Black holes, Holography and Thermodynamics of Gauge Theories

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- Duality between a five-dimensional AdS-Schwarzschild geometry and a four-dimensional thermalized, strongly coupled CFT
- The CFT "lives" on the boundary of AdS
- Many of the deduced properties of the CFT are generic for strongly coupled theories (QCD)
- Interesting example: Hydrodynamic properties of CFTs on flat or Bjorken geometries $(\eta/s = 1/4\pi)$
- AdS/CFT for a LFRW boundary
- Thermodynamic properties of CFT on cosmological backgrounds
- Relevant five-dimensional geometry: AdS-Schwazschild in isotropic coordinates

Outline

- AdS-Schwazschild with static boundary
- Time-dependent boundary
- Temperature
- Entropy
- Comments

P. Apostolopoulos, G. Siopsis, N. T. :
arxiv:0809.3505[hep-th], Phys. Rev. Lett. 102 (2009) 151301

N. T. :
arxiv:0905.2763[hep-th], JHEP 1003 (2010) 040

AdS-Schwarzschild

- Metric in Schwarzschild coordinates:

$$ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega_k^2\,, \qquad f(r) = r^2 + k - \mu/r^2\,, \quad (1)$$

where $k = +1, 0, 1.$ for spherical, flat and hyperbolic horizons.
Hawking temperature and mass of the BH:

$$T = \frac{2r_e^2 + k}{2\pi r_e}\,, \qquad E = \frac{3V_k}{16\pi G_5} r_e^2 (r_e^2 + k)\,, \quad (2)$$

where r_e is the radius of the event horizon ($f(r_e) = 0$).

- $k = 1$ (spherical horizon):
  - For $\mu \gg 1$, we have $T \sim \mu^{1/4}/\pi$.
  - For $\mu \ll 1$, we have $T \sim 1/(2\pi\,\mu^{1/2})$.
  - No black holes with T below $\sqrt{2}/\pi$.
  - For larger T, the lower-mass black hole is unstable.
  - The larger-mass solution is dual to the high-temperature deconfined phase of the gauge theory (Witten).

- Metric in isotropic coordinates:

$$z^4 = \frac{16}{k^2+4\mu}\,\frac{r^2+\frac{k}{2}-r\sqrt{f(r)}}{r^2+\frac{k}{2}+r\sqrt{f(r)}}\,. \qquad (3)$$

- Invert:

$$r^2 = \frac{\alpha+\beta z^2+\gamma z^4}{z^2}\,, \quad \alpha = 1\,, \quad \beta = -\frac{k}{2}\,, \quad \gamma = \frac{k^2+4\mu}{16}\,. \qquad (4)$$

- The metric becomes

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \frac{\left(1-\gamma z^4\right)^2}{1+\beta z^2+\gamma z^4}dt^2 + \left(1+\beta z^2+\gamma z^4\right)d\Omega_k^2\right]\,. \qquad (5)$$

- For the Schwazschild geometry, the isotropic coordinates cover the two regions of the Kruskal-Szekeres plane outside the horizons (Einstein-Rosen bridge).
- The same happens for (τ, z) for AdS-Schwarzschild. The region (r_e, ∞) of r is covered twice by z taking values in $(0, \infty)$.

Temperature

- The temperature of the CFT is identified with the Hawking temperature of the BH.
- It can be calculated by switching to Euclidean time and eliminating the conical singularity at the horizon ($z_e = \gamma^{-1/4}$).
- T is given by

$$T = \frac{1}{\sqrt{2}\,\pi} \left(\frac{k^2 + 4\mu}{(k^2 + 4\mu)^{1/2} - k} \right)^{1/2} . \qquad (6)$$

Energy and pressure

- For a metric of the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{\mu\nu}dx^\mu dx^\nu\right], \quad g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \dots \tag{7}$$

the stress-energy tensor of the CFT is (Skenderis)

$$T^{(CFT)}_{\mu\nu} = \frac{1}{4\pi G_5}\Big\{ g^{(4)} - \frac{1}{2}g^{(2)}g^{(2)} + \frac{1}{4}\mathrm{Tr}\left[g^{(2)}\right] g^{(2)} - \frac{1}{8}\left(\left(\mathrm{Tr}\left[g^{(2)}\right]\right)^2 - \mathrm{Tr}\left[g^{(2)}g^{(2)}\right]\right) g^{(0)}\Big\}_{\mu\nu} \tag{8}$$

- This gives for a static background

$$T^{(CFT)}_{tt} = 3T^{(CFT)}_{ii} = \frac{3\left(k^2 + 4\mu\right)}{64\pi G_5}. \tag{9}$$

- The energy $E = T^{(CFT)}_{tt} V_k$ is larger than the mass of the black hole by a constant (Casimir energy) for a curved horizon ($k \neq 0$).

Entropy

- An intuitive derivation
  - Consider an infinitesimal variation of the parameter μ.
  - The volume V_k of the boundary is not affected.
  - The variation of μ generates a variation of the internal energy E of the system that can be attributed to a change of its entropy S.
  - Assuming that the process takes place sufficiently slowly, we have $dE = TdS$. A simple integration gives the entropy:

$$S = \frac{V_k}{4G_5} r_e^3. \tag{10}$$

  - The entropy is proportional to the surface of the event horizon.

LFRW boundary

- Consider a boundary with the form of a LFRW spacetime

$$g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)d\Omega_k^2 \; . \tag{11}$$

- The AdS-Schwarzschild metric can be written as

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \mathcal{N}^2(\tau,z)d\tau^2 + \mathcal{A}^2(\tau,z)d\Omega_k^2\right] \tag{12}$$

$$\mathcal{A}^2 = \alpha(\tau) + \beta(\tau)z^2 + \gamma(\tau)z^4 \; , \quad \mathcal{N} = \frac{\dot{\mathcal{A}}}{\dot{a}} \tag{13}$$

$$\alpha = a^2 \; , \qquad \beta = -\frac{\dot{a}^2 + k}{2} \; , \qquad \gamma = \frac{(\dot{a}^2 + k)^2 + 4\mu}{16a^2} \; . \tag{14}$$

Horizons

- The difference with the static case is that now the coordinate z spans a larger part of the Schwarzschild geometry.
- We have

$$(r')^2 = \frac{\dot{a}^2 + f(r)}{z^2}. \tag{15}$$

$\partial r/\partial z$ vanishes behind the static event horizon, at

$$z_m^2(\tau) = \frac{4a^2(\tau)}{\left((\dot{a}^2 + k)^2 + 4\mu\right)^{1/2}}. \tag{16}$$

- The region (r_m, ∞) of r is covered twice by the coordinate z taking values in $(0, \infty)$ (Einstein-Rosen bridge).
- An important surface is defined by $\mathcal{N}(\tau, z_a(\tau)) = 0$. It has

$$z_a^2(\tau) = \frac{4a^2(\tau)}{a\ddot{a} + \left((\dot{a}^2 - a\ddot{a} + k)^2 + 4\mu\right)^{1/2}}. \tag{17}$$

- Example: $a(\tau) = \lambda\tau$
  We have

$$r_m^2 = r_a^2 = \frac{1}{2}\left[-\tilde{k} + \left(\tilde{k}^2 + 4\mu\right)^{1/2}\right], \qquad (18)$$

  where $\tilde{k} = k + \lambda^2$. The static event horizon has

$$r_e^2 = \frac{1}{2}\left[-k + \left(k^2 + 4\mu\right)^{1/2}\right]. \qquad (19)$$

  It can be checked that $r_m = r_a \leq r_e$.
- For $\lambda = 0$ all three surfaces defined by r_m, r_a and r_e coincide.

- Apparent event horizon: Vanishing expansion of outgoing null geodesics.
- The out/ingoing null geodesics obey $(dz(\tau)/d\tau)_\pm = \mp\mathcal{N}(\tau,z)$ and define a surface of areal radius $A(\tau,z(\tau))/z(\tau) = r(\tau)$.
- The growth of the volume of this surface is proportional to the total time derivative of r along the light path, i.e. to

$$\left(\frac{dr}{d\tau}\right)_\pm = \dot{r} + r'\left(\frac{dz}{d\tau}\right)_\pm = \mathcal{N}\left(\frac{\dot{a}}{z} \mp r'\right), \qquad (20)$$

- The expansion of outgoing null geodesics vanishes on the surface parametrized by $z_a(\tau)$, for which $\mathcal{N} = 0$.

General expression for the temperature

- A thermalized system fluctuates at the microscopic level with a characteristic time scale of order $1/T$. For strongly coupled theories, this scale determines the interaction rates that keep the system thermalized.
- At the macroscopic level, the system (e.g. the Universe) may also evolve with a different, much longer, characteristic time scale.
- A temperature T can be assigned to the AdS-Schwarzschild solution with a time-dependent boundary when the variation of the scale factor is negligible at time intervals of order $1/T$.
- This requires $T \gg \dot{a}/a$.
- We can calculate the temperature as for the static case assuming that $a(\tau)$ and its time derivatives are constant.
- For $\mu \neq 0$ we have (with $\mathcal{A}_a = \mathcal{A}(\tau, z_a)$)

$$T(\tau) = \frac{1}{2\pi} \left| \frac{4a - \ddot{a} z_a^2}{\mathcal{A}_a z_a} \right| . \tag{21}$$

Zero acceleration

- For $a(\tau) = \lambda\tau$ the temperature is (with $\tilde{k} = k + \lambda^2$)

$$T = \frac{1}{\sqrt{2}\,\pi a} \left(\frac{\tilde{k}^2 + 4\mu}{\left(\tilde{k}^2 + 4\mu\right)^{1/2} - \tilde{k}} \right)^{1/2} . \qquad (22)$$

- The temperature is redshifted by the scale factor $a(\tau)$.
- The proportionality constant is not just the temperature in the static case. The two expressions differ by the change of the effective curvature $k \to \tilde{k} = k + \lambda^2$.
- This modification is natural as the total curvature of the boundary metric is proportional to $\tilde{k}$ for $a = \lambda\tau$.
- For sufficiently large λ we have $\tilde{k} > 0$ for any value of k. The behavior similar to that of a CFT on a sphere.
- The temperature diverges for $\lambda^4 \gg \mu$. This is analogous to the divergence of the temperature for a static background with $k = 1$ and $\mu \to 0$ (unstable configuration).

Non-zero acceleration

- Consider $a = \tau^\nu$ and constant ν for large τ. Also concentrate on the case $k = 0$.
- For $0 < \nu < 1$ the expansion is decelerating and for $\tau \to \infty$ we always have $\dot{a}^4 \ll \mu$. The curvature of the boundary geometry becomes negligible relative to the thermal energy of the CFT. In the same limit the apparent horizon approaches the event horizon. Ta becomes equal to the static temperature.
- For $\nu > 1$ the expansion is accelerating and at late times we have $\dot{a}^4 \gg \mu$. The apparent horizon deviates strongly from the event horizon and r_a eventually approaches zero. The product Ta diverges asymptotically for $\tau \to \infty$. The regime $\dot{a}^4 \gg \mu$ is equivalent to the $\mu \to 0$ limit for the static case with $k = 1$. For $\nu > 1$ the solution always approaches this regime at late times.

- Apart from the rescaling by a, there are two qualitatively different types of evolution.
  1. For $\nu < 1$ the CFT corresponds to a black hole with a mass that grows relative to the scale of the curvature induced by the expansion.
  2. For $\nu > 1$ the effective mass of the black hole seems to diminish and eventually vanish for $\tau \to \infty$.
- More precisely, the two quantities that characterize the different types of evolution are the Casimir and the thermal energy of the CFT. For $\nu < 1$ the Casimir energy becomes negligible at late times, while for $\nu > 1$ it dominates over the thermal energy.
- The deconfined phase of the CFT is dual to the large-mass solution with the same temperature. It seems reasonable to interpret the black-hole configuration with an accelerating boundary as dual to a CFT in the deconfined phase on an accelerating FLRW background geometry.
- It is also likely that such a configuration is unstable or unphysical. The form of the entropy gives more indications of this instability.

dS boundary

- For $a = \exp(H\tau)$, $k = 0$ we have a deSitter (dS) boundary.
- For $\mu \neq 0$ and large τ, the temperature quickly approaches the value $T = H/(\sqrt{2}\,\pi)$. This differs from the standard dS temperature by a factor $\sqrt{2}$.
- The configuration with $\mu \neq 0$ on a background with $a = \exp(H\tau)$ cannot evolve continuously to pure dS space.
- Set $a = \exp(H\tau)$, $k = 0$, $\mu = 0$ directly in the metric. This gives $\mathcal{N}(\tau, z) = 1 - H^2 z^2/4$. Despite the absence of a black hole, a conical singularity still exists at $z_a = 2/H$ for periodic Euclidean time.
- The location of the singularity is τ-independent for a dS boundary. No assumptions are needed about the relative size of T and H.
- The singularity can be eliminated for an appropriate value of the temperature. This gives $T = H/(2\pi)$.

Stress-energy tensor

- The stress-energy tensor of the dual CFT for a cosmological boundary is determined via holographic renormalization:

$$\langle (T^{(CFT)})_{\tau\tau} \rangle = \frac{3}{64\pi G_5} \, \frac{(\dot{a}^2+k)^2+4\mu}{a^4} \tag{23}$$

$$\langle (T^{(CFT)})^i_i \rangle = \frac{(\dot{a}^2+k)^2+4\mu-4a\ddot{a}(\dot{a}^2+k)}{64\pi G_5 a^4} \,, \tag{24}$$

- The conformal anomaly is

$$g^{(0)\mu\nu}\langle T^{(CFT)}_{\mu\nu}\rangle = -\frac{3\ddot{a}(\dot{a}^2+k)}{16\pi G_5 a^3} \,. \tag{25}$$

- The Casimir energy density is $\sim (\dot{a}^2+k)^2/a^4$ and reflects the total curvature of the boundary metric. For $\dot{a}^4 \gtrsim \mu$ it becomes comparable to or dominates over the thermal energy $\sim \mu/a^4$ of the CFT.

- The boundary geometry can be made dynamical if one introduces an Einstein term for the boundary metric and employs mixed boundary conditions.
- The resulting Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_4}{3}\left\{\frac{1}{16\pi G_5}\left[\frac{\left(\dot{a}^2+k\right)^2}{a^4} + \frac{4\mu}{a^4}\right] + \rho\right\} . \quad (26)$$

Entropy

- We generalize the intuitive derivation of the entropy in the static case.
- Consider an infinitesimal adiabatic variation of μ that takes place within a time interval that is sufficiently small for the evolution of $a(\tau)$ to be negligible. In contrast to the determination of the temperature, the required time for the variation can be made arbitrarily small by sending $d\mu \to 0$.
- The fundamental relation $dE + pdV = TdS$ can be employed for the determination of the entropy. The volume $a^3 V_k$ of the boundary remains constant, while the temperature is a function of μ (and a, $\dot{a}$, $\ddot{a}$).

- For the engropy of the CFT we find

$$S = \frac{V_k}{4\,G_5}\left(\frac{\mathcal{A}_a}{z_a}\right)^3 - \frac{3V_k}{32\,G_5}\frac{\left(\dot{a}^2+k\right)\ddot{a}}{a}\int^{z_a}\mathcal{A}_a(z)dz + F(a,\dot{a},\ddot{a}), \quad (27)$$

where

$$\mathcal{A}_a(z_a) = \left(2a^2 - \frac{\dot{a}^2+k+a\ddot{a}}{2}z_a^2 + \frac{(\dot{a}^2+k)\ddot{a}}{8a}\,z_a^4\right)^{1/2}. \quad (28)$$

- z_a is the location of the apparent horizon in isotropic coordinates, and $\mathcal{A}_a/z_a$ in Schwarzschild coordinates.
- $F(a,\dot{a},\ddot{a})$ is fixed by requiring that the entropy vanish for $\mu = 0$.

Specific cases

$a(\tau) = \lambda\tau$ (zero acceleration)

- The entropy is

$$S = \frac{V_k}{4G_5}\left(\frac{\mathcal{A}_a}{z_a}\right)^3 = \frac{V_k}{4G_5} r_a^3. \tag{29}$$

The areal distance of the apparent horizon r_a is constant:

$$r_a^2 = \frac{1}{2}\left[-\tilde{k} + \left(\tilde{k}^2 + 4\mu\right)^{1/2}\right], \tag{30}$$

where $\tilde{k} = k + \lambda^2$.

- The temperature is

$$T = \frac{1}{\sqrt{2}\,\pi a}\left(\frac{\tilde{k}^2 + 4\mu}{\left(\tilde{k}^2 + 4\mu\right)^{1/2} - \tilde{k}}\right)^{1/2}. \tag{31}$$

- For decelerating expansion ($\ddot{a} < 0$) the total entropy at late times is proportional to the area of the apparent horizon. The apparent horizon approaches the event horizon, while its area increases. Asymptotically, the two become identical. In this limit, the temperature and entropy density scale with simple powers of a.
- Example: $a = \lambda \tau^{\nu}$ with $\nu = 1/2$
  The leading terms in the late-time expansion are

$$S = \frac{V_k}{4G_5}\left[\mu^{3/4}\left(1 - \frac{3}{16}\frac{1}{\sqrt{\mu}\,\tau} + \frac{15}{512}\frac{1}{\mu\tau^2}\cdots\right) + \frac{3}{64}\frac{1}{\mu^{1/4}\,\tau^2}\left(1 + \frac{1}{16}\frac{1}{\sqrt{\mu}\,\tau} - \frac{29}{2560}\frac{1}{\mu\tau^2}\cdots\right)\right] \tag{32}$$

$$T = \frac{\mu^{1/4}}{\pi a}\left(1 + \frac{1}{16}\frac{1}{\sqrt{\mu}\tau} + \frac{15}{512}\frac{1}{\mu\tau^2}\cdots\right) \tag{33}$$

- For accelerating expansion ($\ddot{a} > 0$) the CFT entropy decreases with time. This is a sign that the corresponding configuration is unstable or unphysical. Our interpretation is that this configuration corresponds to a high-temperature CFT in the deconfined phase on an accelerating background. This must be unstable relative to the confined phase.
- Example: $a = \lambda \tau^{\nu}$ with $\nu = 3/2$
  The leading terms are

$$S = \frac{V_k}{4G_5}\left[\frac{8}{27}\frac{\mu^{3/2}}{\tau^{3/2}}\left(1 - \frac{2}{9}\frac{\mu}{\tau^2} - \frac{14}{243}\frac{\mu^2}{\tau^4}\dots\right) + \frac{4}{27}\frac{\mu^{3/2}}{\tau^{3/2}}\left(1 - \frac{58}{45}\frac{\mu}{\tau^2} + \frac{2878}{1701}\frac{\mu^2}{\tau^4}\dots\right)\right]. \quad (34)$$

Comments

- The Schwarzschild coordinate r and the isotropic coordinate z are related through $r = a(\tau)/z$. The five-velocity of an observer at small fixed z (near the boundary) is $(z/a, \dot{a}, \vec{0})$. The temperature seen by such an observer is not just the redshifted static temperature (Unruh effect).
- When are the corrections relevant in the real world?
  For $H^4 \gg \rho_{CFT}$. Assume $H^2 \sim \bar{\rho}/M_{Pl}^2$, where $\bar{\rho}$ is the energy density that drives the expansion. Then

$$\rho_{CFT} \ll \frac{\bar{\rho}}{M_{Pl}^4}\bar{\rho}. \tag{35}$$