scalar mode that survives in the decoupling limit of the DGP model (Dvali, Gabadadze, Porrati).
- The action contains a higher-derivative term, cubic in the field $\pi(x)$, with a dimensionful coupling that sets the scale $\Lambda$ at which the theory becomes strongly coupled.
- $\Lambda \sim (m^2 M_{\rm Pl})^{1/3}$ with $m \sim H \sim M_5^3/M_{\rm Pl}^2$
- The action is invariant under the Galilean transformation $\pi(x) \to \pi(x) + b_\mu x^\mu + c$, up to surface terms.
- In the Galileon theory additional terms can also be present, but the theory is ghost-free: EOM is second order (Nicolis, Rattazzi, Trincherini).
- Nonlinearities become important below the Vainshtein radius $r_V \sim (M/\Lambda^3 M_{\rm Pl})^{1/3}$.
- Does this contruction survive quantum corrections?

- The DBI action corresponds to the simplest term of a theory of embedded surfaces.
- The effective theory of embedded surfaces can be used in order to reproduce the Galileon theory at low energies (de Rham, Tolley).
- How does the theory behave under renormalization?
- There is a connection with asymptotic safety in gravity.

## Outline

- A numerical study of classicalization.
- Classical solutions of higher-derivative theories that describe surfaces embedded in Minkowski space.
- Branes with throats or shock fronts. Brane annihilation. Classicalization?
- Connection with solutions of the Galileon theory. Vainshtein mechanism.
- Renormalization of the cubic Galileon theory.
- Suppression of quantum corrections by the Vainshtein mechanism.
- Renormalization of theories that describe surfaces.
- Renormalization-group evolution and asymptotic safety.

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## DBI model

- Lagrangian density ($\lambda = \pm L_*^4$)

  $\mathcal{L} = -\frac{1}{\lambda}\sqrt{1-\lambda\left(\partial_\mu\phi\right)^2}$ or $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{\lambda}{8}\left(\left(\partial_\mu\phi\right)^2\right)^2$
- Equation of motion

$$\partial^\mu\left[\partial_\mu\phi/\sqrt{1-\lambda\left(\partial_\nu\phi\right)^2}\right] = 0.$$

- Idealized scattering process: collapsing spherical wavepacket

$$\phi_0(t,r) = \frac{A}{r}\exp\left[-\frac{\left(r+t-r_0\right)^2}{a^2}\right].$$

- Perturbation theory (Dvali, Pirtskhalava): strong deformation at the classicalization radius

$$r_* \sim L_*\left(A^2L_*/a\right)^{1/3}.$$

- We have $r_* \gg L_*$ when the center-of-mass energy $\sqrt{s} \sim A^2/a$ is much larger than $1/L_*$.

## Alternative point of view

- With spherical symmetry, the equation of motion is

$$\left(1+\lambda\phi_r^2\right)\phi_{tt}-\left(1-\lambda\phi_t^2\right)\phi_{rr}-2\lambda\phi_r\phi_t\,\phi_{tr}=\frac{2\phi_r}{r}\left(1-\lambda\phi_t^2+\lambda\phi_r^2\right).$$

- This is a quasilinear second-order partial differential equation

$$\mathcal{A}(\phi_t,\phi_r)\,\phi_{tt}+\mathcal{B}(\phi_t,\phi_r)\,\phi_{tr}+\mathcal{C}(\phi_t,\phi_r)\,\phi_{rr}=\mathcal{D}(\phi_t,\phi_r,r),$$

with discriminant

$$\Delta=\frac{1}{4}(\mathcal{B}^2-4\mathcal{A}\mathcal{C})=1-\lambda\phi_t^2+\lambda\phi_r^2.$$

- $\Delta>0$: hyperbolic, $\Delta=0$: parabolic, $\Delta<0$: elliptic.
- Hyperbolic equations admit wave-like solutions, while elliptic ones do not support propagating solutions. Equations whose discriminant can change sign are of mixed type.
- If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are evaluated for the unperturbed configuration, the discriminant can vanish or change sign in the vicinity of the classicalization radius.

## Possible problems

- At some stage the solution develops a shock front. From this point on, the numerical integration cannot be continued, as the evolution of the shock depends on additional physical assumptions about its nature (discontinuities in the field configuration, or its derivatives).
- At some time a real solution ceases to exist within a certain range of $r$. This possibility is also apparent in exact analytical solutions.
- The equation of motion switches type within a range of $r$. When it becomes elliptic, its solution requires (Dirichlet or Neumann) boundary conditions on a closed contour. The scattering problem that we are considering cannot provide such conditions, as it is set up through Cauchly boundary conditions at the initial time. Boundary conditions on a closed contour would require the values of $\phi$ or its derivatives at times later than the time of interest.

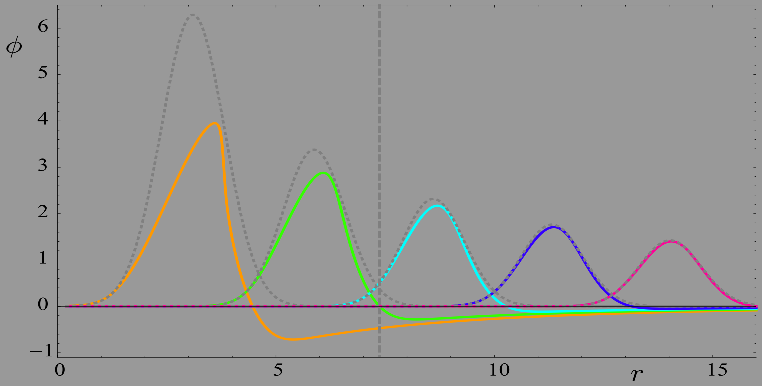


**Figure:** The nonlinear wavepacket at various times (solid lines) vs. the linear wavepacket (dotted lines), in the context of the DBI model with $\lambda = 1$. The initial wavepacket has $A = 20$, $a = 1$. The vertical dashed line denotes the classicalization radius.

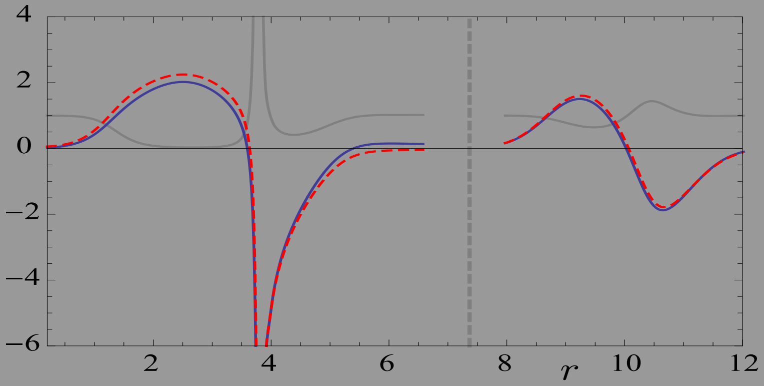


**Figure:** The derivatives $\phi_t$ (dashed) and $\phi_r$ (solid) of the nonlinear field, and the discriminant $\Delta$ (solid grey), at two different times, before and after the crossing of the classicalization radius. The model is the DBI model with $\lambda = 1$. The vertical dashed line denotes the classicalization radius.

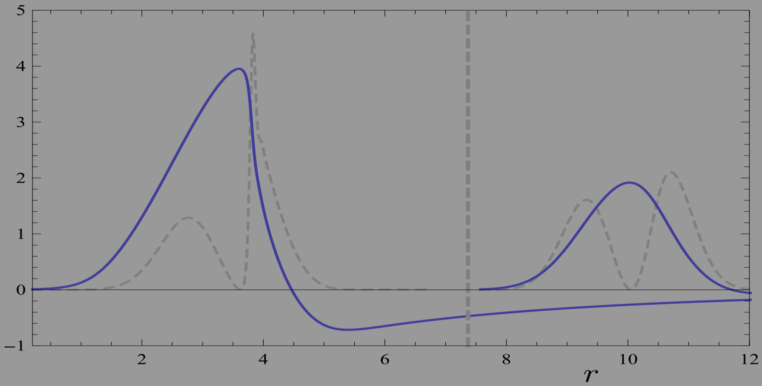


**Figure:** The nonlinear field $\phi$ (solid) and the product $4\pi r^2\rho$, with $\rho$ the energy density (dashed). The model is the DBI model with $\lambda = 1$. The vertical dashed line denotes the classicalization radius. The energy density is multiplied by $5 \times 10^{-4}$.

## Features

- The classicalization radius sets the scale for the onset of significant deformations of a collapsing classical configuration with large energy concentration in a central region.
- Similarity with the Vainshtein mechanism.
- Shock fronts develop during the scattering process at distances comparable to the classicalization radius.
- An observable feature of the classical evolution is the creation of an outgoing field configuration that extends far beyond the classicalization radius. However, the scattering during the classical evolution seems to be minimal.
- Within the DBI model ($\lambda > 0$) the collapsing wavepacket can approach distances $\sim L_* = |\lambda|^{1/4}$ before strong scattering appears.
- Within the "wrong"-sign DBI model ($\lambda < 0$), the scattering problem may not have real solutions over the whole space. What happens in the quantum theory?

## Brane effective action

- Consider a (3+1)-dimensional surface (brane) embedded in (4+1)-dimensional Minkowski space.
- Induced metric in the static gauge: $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \pi \, \partial_\nu \pi$
- Extrinsic curvature: $K_{\mu\nu} = -\partial_\mu \partial_\nu \pi / \sqrt{1 + (\partial\pi)^2}$.
- Leading terms in the effective action (de Rham, Tolley)

$$S_\lambda = -\lambda \int d^4x \sqrt{-g} = -\lambda \int d^4x \sqrt{1 + (\partial\pi)^2}$$

$$S_K = -M_5^3 \int d^4x \sqrt{-g}\, K = M_5^3 \int d^4x \, \left([\Pi] - \gamma^2 [\phi]\right)$$

$$S_R = (M_4^2/2) \int d^4x \sqrt{-g}\, R$$

$$= (M_4^2/2) \int d^4x \, \gamma \left([\Pi]^2 - [\Pi^2] + 2\gamma^2([\phi^2] - [\Pi][\phi])\right)$$

- Notation: $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, $\gamma = 1/\sqrt{-g} = 1/\sqrt{1 + (\partial\pi)^2}$, $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$, square brackets represent the trace, $[\phi^n] \equiv \partial\pi \cdot \Pi^n \cdot \partial\pi$.

## Galileon theory

- The Galileon theory can be obtained in the nonrelativistic limit $(\partial\pi)^2 \ll 1$.
- The action becomes

$$S^{NR} = \int d^4x \left\{ -\frac{\lambda}{2}(\partial\pi)^2 + \frac{M_5^3}{2}(\partial\pi)^2\Box\pi + \frac{M_4^2}{4}(\partial\pi)^2\left((\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2\right)\right\}$$

- Invariant under the Galilean symmetry $\delta\pi = c + v_\mu x^\mu$.
- The term of highest order in the Galileon theory, omitted here, can be obtained by including in the brane action the Gibbons-Hawking-York term associated with the Gauss-Bonnet term of $(4+1)$-dimensional gravity.
- Generalized Galileon or Horndeski (1974) theory: Up to second derivatives in the EOM, but no Galilean symmetry.

## Exact analytical solutions

- DBI action: surface area

$$\mathcal{L} = -\frac{1}{\lambda}\sqrt{1+\lambda\left(\partial_\mu \pi\right)^2}.$$

- Exact solutions ($c > 0$):

$$d\pi/dr = \pm\frac{c}{\sqrt{r^4-\lambda c^2}}.$$

- $\lambda < 0$: Field configuration induced by a $\delta$-function source resulting from the concentration of energy around $r = 0$ (Dvali, Giudice, Gomez, Kehagias).
  Static classicalons: Similar to BIons (Gibbons).
- $\lambda > 0$: The solutions have a square-root singluarity at $r_s = \lambda^{1/4}c^{1/2}$. They can be joined smoothly in a continuous double-valued function of $r$ for $r \geq r_s$: throat connecting two (3)-branes embedded in $(4+1)$-dimensional Minkowski space.
  The field $\pi$ corresponds to the Goldstone mode of the broken translational invariance (Gibbons).

- Exact dynamical solutions $\pi = \pi(z)$, with $z = r^2 - t^2$ , satisfying

$$d\pi/dz = \pm \frac{1}{\sqrt{cz^4 - 4\lambda z}}.$$

- For both signs of $\lambda$, the solutions display square-root singularities at $z = 0$ and at the value $z_s$ that satisfies $z_s^3 = 4\lambda/c \quad (c > 0)$.
- For $\lambda > 0$, the singularity is located at $r_s^2 = t^2 + (4\lambda/c)^{1/3}$.
- Shock fronts associated with meson production (Heisenberg).
- They display strong scattering at a length scale $\sim (4\lambda/c)^{1/6}$. Classicalization?
- These are particular solutions. The general solution of the equation of motion does not display scattering at large length scales.

## Brane picture

- By joining solutions with opposite signs, one can create evolving networks of throats or wormholes, connecting two branes.
- When the throat expands, the worldvolume of the part of the branes that is eliminated reappears as energy distributed over the remaining part of the branes.
- Interpretation: Annihilating branes, bouncing Universe.
- The solutions can be generalized in the context of higher-derivative effective actions that describe surfaces embedded in Minkowski space.

## Equations of motion

- Brane theory

$$\lambda\gamma\left\{([\Pi]-\gamma^2[\phi]\right\}-M_5^3\gamma^2\left\{[\Pi]^2-[\Pi^2]+2\gamma^2\left([\phi^2]-[\Pi][\phi]\right)\right\}$$
$$-\frac{M_4^2}{2}\gamma^3\left\{[\Pi]^3+2[\Pi^3]-3[\Pi][\Pi^2]\right.$$
$$\left.+3\gamma^2\left(2\left([\Pi][\phi^2]-[\phi^3]\right)-\left([\Pi]^2-[\Pi^2]\right)[\phi]\right)\right\}=0.$$

- Galileon theory

$$\lambda[\Pi]-M_5^3\left([\Pi]^2-[\Pi^2]\right)-\frac{M_4^2}{2}\left([\Pi]^3+2[\Pi^3]-3[\Pi][\Pi^2]\right)=0,$$

Notation: $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$, $\gamma=1/\sqrt{-g}=1/\sqrt{1+(\partial\pi)^2}$, $\Pi_{\mu\nu}=\partial_\mu\partial_\nu\pi$, square brackets: trace, $[\phi^n]\equiv\partial\pi\cdot\Pi^n\cdot\partial\pi$.

- They have solutions of the form
- $\pi=\pi(r^2)$
- $\pi=\pi(r^2-t^2)$.

## Solutions $\pi(w)$, with $w = r^2$

- Brane theory

  a) For $M_5 = M_4 = 0$ and $c > 0$

$$\pi_w = \pm \frac{c}{\sqrt{w^3 - 4c^2 w}},$$

  b) For $M_4 = 0$ and $\kappa = 12M_5^3/\lambda$

$$\pi_w = \frac{\pm\sqrt{6}c}{\sqrt{3w^3 + \sqrt{9w^6 \mp 12\kappa c\, w^{9/2}} - 24c^2 w \mp 2\kappa c\, w^{3/2}}}.$$

- Galileon theory

  For $M_4 = 0$ and $\kappa = 12M_5^3/\lambda$

$$\pi_w = \frac{3}{2\kappa}\left(1 - \sqrt{1 \mp \frac{4}{3}\frac{\kappa c}{w^{3/2}}}\right).$$

  Vainshtein mechanism: $r_V \sim (\kappa c)^{1/3}$

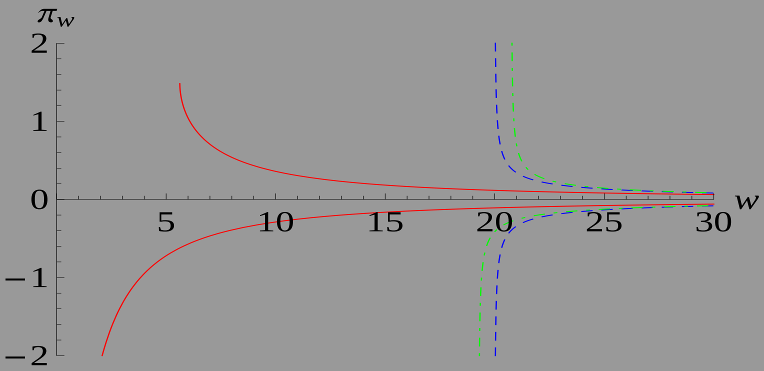


**Figure:** The solution $\pi_w = d\pi/dw$ for: a) The brane theory with $M_4 = M_5 = 0$, $c = 10$ (blue). b) The brane theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 1$, $c = 10$ (green). c) The Galileon theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 1$, $c = 10$ (red).

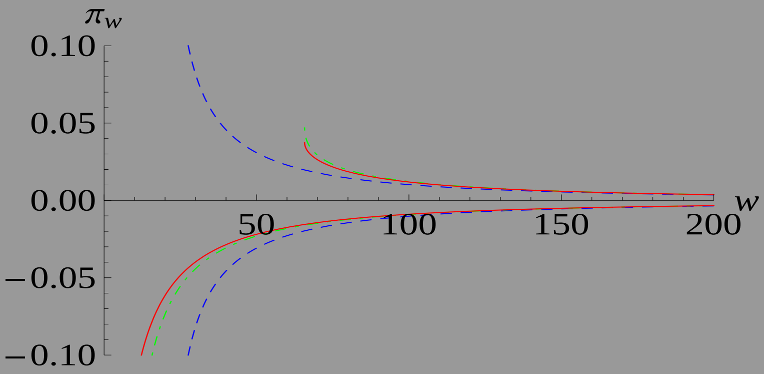


**Figure:** The solution $\pi_w = d\pi/dw$ for: a) The brane theory with $M_4 = M_5 = 0$, $c = 10$ (blue). b) The brane theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 40$, $c = 10$ (green). c) The Galileon theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 40$, $c = 10$ (red).

## Solutions $\pi(z)$, with $z = r^2 - t^2$

- Brane theory
  a) For $M_5 = M_4 = 0$ and $c > 0$

$$\pi_z = \pm \frac{c}{\sqrt{z^4 - 4c^2 z}}.$$

  b) For $M_4 = 0$ and $\kappa = 12M_5^3/\lambda$

$$\pi_z = \frac{\pm\sqrt{2}c}{\sqrt{z^4 + z^3\sqrt{z^2 \mp 2\kappa c} - 8c^2 z \mp \kappa c\, z^2}}.$$

- Galileon theory
  For $M_4 = 0$ and $\kappa = 12M_5^3/\lambda$

$$\pi_z = \frac{1}{\kappa}\left(1 - \sqrt{1 \pm \frac{2\kappa c}{z^2}}\right).$$

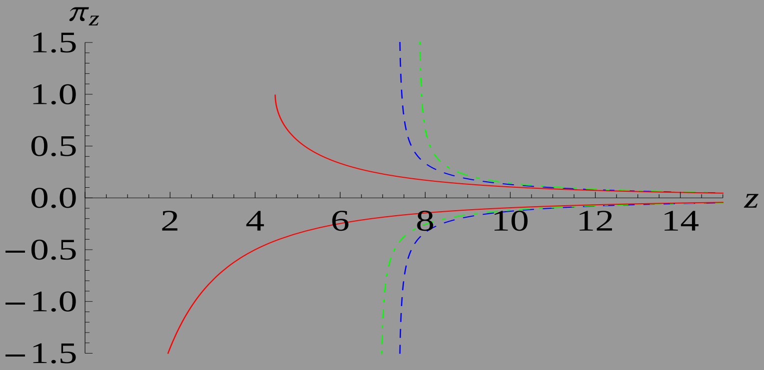


**Figure:** The solution $\pi_z = d\pi/dz$ for: a) The brane theory with $M_4 = M_5 = 0$, $c = 10$ (blue). b) The brane theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 1$, $c = 10$ (green). c) The Galileon theory with $M_4 = 0$, $12M_5^3/\lambda = \kappa = 1$, $c = 10$ (red).

- The throat solutions of the DBI theory can be generalized to solutions of the (quantum corrected) brane theory.
- The Galileon theory reproduces correctly the shape of the throats at large distances, but fails to do so at short distances.
- The solutions of the brane and Galileon theories coincide in the formal limits $\kappa c \to \infty$ with $c$ fixed, or $c \to 0$, with $\kappa c$ fixed.
- Similar solutions exist in the context of the generalized Galileon theory, and in particular in theories with kinetic gravity braiding.
- Possible cosmological applications: brane annihilation, bouncing Universe.

## Renormalization of the Galileon theory

- $$S=\int d^4x\left\{\frac{\mu}{2}(\partial\pi)^2-\frac{\nu}{2}(\partial\pi)^2\Box\pi+\frac{\bar{\kappa}}{4}(\partial\pi)^2\left((\Box\pi)^2-(\partial_\mu\partial_\nu\pi)^2\right)\right\}.$$
- If a momentum cutoff is used, of the order of the fundamental scale $\Lambda$ of the theory, and the couplings are taken of order $\Lambda$, the one-loop effective action of the Galileon theory is, schematically, (Luty, Porrati, Nicolis, Rattazzi)

$$\Gamma_1\sim\int d^4x\sum_m\left[\Lambda^4+\Lambda^2\partial^2+\partial^4\log\left(\frac{\partial^2}{\Lambda^2}\right)\right]\left(\frac{\partial^2\pi}{\Lambda^3}\right)^m.$$

- Non-renormalization of the Galileon couplings (de Rham, Gabadadze, Heisenberg, Pirtskhalava, Hinterbichler, Trodden, Wesley).
- Explicit one-loop calculation using dimensional regularization (Paula Netto, Shapiro).

## One-loop corrections to the cubic Galileon

- Tree-level action in Euclidean $d$-dimensional space

$$S = \int d^d x \left\{ \frac{\mu}{2}(\partial\pi)^2 - \frac{\nu}{2}(\partial\pi)^2 \Box\pi \right\}.$$

- Field fluctuation $\delta\pi$ around the background $\pi$. The quadratic part is

$$S^{(2)} = \int d^d x \left\{ -\frac{\mu}{2}\delta\pi\Box\delta\pi + \frac{\nu}{2}\delta\pi \left[ 2(\Box\pi)\Box\delta\pi - 2(\partial^\mu\partial^\nu\pi)\partial_\mu\partial_\nu\delta\pi \right] \right\}.$$

- Define

$$K = -\mu\Box \qquad \Sigma_1 = 2\nu(\Box\pi)\,\Box \qquad \Sigma_2 = -2\nu(\partial_\mu\partial_\nu\pi)\,\partial^\mu\partial^\nu$$

- One-loop contribution to the effective action

$$\Gamma_1 = \frac{1}{2}\mathrm{tr}\,\log\left(K + \Sigma_1 + \Sigma_2\right) = \frac{1}{2}\mathrm{tr}\,\log\left(1 + \Sigma_1 K^{-1} + \Sigma_2 K^{-1}\right) + \mathcal{N}.$$

- Expanding the logarithm

$$\mathrm{tr}\left(\Sigma_1 K^{-1}\Sigma_1 K^{-1}\right) = 4\frac{\nu^2}{\mu^2}(2\pi)^d \int d^dk\, k^4 \tilde{\pi}(k)\tilde{\pi}(-k) \int \frac{d^dp}{(2\pi)^d}$$

$$\mathrm{tr}\left(\Sigma_1 K^{-1}\Sigma_2 K^{-1}\right) = -4\frac{\nu^2}{\mu^2}(2\pi)^d \int d^dk\, k^4 \tilde{\pi}(k)\tilde{\pi}(-k)\frac{1}{d} \int \frac{d^dp}{(2\pi)^d}$$

$$\begin{aligned}\mathrm{tr}\left(\Sigma_2 K^{-1}\Sigma_2 K^{-1}\right) = 4\frac{\nu^2}{\mu^2}(2\pi)^d \int d^dk\, \tilde{\pi}(k)\tilde{\pi}(-k)\Bigg\{ &\frac{3}{d(d+2)}k^4 \int \frac{d^dp}{(2\pi)^d} \\ &+\frac{(d-8)(d-1)}{d(d+2)(d+4)}k^6 \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2} \\ &-\frac{(d-24)(d-2)(d-1)}{d(d+2)(d+4)(d+6)}k^8 \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^4}\Bigg\}.\end{aligned}$$

- Putting everything together, we obtain in position space, the one-loop correction to the effective action

$$\Gamma_1 = \frac{\nu^2}{\mu^2}\int d^dx\, \pi(x)\Bigg\{ -\frac{d^2-1}{d(d+2)}\left(\int \frac{d^dp}{(2\pi)^d}\right)\Box^2 + \frac{(d-8)(d-1)}{d(d+2)(d+4)}\left(\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\right)\Box^3 + \frac{(d-24)(d-2)(d-1)}{d(d+2)(d+4)(d+6)}\left(\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^4}\right)\Box^4 \Bigg\}\pi(x).$$

- The momentum integrals are defined with UV and IR cutoffs.
- If dimensional regularization near $d=4$ is used, the first two terms are absent. The third one corresponds to a counterterm $\sim 1/\epsilon$ (Paula Netto, Shapiro).
- No corrections to the Galileon couplings.
- Terms outside the Galileon theory are generated.

## Heat-kernel approach around a non-trivial background

- Our task is to evaluate the one-loop effective action

$$\Gamma_1 = \frac{1}{2}\mathrm{tr}\log\Delta$$

with $(\mu = 0)$ $\quad \Delta = -\Box + 2\nu\,(\Box\pi)\,\Box - 2\nu\,(\partial_\mu\partial_\nu\pi)\,\partial^\mu\partial^\nu$
around the background $(w = r^2)$

$$\pi'_{cl}(w) = \frac{1}{8\nu}\left(1 - \sqrt{1 + \frac{16\nu c}{w^{3/2}}}\right).$$

- The propagation of classical fluctuations in suppressed below the Vainshtein radius $r_V \sim (\nu c)^{1/3}$, where $\nu\Box\pi_{cl} \sim (r_V/r)^{3/2} \gg 1$.
- What about the quantum fluctuations?

- Calculate the heat kernel

$$h(x,x',\epsilon) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx'} e^{-\epsilon\Delta} e^{ikx}$$

- The one-loop effective action can be obtained as

$$\Gamma_1 = -\frac{1}{2}\int_{1/\Lambda^2}^{\infty} \frac{d\epsilon}{\epsilon} \int d^4x\, h(x,x,\epsilon).$$

- The diagonal part of the heat kernel becomes

$$\begin{aligned} h(x,x,\epsilon) \;&=\; \int \frac{d^4k}{(2\pi)^4}\frac{1}{\epsilon^2}\exp\Big\{-k^2+2i\sqrt{\epsilon}k^\mu\partial_\mu+\epsilon\Box \\ &+2\nu\Box\pi\left(k^2-2i\sqrt{\epsilon}k^\mu\partial_\mu-\epsilon\Box\right) \\ &-2\nu\partial_\mu\partial_\nu\pi\left(k^\mu k^\nu-2i\sqrt{\epsilon}k^\mu\partial^\nu-\epsilon\partial^\mu\partial^\nu\right)\Big\} \end{aligned}$$

- Expand in $\epsilon$ and $\nu$.
- The leading perturbative result is reproduced:

$$h(x,x,\epsilon)=\frac{15}{32\pi^2\epsilon^2}\nu^2(\Box\pi)^2$$

$$\Gamma_1^{(2)}=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{d\epsilon}{\epsilon}\int d^4x\, h(x,x,\epsilon)=-\frac{15}{128\pi^2}\nu^2\Lambda^4\int d^4x\,(\Box\pi)^2.$$

- Pertubation theory:

$$\Gamma_1 \sim \int d^4x \sum_m \left[ \Lambda^4 + \Lambda^2 \partial^2 + \partial^4 \log\left(\frac{\partial^2}{\Lambda^2}\right)\right] \left(\nu \partial^2 \pi\right)^m .$$

- Split the field as $\pi = \pi_{cl} + \delta\pi$.
- The action includes terms $\sim \nu^2 \Lambda^4 (\nu \Box \pi_{cl})^n (\Box \delta\pi)^2$
- But $\nu \Box \pi_{cl} \sim (r_V/r)^{3/2} \gg 1$ below the Vainshtein radius.

## Modified heat kernel

- The exponent of the heat-kernel is ( $\pi = \pi_{cl} + \delta\pi$)

$$
\begin{aligned}
F &= -G_{\mu\nu}k^\mu k^\nu - (1-2\nu\Box\pi_{cl})D_\epsilon(k) + 2\nu\partial_\mu\partial_\nu\pi_{cl}\, L_\epsilon^{\mu\nu}(k) \\
&\quad +2\nu\Box\delta\pi\left(k^2 + D_\epsilon(k)\right) + 2\nu\partial_\mu\partial_\nu\delta\pi\left(-k^\mu k^\nu + L_\epsilon^{\mu\nu}(k)\right)
\end{aligned}
$$

with the "metric" $G_{\mu\nu} = g_{\mu\nu} - 2\nu\Box\pi_{cl}\, g_{\mu\nu} + 2\nu\partial_\mu\partial_\nu\pi_{cl}$ and

$$
\begin{aligned}
D_\epsilon(k) &= -2i\sqrt{\epsilon}k^\mu\partial_\mu - \epsilon\Box \\
L_\epsilon^{\mu\nu}(k) &= 2i\sqrt{\epsilon}k^\mu\partial^\nu + \epsilon\partial^\mu\partial^\nu .
\end{aligned}
$$

- Make the "metric" $G_{\mu\nu}$ trivial by rescaling $k^\mu = S^\mu_{\ \nu}k'^\nu$, with

$$
S^\mu_{\ \rho}G_{\mu\nu}S^\nu_{\ \sigma} = g_{\rho\sigma} .
$$

- The most divergent term quadratic in $\delta\pi$ in the heat kernel is

$$
\begin{aligned}
h(x,x,\epsilon) &= \int \frac{d^4k}{(2\pi)^4}(\det S)\frac{1}{2\epsilon^2}e^{-k^2}\Big(2\nu\Box\delta\pi(Sk)^2 \\
&\quad +2\nu\partial_\mu\partial_\nu\delta\pi\left(-Sk^\mu Sk^\nu\right)\Big)^2 .
\end{aligned}
$$

- On the background that realizes the Vainshtein mechanism

$$\Gamma_1^{(2)} = -\frac{1}{128\pi^2}\nu^2\Lambda^4\int d^4x\left(\left(\Box\delta\pi\right)^2 P(r^2) - 2(\Box\delta\pi)(\partial_\mu\partial_\nu\delta\pi)\,V^{\mu\nu}(r^2) + (\partial_\mu\partial_\nu\delta\pi)\,(\partial_\rho\partial_\sigma\delta\pi)\;W^{\mu\nu\rho\sigma}(r^2)\right).$$

with $P(r^2), V^{\mu\nu}(r^2), W^{\mu\nu\rho\sigma}(r^2) \sim (r/r_V)^6$ and $r_v \sim (\nu c)^{1/3}$.

- The general structure of the effective action is

$$\Gamma_1^{(2)} = \nu^2 \int d^4x$$
$$\Bigg[\Lambda^4 \left( c_0 \frac{r^6}{R_V^6} \left(\delta\pi\partial^4\delta\pi\right)\right)$$
$$+\Lambda^2 \left( c_{1a}\frac{r^{5/2}}{R_V^{9/2}} \left(\delta\pi\partial^4\delta\pi\right) + c_{1b}\frac{r^{7/2}}{R_V^{9/2}} \left(\delta\pi\partial^5\delta\pi\right) + c_{1c}\frac{r^{9/2}}{R_V^{9/2}} \left(\delta\pi\partial^6\delta\pi\right)\right)$$
$$+\log(\Lambda/\mu)\Bigg( c_{2a}\frac{1}{rR_V^3} \left(\delta\pi\partial^4\delta\pi\right) + c_{2b}\frac{1}{R_V^3} \left(\delta\pi\partial^5\delta\pi\right)$$
$$+c_{2c}\frac{r}{R_V^3} \left(\delta\pi\partial^6\delta\pi\right) + c_{2d}\frac{r^2}{R_V^3} \left(\delta\pi\partial^7\delta\pi\right) + c_{2e}\frac{r^3}{R_V^3} \left(\delta\pi\partial^8\delta\pi\right)\Bigg)\Bigg].$$

## Brane effective action

- Leading terms in the effective action (Euclidean space)

$$S_\mu = \mu \int d^4x \sqrt{g} = \mu \int d^4x \sqrt{1+(\partial\pi)^2}$$

$$S_\nu = \nu \int d^4x \sqrt{g}\, K = -\nu \int d^4x \ \left([\Pi] - \gamma^2[\phi]\right)$$

$$S_\kappa = (\kappa/2) \int d^4x \sqrt{g}\, K^2 = (\kappa/2) \int d^4x \sqrt{g} \left([\Pi] - \gamma^2[\phi]\right)^2$$

$$S_{\bar\kappa} = (\bar\kappa/2) \int d^4x \sqrt{g}\, R$$

$$= (\bar\kappa/2) \int d^4x\, \gamma \left([\Pi]^2 - [\Pi^2] + 2\gamma^2([\phi^2] - [\Pi][\phi])\right)$$

- The action $S_\lambda + S_\nu + S_{\bar\kappa}$ belongs to the generalized Galileon (Horndeski) class. It reduces to the Galileon theory in the nonrelativistic limit.
- The first Gauss-Codazzi equation gives $R = K^2 - K^{\mu\nu}K_{\mu\nu}$.
- The term $S_\kappa$ becomes $\sim \pi\Box^2\pi$ in the nonrelativistic limit $(\partial\pi)^2 \ll 1$. This term is not included in the Galileon theory.

## One-loop corrections to the brane action

- Brane theory with $\mu = \mu_0$, $\nu = \nu_0$, $\kappa = \bar{\kappa} = 0$
- The one-loop correction is

$$\Gamma_1 = \frac{1}{2}\mathrm{tr}\log\left(\frac{\delta^2 S}{\delta\pi^2}\right),$$

with

$$\frac{\delta^2 S}{\delta\pi^2} = \mu_0\Delta + \nu_0 V^{\mu\nu}\nabla_\mu\nabla_\nu + \mu_0 U + \mathcal{O}(K^4, \nabla K),$$

- Covariant derivatives are evaluated with the induced metric $g^{\mu\nu}$. $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$, $V^{\mu\nu} = 2(K^{\mu\nu} - Kg^{\mu\nu})$, $U = K^2 - K^{\mu\nu}K_{\mu\nu} = R$.
- Expanding the logarithm

$$\begin{aligned}\Gamma_1 \quad = \quad & \frac{1}{2}\mathrm{tr}\log(\mu_0\Delta) + \frac{1}{2}\frac{\nu_0}{\mu_0}\mathrm{tr}\left(\frac{1}{\Delta}V^{\mu\nu}\nabla_\mu\nabla_\nu\right) + \frac{1}{2}\mathrm{tr}\left(\frac{1}{\Delta}U\right)\\ & -\frac{1}{4}\frac{\nu_0^2}{\mu_0^2}\mathrm{tr}\left(\frac{1}{\Delta}V^{\mu\nu}\nabla_\mu\nabla_\nu\frac{1}{\Delta}V^{\alpha\beta}\nabla_\alpha\nabla_\beta\right) + \mathcal{O}(K^4, \nabla K).\end{aligned}$$

- Evaluation of the traces with heat kernel techniques

$$\mathrm{tr}\log(\mu_0\Delta) = \left(\int \frac{d^dp}{(2\pi)^d}\ln\left(\mu_0 p^2\right)\right)\int d^dx\sqrt{g}$$
$$+\frac{d-2}{12}\left(\int \frac{d^dp}{(2\pi)^d}\frac{\ln\left(\mu_0 p^2\right)}{p^2}\right)\int d^dx\sqrt{g}R$$

$$\mathrm{tr}\left(\frac{1}{\Delta}V^{\mu\nu}\nabla_\mu\nabla_\nu\right) = \frac{d-1}{d}\left(\int \frac{d^dp}{(2\pi)^d}\right)\int d^dx\sqrt{g}K$$

$$\mathrm{tr}\left(\frac{1}{\Delta}U\right) = \left(\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\right)\int d^dx\sqrt{g}R$$

$$\mathrm{tr}\left(\frac{1}{\Delta}V^{\mu\nu}\nabla_\mu\nabla_\nu\frac{1}{\Delta}V^{\alpha\beta}\nabla_\alpha\nabla_\beta\right) = \frac{4(d^2-1)}{d(d+2)}\left(\int \frac{d^dp}{(2\pi)^d}\right)\int d^dx\sqrt{g}K^2$$
$$-\frac{8}{d(d+2)}\left(\int \frac{d^dp}{(2\pi)^d}\right)\int d^dx\sqrt{g}R\,.$$

- The couplings at one-loop level are

$$
\begin{aligned}
\mu &= \mu_0 + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln\left(\mu_0 p^2\right) \\
\nu &= \nu_0 + \frac{d-1}{2d}\frac{\nu_0}{\mu_0}\int \frac{d^d p}{(2\pi)^d} \\
\kappa &= -\frac{2(d^2-1)}{d(d+2)}\frac{\nu_0^2}{\mu_0^2}\int \frac{d^d p}{(2\pi)^d} \\
\bar{\kappa} &= \frac{4}{d(d+2)}\frac{\nu_0^2}{\mu_0^2}\int \frac{d^d p}{(2\pi)^d} + \int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2} + \frac{d-2}{12}\int \frac{d^d p}{(2\pi)^d}\frac{\ln\left(\mu_0 p^2\right)}{p^2}.
\end{aligned}
$$

- Terms outside the Galileon theory are generated.
- The couplings of the brane (generalized Galileon) theory are renormalized.

## Renormalization-group evolution

- We use the Wilsonian (exact) renormalization group.
- Heat-kernel techniques.
- The evolution equations for the couplings take the form

$$\partial_t \mu_k = \frac{k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)} \frac{2\kappa_k k^2+\mu_k}{\kappa_k k^2+\mu_k}$$

$$\partial_t \nu_k = -\frac{k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}(d-1)\frac{(2\kappa_k k^2+\mu_k)\nu_k}{(\kappa_k k^2+\mu_k)^2}$$

$$\partial_t \kappa_k = \frac{2k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d+4}{4}\frac{(2\kappa_k k^2+\mu_k)\kappa_k}{(\kappa_k k^2+\mu_k)^2}+\frac{4(d^2-1)}{d+4}\frac{(2\kappa_k k^2+\mu_k)\nu_k^2}{(\kappa_k k^2+\mu_k)^3}\right\}$$

$$\partial_t \bar{\kappa}_k = \frac{k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+2)}{12}\frac{2\kappa_k k^2+\mu_k}{(\kappa_k k^2+\mu_k)k^2}-\frac{16}{d+4}\frac{(2\kappa_k k^2+\mu_k)\nu_k^2}{(\kappa_k k^2+\mu_k)^3}\right.$$

$$\left.-\left[(d+2)\frac{\mu_k}{k^2}+2d\kappa_k+\frac{3(d-2)}{2}\bar{\kappa}_k\right]\frac{2\kappa_k k^2+\mu_k}{(\kappa_k k^2+\mu_k)^2}\right\}.$$

- We can obtain the $\beta$-functions of $\kappa_k$, $\bar{\kappa}_k$ for two-dimensional fluid membranes for which the volume (now area) term is considered subleading.
- We set $d = 2$, $\mu_k = \nu_k = 0$ and obtain

$$\partial_t \kappa_k = \frac{3}{4\pi}, \qquad \partial_t \bar{\kappa}_k = -\frac{5}{6\pi}. \tag{1}$$

- These expressions reproduce known results (Polyakov, Kleinert, Forster) for the renormalization of the bending and Gaussian rigidities of fluctuating membranes in a three-dimensional bulk space.

## Asymptotic safety

- Consider the theory with $d = 4$, $\nu = \kappa = 0$. It includes a cosmological constant and an Einstein term.
- Define the dimensionless cosmological and Newton's constants through

$$\frac{\mu_k}{k^4} = \frac{\Lambda_k}{8\pi G_k}, \qquad \frac{\bar{\kappa}_k}{k^2} = -\frac{1}{8\pi G_k}. \tag{2}$$

- Their scale dependence is given by

$$\partial_t \Lambda_k = -2\Lambda_k + \frac{1}{6\pi} G_k (3 - 2\Lambda_k) \tag{3}$$

$$\partial_t G_k = 2G_k + \frac{1}{12\pi} \frac{G_k^2}{\Lambda_k} (3 - 4\Lambda_k). \tag{4}$$

- This system of equations has two fixed points at which the $\beta$-functions vanish:
  a) the Gaussian one, at $\Lambda_k = G_k = 0$, and
  b) a nontrivial one, at $\Lambda_k = 9/8$, $G_k = 18\pi$.
- The flow diagram is similar to the scenario of asymptotic safety.

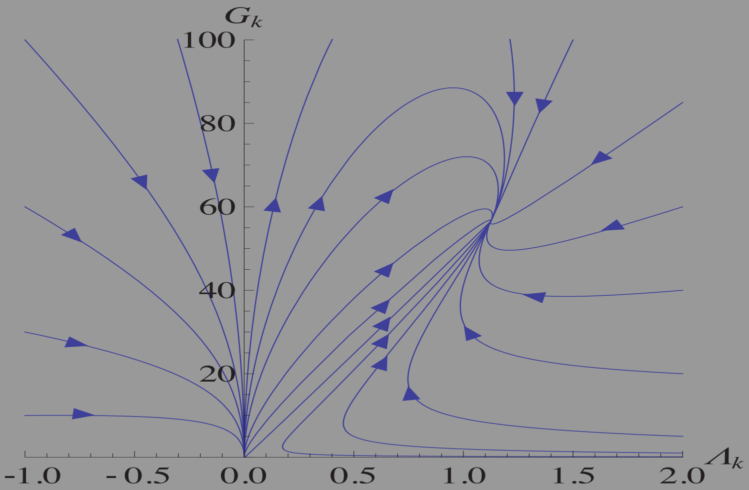


**Figure:** The flow diagram.

## Conclusions

- The DBI action and its generalizations have exact classical solutions that can be interpreted as shock fronts that scatter at length scales much larger than the fundamental scale (classicalization). This is possible for specific initial conditions.
- The same solution can be interpreted as wormholes or throats connecting a pair of branes.
- They can also be viewed as bouncing Universe solutions. Cosmological applications?
- The couplings of the Galileon theory do not get renormalized. However, the Galileon theory is not stable under quantum corrections. Additional terms are generated.
- Quantum corrections are suppressed below the Vainshtein radius.
- The nonrenormalization of couplings is not a feature of the generalized Galileon theories.
- The brane theory displays RG evolution very similar to that in the asymptotic safety scenario.