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The q-Factorial Moments of Discrete q-Distributions and a Characterization of the Euler Distribution

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ABSTRACT The classical discrete distributions binomial, geometric and negative binomial are defined on the stochastic model of a sequence of independent and identical Bernoulli trials. The Poisson distribution may be defined as an approximation of the binomial (or negative binomial) distribution. The corresponding q-distributions are defined on the more general stochastic model of a sequence of Bernoulli trials with probability of success at any trial depending on the number of trials. In this paper targeting to the problem of calculating the moments of q-distributions, we introduce and study q-factorial moments, the calculation of which is as ease as the calculation of the factorial moments of the classical distributions. The usual factorial moments are connected with the q-factorial moments through the q-Stirling numbers of the first kind. Several examples, illustrating the method, are presented. Further, the Euler distribution is characterized through its q-factorial moments.

Keywords and phrases: q-distributions, q-moments, q-Stirling numbers

3.1 INTRODUCTION

Consider a sequence of independent Bernoulli trials with probability of success at the ith trial p_i, $i = 1, 2, \ldots$. The study of the distribution of the number X_n of successes up to the nth trial, as well as the closely related to it distribution of the number Y_k of trials until the occurrence of the kth success, have attracted special attention. In the particular case $p_i = \theta q^{i-1}/(1 + \theta q^{i-1})$, $i = 1, 2, \ldots$, $0 < q < 1$, $\theta > 0$, the distribution of the random variable X_n, called q-binomial distribution, has been studied by Kemp and Newton (1990) and Kemp and Kemp (1991). The q-binomial distribution, for $n \to \infty$, converges to a q-analog of the Poisson distribution, called Heine distribution. This distribution was introduced and examined by Benkherouf and Bather (1988). Kemp (1992a,b) further studied the Heine distribution. In the case $p_i = 1 - \theta q^{i-1}$, $i = 1, 2, \ldots$, $0 < q < 1$, $0 < \theta < 1$, the distribution of the random variable Y_k is called q-Pascal distribution. A stochastic model described by Dunkl (1981) led to the particular case $\theta = q^{m-k+1}$ of this distribution. This distribution also studied by Kemp (1998) is called absorption distribution. For $k \to \infty$, the distribution of the number of failures until the occurrence of the kth success $W_k = Y_k - k$ converges to another q-analog of the Poisson distribution, called Euler distribution. This distribution was studied by Benkherouf and Bather (1988) and Kemp (1992a,b). Kemp (2001) characterized the absorption distribution as the conditional distribution of a q-binomial distribution given the sum of a q-binomial and a Heine distribution with the same argument parameter.

In the present paper, we propose the introduction of q-factorial moments for q-distributions. These moments, apart from the interest in their own, may be used as an intermediate step in the evaluation of the usual moments of the q-distributions. In this respect, an expression of the usual factorial moments in terms of the q-factorial moments is derived. Several examples, illustrating the method, are presented and a characterization of the Euler distribution through its q-factorial moments is derived.

3.2 q-NUMBERS, q-FACTORIALS AND q-STIRLING NUMBERS

Let $0 < q < 1$, x a real number and k a positive integer. The number $[x]_q = (1 - q^x)/(1 - q)$ is called *q-real number*. In particular, $[k]_q$ is called q-positive integer. The factorial of the q-number $[x]_q$ of order k, which is defined by

$$[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k}$$

is called *q-factorial of x of order k*. In particular $[k]_q! = [1]_q[2]_q \cdots [k]_q$ is called *q-factorial of k*. The *q-binomial coefficient* is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q!} = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Note that

$$\lim_{q\to 1} \begin{bmatrix} x \\ k \end{bmatrix}_q = \binom{x}{k}.$$

The *q-binomial* and the *negative q-binomial* expansions are expressed as

$$\prod_{i=1}^{n}(1 + tq^{i-1}) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k, \tag{3.2.1}$$

and

$$\prod_{i=1}^{n}(1-tq^{i-1})^{-1}=\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k,\ |t|<1, \tag{3.2.2}$$

respectively. In general, the transition of an expression to a q-analog is not unique. Other *q-binomial* and *negative q-binomial expansions*, useful in the sequel, are the following

$$(1-(1-q)[t]_q)^n=(q^t)^n=\sum_{k=0}^{n}(-1)^k q^{k(k-1)/2}(1-q)^k\begin{bmatrix} n \\ k \end{bmatrix}_q [t]_{k,q} \tag{3.2.3}$$

and

$$(1-(1-q)[t]_q)^{-n}=(q^t)^{-n}=\sum_{k=0}^{\infty}q^{-nk}(1-q)^k\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q [t]_{k,q}. \tag{3.2.4}$$

Also, useful are the following two *q-exponential functions*:

$$e_q(t)=\prod_{i=1}^{\infty}(1-(1-q)q^{i-1}t)^{-1}=\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!},\ |t|<1/(1-q), \tag{3.2.5}$$

$$E_q(t)=\prod_{i=1}^{\infty}(1+(1-q)q^{i-1}t)=\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{t^k}{[k]_q!},\ -\infty<t<\infty, \tag{3.2.6}$$

with $e_q(t)E_q(-t)=1$. The nth order q-factorial $[t]_{n,q}$ is expanded into powers of the q-number $[t]_q$ and inversely as follows

$$[t]_{n,q}=q^{-n(n-1)/2}\sum_{k=0}^{n}s_q(n,k)[t]_q^k,\ n=0,1,\ldots, \tag{3.2.7}$$

$$[t]_q^n=\sum_{k=0}^{n}q^{k(k-1)/2}S_q(n,k)[t]_{k,q},\ n=0,1,\ldots. \tag{3.2.8}$$

The coefficients $s_q(n,k)$ and $S_q(n,k)$ are called *q-Stirling numbers of the first and second kind*, respectively. Closed expressions, recurrence relations and other properties of these numbers are examined by Carlitz (1933, 1948) and Gould (1961).

3.3 q-FACTORIAL MOMENTS

The calculation of the mean and the variance and generally the calculation of the moments of a discrete q-distribution is quite difficult. Several techniques have been used for the calculation of the mean and the variance of particular q-distributions. The general method of evaluation of moments by differentiating the probability generating function, used by Kemp (1992a, 1998), is bounded to the calculation of the first two moments. This limited applicability is due to the inherent difficulties in the differentiation of the hypergeometric series. We propose the introduction of the q-factorial moments of q-distributions, the calculation of which is as ease as that of the usual factorial moments of the classical discrete distributions.

Definition 3.3.1 Let X be a nonnegative integer valued random variable with probability mass function $f(x) = P(X = x)$, $x = 0, 1, \ldots$.

(a) The mean of the rth order q-factorial $[X]_{r,q}$,

$$E([X]_{r,q}) = \sum_{x=r}^{\infty} [x]_{r,q} f(x), \tag{3.3.1}$$

provided it exists, is called rth order *(*descending*)* q-factorial moment of the random variable X.

(b) The mean of the rth order ascending q-factorial $[X + r - 1]_{r,q}$,

$$E([X + r - 1]_{r,q}) = \sum_{x=1}^{\infty} [x + r - 1]_{r,q} f(x), \tag{3.3.2}$$

provided it exists, is called rth order ascending q-factorial moment of the random variable X.

The usual factorial moments are expressed in terms of the q-factorial moments, through the q-Stirling number of the first kind, in the following theorem.

Theorem 3.3.1 *Let $E([X]_{r,q})$ and $E([X+r-1]_{r,q})$ be the rth order descending and ascending q-factorial moments, $r = 1, 2, \ldots$, respectively, of a nonnegative integer valued random variable X. Then*

$$E[(X)_m] = m! \sum_{r=m}^{\infty} (-1)^{r-m} s_q(r, m) \frac{(1 - q)^{r-m}}{[r]_q!} E([X]_{r,q}), \tag{3.3.3}$$

and

$$E[(X + m - 1)_m] = m! \sum_{r=m}^{\infty} q^{-\binom{r}{2}} s_q(r, m) \frac{(1 - q)^{r-m}}{[r]_q!} E(q^{-rX}[X + r - 1]_{r,q}), \tag{3.3.4}$$

provided the series are convergent. The coefficient $s_q(r,k)$ is the q-Stirling number of the first kind.

Proof. According to Newton's binomial formula, for x nonnegative integer, we have

$$(1-(1-q)[t]_q)^x = \sum_{k=0}^{x} (-1)^k \binom{x}{k} (1-q)^k [t]_q^k$$

while, from (3.2.3) and (3.2.7) we get

$$(1-(1-q)[t]_q)^x = \sum_{k=0}^{x} \left\{ \sum_{r=k}^{x} (-1)^r (1-q)^r s_q(r,k) \begin{bmatrix} x \\ r \end{bmatrix}_q \right\} [t]_q^k$$

and so

$$\binom{x}{k} = \sum_{r=k}^{x} (-1)^{r-k} (1-q)^{r-k} s_q(r,k) \begin{bmatrix} x \\ r \end{bmatrix}_q .$$

Multiplying both members of this expression by the probability mass function $f(x)$ of the random variable X and summing for all $x = 0, 1, \ldots$, we deduce, according to (3.3.1), the required expression (3.3.3).

Similarly, expanding both members of (3.2.4) into powers of $[t]_q$ by the aid of Newton's negative binomial formula and expression (3.2.7) and taking expectations in the resulting expression, (3.3.4) is deduced. □

Note that Dunkl (1981), starting from Newton's polynomial expression of a function in terms of divided differences at certain points and letting the function to be the binomial coefficient $\binom{x}{k}$ and the points to be the q-numbers $[r]_q$, $r = 0, 1, \ldots, x$, first derived expression (3.3.3).

In the following examples the q-factorial moments and the usual factorial moments of several discrete q-distributions are evaluated.

Example 3.3.1 *q-binomial distribution.* Consider a sequence of independent Bernoulli trials with probability of success at the ith trial $p_i = \theta q^{i-1}/(1+\theta q^{i-1})$, $i = 1, 2, \ldots$, $0 < q < 1$, $\theta > 0$. The probability mass function of the number X_n of successes up to the nth trial is given by

$$f_{X_n}(x) = \begin{bmatrix} n \\ x \end{bmatrix}_q q^{x(x-1)/2} \theta^x \prod_{i=1}^{n} (1+\theta q^{i-1})^{-1},\ x = 0, 1, \ldots, n,$$

with $0 < q < 1$, $\theta > 0$. The rth order q-factorial moment of the random variable X_n, according to Definition 3.3.1, is given by the sum

$$E([X_n]_{r,q}) = \frac{1}{\prod_{i=1}^{n}(1+\theta q^{i-1})} \sum_{x=r}^{n} [x]_{r,q} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{x(x-1)/2} \theta^x$$

and since

$$[x]_{r,q}\left[\begin{array}{c} n \\ x \end{array}\right]_q = [n]_{r,q}\left[\begin{array}{c} n-r \\ x-r \end{array}\right]_q, \quad \binom{x}{2} = \binom{x-r}{2} + \binom{r}{2} + r(x-r),$$

it is written as

$$E([X]_n]_{r,q}) = \frac{[n]_{r,q} q^{r(r-1)/2}\theta^r}{\prod_{i=1}^{n}(1+\theta q^{i-1})} \sum_{x=r}^{n} q^{(x-r)(x-r-1)/2} \left[\begin{array}{c} n-r \\ x-r \end{array}\right]_q (\theta q^r)^{x-r}$$

and by the q-binomial formula (3.2.1), reduces to

$$E([X_n]_{r,q}) = \frac{[n]_{r,q} q^{r(r-1)/2}\theta^r}{\prod_{i=1}^{r}(1+\theta q^{i-1})}.$$

The kth order factorial moment of the random variable X_n, according to Theorem 3.3.1, is given by

$$E[(X_n)_k] = k! \sum_{r=k}^{n} (-1)^{r-k} s_q(r,k) \frac{(1-q)^{r-k} q^{r(r-1)/2}\theta^r}{\prod_{i=1}^{r}(1+\theta q^{i-1})} \left[\begin{array}{c} n \\ r \end{array}\right]_q.$$

Example 3.3.2 *Heine distribution.* The probability mass function of the q-binomial distribution for $n \to \infty$, converges to the probability mass function of the Heine distribution

$$f_X(x) = e_q(-\lambda)\frac{q^{x(x-1)/2}\lambda^x}{[x]_q!}, \; x = 0, 1, \ldots,$$

with $0 < q < 1$, $\lambda > 0$, where $\lambda = \theta/(1-q)$ and $e_q(-\lambda) = \prod_{i=1}^{\infty}(1+\lambda(1-q)q^{i-1})^{-1}$ is the q-exponential function (3.2.5). The rth order q-factorial moment of the random variable X is given by

$$\begin{aligned} E([X]_{r,q}) &= e_q(-\lambda)\sum_{x=r}^{\infty}[x]_{r,q}\frac{q^{x(x-1)/2}\lambda^x}{[x]_q!} \\ &= q^{r(r-1)/2}\lambda^r e_q(-\lambda)\sum_{x=r}^{\infty}\frac{q^{(x-r)(x-r-1)/2}(\lambda q^r)^{x-r}}{[x-r]_q!} \end{aligned}$$

and since

$$\sum_{x=r}^{\infty}\frac{q^{(x-r)(x-r-1)/2}(\lambda q^r)^{x-r}}{[x-r]_q!} = E_q(\lambda q^r) = \prod_{i=1}^{\infty}(1+\lambda(1-q)q^{r+i-1})$$

it reduces to

$$E([X]_{r,q}) = \frac{q^{r(r-1)/2}\lambda^r}{\prod_{i=1}^{r}(1+\lambda(1-q)q^{i-1})}.$$

Further, by Theorem 3.3.1,

$$E[(X)_k] = k!\sum_{r=k}^{\infty}(-1)^{r-k}s_q(r,k)\frac{(1-q)^{r-k}q^{r(r-1)/2}}{\prod_{i=1}^{r}(1+\lambda(1-q)q^{i-1})}\cdot\frac{\lambda^r}{[r]_q!}.$$

Example 3.3.3 *q-Pascal distribution.* Consider a sequence of independent Bernoulli trials with probability of success at the ith trial $p_i = 1-\theta q^{i-1}$, $i = 1,2,\ldots$, $0<q<1$, $0<\theta<1$. The probability mass function of the number Y_k of trials until the occurrence of the kth success is given by

$$f_{Y_k}(y) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}_q \theta^{y-k}\prod_{i=1}^{k}(1-\theta q^{i-1}),\ y = k, k+1, \ldots,$$

with $0<q<1$, $0<\theta<1$. The rth order ascending q-factorial moment of the random variable Y_k, according to Definition 3.3.1, is given by the sum

$$E([Y_k+r-1]_{r,q}) = \frac{1}{\prod_{i=1}^{k}(1-\theta q^{i-1})^{-1}}\sum_{y=k}^{\infty}[y+r-1]_{r,q}\begin{bmatrix} y-1 \\ k-1 \end{bmatrix}_q \theta^{y-k}$$

and since

$$[y+r-1]_{r,q}\begin{bmatrix} y-1 \\ k-1 \end{bmatrix}_q = [k+r-1]_{r,q}\begin{bmatrix} y+r-1 \\ k+r-1 \end{bmatrix}_q,$$

it is written as

$$E([Y_k+r-1]_{r,q}) = \frac{[k+r-1]_{r,q}}{\prod_{i=1}^{k}(1-\theta q^{i-1})^{-1}}\sum_{y=k}^{\infty}\begin{bmatrix} y+r-1 \\ k+r-1 \end{bmatrix}_q \theta^{y-k}.$$

Thus, by the q-negative binomial formula (3.2.2),

$$E([Y_k+r-1])_{r,q}) = \frac{[k+r-1]_{r,q}}{\prod_{i=1}^{r}(1-\theta q^{k+i-1})}.$$

Similarly

$$E(q^{-rY_k}[Y_k+r-1])_{r,q}) = \frac{[k+r-1]_{r,q}q^{-kr}}{\prod_{i=1}^{r}(1-\theta q^{-r+i-1})}$$

and by Theorem 3.3.1,

$$E[(X+m-1)_m] = m!\sum_{r=m}^{\infty}q^{-kr-\binom{r}{2}}s_q(r,m)\frac{(1-q)^{r-m}}{\prod_{i=1}^{r}(1-\theta q^{-r+i-1})}\begin{bmatrix} k+r-1 \\ r \end{bmatrix}_q.$$

Example 3.3.4 *Euler distribution.* The probability mass function of the number of failures until the occurrence of the kth success $W_k = Y_k - k$ is given by

$$f_{W_k}(w) = \begin{bmatrix} k+w-1 \\ w \end{bmatrix}_q \theta^w \prod_{i=1}^{k}(1-\theta q^{i-1}),\ w = 0, 1, \ldots.$$

This distribution, which may be called q-negative binomial distribution, for $k \to \infty$, converges to the Euler distribution with probability mass function

$$f_X(x) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!},\ x = 0, 1, \ldots,$$

with $0 < q < 1$, $0 < \lambda < 1/(1-q)$, where $\lambda = \theta/(1-q)$ and $E_q(-\lambda) = \prod_{i=1}^{\infty}(1-\lambda(1-q)q^{i-1})$ is the q-exponential function (3.2.6). The rth order q-factorial moment of the random variable X is given by

$$E([X]_{r,q}) = E_q(-\lambda)\sum_{x=r}^{\infty}[x]_{r,q}\frac{\lambda^x}{[x]_q!} = \lambda^r E_q(-\lambda)\sum_{x=r}^{\infty}\frac{\lambda^{x-r}}{[x-r]_q!}$$

and since, by (3.2.5),

$$\sum_{x=r}^{\infty}\frac{\lambda^{x-r}}{[x-r]_q!} = e_q(\lambda) = \frac{1}{E_q(-\lambda)},$$

it follows that

$$E([X]_{r,q}) = \lambda^r.$$

Further, by Theorem 3.3.1,

$$E[(X)_k] = k!\sum_{r=k}^{\infty}(-1)^{r-k}s_q(r,k)\frac{(1-q)^{r-k}\lambda^r}{[r]_q!}.$$

3.4 A CHARACTERIZATION OF THE EULER DISTRIBUTION

Consider a family of nonnegative integer valued random variables $\{X_\lambda, 0 < \lambda < \rho \le \infty\}$ having a power series distribution with probability mass function

$$f(x;\lambda) = \frac{a(x)\lambda^x}{g(\lambda)},\ x = 0, 1, \ldots,\ 0 < \lambda < \rho \tag{3.4.1}$$

and series function

$$g(\lambda) = \sum_{x=0}^{\infty} a(x)\lambda^x,\ 0 < \lambda < \rho.$$

It is well-known that the mean-variance equality

$$E(X_\lambda) = Var(X_\lambda) \quad \text{for all} \quad \lambda \in (0,\rho)$$

characterizes the Poisson family of distributions [see Kosambi (1949) and Patil (1962)]. Note that the requirement that this equality holds for all $\lambda \in (0,\rho)$ has been overlooked by some authors [see Sapatinas (1994) for details]. This requirement may be relaxed by weaker ones; e.g., it suffices to verify it for all $\lambda \in I$, where I is any nondegenerate subinterval of $(0,\rho)$. A q-analogue to the Kosambi-Patil characterization for the Euler distribution is derived in the following theorem.

Theorem 3.4.1 *Assume that a family of nonnegative integer valued random variables $\{X_\lambda, 0 < \lambda < \rho \le \infty\}$ obeys a power series distribution with probability mass function* (3.4.1). *Then, X_λ has an Euler distribution if and only if*

$$E([X_\lambda]_{2,q}) = [E([X_\lambda]_q)]^2 \tag{3.4.2}$$

for all $\lambda \in (0,\rho)$.

Proof. Assume first that (3.4.2) holds for all $\lambda \in (0,\rho)$. Then

$$\frac{\lambda^2}{g(\lambda)} \sum_{x=0}^{\infty} [x+1]_q [x+2]_q a(x+2)\lambda^x = \frac{\lambda^2}{[g(\lambda)]^2} \left[\sum_{x=0}^{\infty} [x+1]_q a(x+1)\lambda^x \right]^2$$

which, using the series $g(\lambda) = \sum_{x=0}^{\infty} a(x)\lambda^x$, may be written as

$$\left[\sum_{x=0}^{\infty} a(x)\lambda^x \right] \left[\sum_{x=0}^{\infty} [x+1]_q [x+2]_q a(x+2)\lambda^x \right] = \left[\sum_{x=0}^{\infty} [x+1]_q a(x+1)\lambda^x \right]^2$$

or equivalently as

$$\begin{aligned} \textstyle\sum_{x=0}^{\infty} \quad & \left\{ \sum_{k=0}^{x} [k+1]_q [k+2]_q a(k+2) a(x-k) \right\} \lambda^x \\ &= \sum_{x=0}^{\infty} \left\{ \sum_{k=0}^{x} [k+1]_q [x-k+1]_q a(k+1) a(x-k+1) \right\} \lambda^x . \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{k=0}^{x} [k+1]_q [k+2]_q a(k+2) a(x-k) \\ &\qquad = \sum_{k=0}^{x} [k+1]_q [x-k+1]_q a(k+1) a(x-k+1), \end{aligned} \tag{3.4.3}$$

for $x = 0, 1, \ldots$. Setting $x = 0$ it follows that

$$[2]_q! a(2) a(0) = [a(1)]^2.$$

It is easy to see that if $a(0) = 0$ then $a(1) = 0$ and using (3.4.3) it follows that $a(x) = 0$ for all $x = 0, 1, \ldots$, which is a contradiction to the assumption that X_λ obeys a power series distribution. Thus $a(0) \neq 0$, and without any loss of generality we may assume that $a(0) = 1$. Therefore

$$a(2) = a^2/[2]_q!,$$

where $a = a(1) > 0$. Further, setting $x = 1$ it follows that

$$[2]_q! a(2) a(1) + [3]_q! a(3) a(0) = 2[2]_q! a(2) a(1)$$

and

$$a(3) = a^3/[3]_q!.$$

Suppose that

$$a(k) = a^k/[k]_q!, \; k = 0, 1, \ldots, x+1.$$

Then

$$[x+1]_q [x+2]_q a(x+2) a(0) + \sum_{k=0}^{x-1} [k+1]_q [k+2]_q a(k+2) a(x-k)$$
$$= [x+1]_q [1]_q a(x+1) a(1) + \sum_{k=0}^{x-1} [k+1]_q [x-k+1]_q a(k+1) a(x-k+1)$$

and since

$$\sum_{k=0}^{x-1} [k+1]_q [k+2]_q a(k+2) a(x-k) = a^{x+2} \sum_{k=0}^{x-1} \frac{1}{[k]_q! [x-k]_q!},$$

$$\sum_{k=0}^{x-1} [k+1]_q [x-k+1]_q a(k+1) a(x-k+1) = a^{x+2} \sum_{k=0}^{x-1} \frac{1}{[k]_q! [x-k]_q!},$$

it follows that

$$[x+2]_q a(x+2) = a \cdot a(x+1)$$

and so

$$a(x+2) = a^{x+2}/[x+2]_q!.$$

Thus

$$a(x) = a^x/[x]_q!, \quad x = 0, 1, \ldots.$$

Further, the series function, by (3.2.5), is given by

$$g(\lambda) = \sum_{x=0}^{\infty} a(x)\lambda^x = \sum_{x=0}^{\infty} \frac{(a\lambda)^x}{[x]_q!} = e_q(a\lambda) = \frac{1}{E_q(-a\lambda)}$$

for $0 < q < 1$, $0 < a\lambda < 1/(1-q)$, and so the random variable X_λ has an Euler distribution with probability mass function

$$f(x;\lambda) = E_q(-a\lambda)\frac{(a\lambda)^x}{[x]_q!},\ x = 0, 1, \ldots,$$

with $0 < q < 1$, $0 < a\lambda < 1/(1-q)$. Finally, according to Example 3.3.4, the q-factorial moments of the Euler distribution satisfy relation (3.4.2). □

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REFERENCES

Benkherouf, L. and Bather, J. A. (1988). Oil exploration: sequential decisions in the face of uncertainty. *Journal of Applied Probability*, **25**, 529–543.

Blomqvist, N. (1952). On an exhaustion process. *Skandinavisk Akktuarietidskrift*, **36**, 201–210.

Carlitz, L. (1933). On Abelian fields. *Transactions of the American Mathematical Society*, **35**, 122–136.

Carlitz, L. (1948). q-Bernoulli numbers and polynomials. *Duke Mathematical Journal*, **15**, 987–1000.

Dunkl, C. F. (1981). The absorption distribution and the q-binomial theorem. *Communications in Statistics—Theory and Methods*, **10**, 1915–1920.

Gould, H. W. (1961). The q-Stirling numbers of the first and second kinds. *Duke Mathematical Journal*, **28**, 281–289.

Kemp, A. (1998). Absorption sampling and the absorption distribution. *Journal of Applied Probability*, **35**, 489–494.

Kemp, A. (2001). A characterization of a distribution arising from absorption sampling. In *Probability and Statistical Models with Applications* (Eds., Ch. A. Charalambides, M. V. Koutras and N. Balakrishnan), pp. 239–246, Chapman & Hall/CRC Press, Boca Raton, Florida.

Kemp, A. (1992a). Heine–Euler extensions of Poisson distribution. *Communications in Statistics—Theory and Methods*, **21**, 791–798.

Kemp, A. (1992b). Steady-state Markov chain models for Heine and Euler distributions. *Journal of Applied Probability*, **29**, 869–876.

Kemp, A. and Kemp, C. D. (1991). Weldon's dice date revisited. *The American Statistician*, **45**, 216–222.

Kemp, A. and Newton, J. (1990). Certain state-dependent processes for dichotomized parasite populations. *Journal of Applied Probability*, **27**, 251–258.

Kosambi, D. D. (1949). Characteristic properties of series distributions. *Proceedings of the National Institute for Science, India*, **15**, 109–113.

Patil, G. P. (1962). Certain properties of the generalized power series distribution. *Annals of the Institute of Statistical Mathematics*, **14**, 179–182.

Sapatinas, T. (1994). Letter to the editor. A remark on P. C. Consul's (1990) counterexample: "New class of location-parameter discrete probability distributions and their characterizations". [*Communications in Statistics—Theory and Methods*, **19** (1990), 4653–4666] with a rejoinder by Consul. *Communications in Statistics—Theory and Methods*, **23**, 2127–2130.