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In paper [1] (see also the Technical Report [2]) we obtained a class of discrete variance bounds. These bounds are the discrete analogues of the corresponding continuous ones, given in [4], [5] and [3].

In Remark 3.1 of [1] it is asserted that

The discrete bound can easily yield the continuous, while there is no a simple way to obtain the discrete bound from the continuous one.

A sketch for the verification of this fact is given in Remark 3.1 of [1]. The purpose of this note is to provide the necessary technical details.

Details used in Remark 3.1 of [1]

1. Discrete implies continuous [Remark 3.1(a) of [1]]

For $\delta > 0$, $a \in \mathbb{R}$, and for any suitable function h, define

$$I_k(a,\delta;h) = \int_a^{a+\delta}\int_{t_1}^{t_1+\delta}\cdots\int_{t_{k-1}}^{t_{k-1}+\delta} h(t_k)dt_k\ldots dt_2dt_1. \tag{1}$$

If $g:[a,a+k\delta]\longrightarrow\mathbb{R}$ has k continuous derivatives in $[a,a+k\delta]$, it is easy to see that

$$I_k(a,\delta;g^{(k)}) = \sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}g(a+\delta s). \tag{2}$$

Thus, if $g:[0,1]\longrightarrow\mathbb{R}$ has k continuous derivatives in $[0,1]$, then for $h_N(j) = g(j/N)$, $j=1,2,\ldots,N$, we get

$$\Delta^k h_N(j) = I_k(j/N,1/N;g^{(k)}),\quad j=1,2,\ldots,N-k. \tag{3}$$

Also, if c is a constant, then

$$\int_a^{a+\delta}\int_{t_1}^{t_1+\delta}\cdots\int_{t_{k-1}}^{t_{k-1}+\delta} c\,dt_k\ldots dt_2dt_1 = c\delta^k,$$

and therefore, for any $j\in\{1,2,\ldots,N-k\}$,

$$\begin{aligned}&g^{(k)}(j/N) - N^k\Delta^k h_N(j)\\ &\quad= N^k\int_{j/N}^{(j+1)/N}\int_{t_1}^{t_1+1/N}\cdots\int_{t_{k-1}}^{t_{k-1}+1/N}\left[g^{(k)}(j/N)-g^{(k)}(t_k)\right]dt_k\ldots dt_2dt_1.\end{aligned} \tag{4}$$

Relation (4) shows that if g has n continuous derivatives $g', g'', \ldots, g^{(n)}$ in $[0,1]$, then for any fixed $k \in \{0, 1, \ldots, n\}$,

$$\max_{j\in\{1,2,\ldots,N-k\}} |g^{(k)}(j/N) - N^k \Delta^k h_N(j)| \to 0, \quad \text{as} \quad N \to \infty. \tag{5}$$

Now, using (5), it is easy to verify that for any such g,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N-k} \frac{[j]_k}{N^k} \frac{(N-j)_k}{N^k} (N^k \Delta^k h_N(j))^2 = \int_0^1 t^k (1-t)^k (g^{(k)}(t))^2 dt, \quad k = 0, 1, \ldots, n, \tag{6}$$

(see Remark 3.1(a) of [1]) and, of course,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h_N(j) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} g(j/N) = \int_0^1 g(t) dt. \tag{7}$$

The discrete inequality suggested by Corollary 3.1(a) of [1] (without the remainder term R_{n+1}), applied to the function $h_N(j) = g(j/N)$, shows that the following inequality holds:

$$(-1)^n \left[\left(\frac{1}{N} \sum_{j=1}^{N} g(j/N) \right)^2 - \sum_{k=0}^{n} \frac{(-1)^k}{k!(k+1)!} \left\{ \frac{1}{N} \sum_{j=1}^{N-k} \frac{[j]_k}{N^k} \frac{(N-j)_k}{N^k} (N^k \Delta^k h_N(j))^2 \right\} \right] \le 0. \tag{8}$$

Thus, taking limits as $N \to \infty$ in (8), and using (6) and (7), we obtain the continuous Mohr and Noll inequality given in [4], namely,

$$(-1)^n \left[\left(\int_0^1 g(t) dt \right)^2 - \sum_{k=0}^{n} \frac{(-1)^k}{k!(k+1)!} \left\{ \int_0^1 t^k (1-t)^k (g^{(k)}(t))^2 dt \right\} \right] \le 0, \tag{9}$$

provided that g has n continuous derivatives $g', g'', \ldots, g^{(n)}$ in $[0,1]$. Observe that for any such g, the functions $t^k(1-t)^k(g^{(k)}(t))^2$, $t \in [0,1]$, $k = 0, 1, \ldots, n$, are (uniformly) continuous and thus, bounded, so that the condition

$$\sum_{k=0}^{n} \int_0^1 t^k (1-t)^k (g^{(k)}(t))^2 dt < \infty \tag{10}$$

is automatically satisfied. Therefore, the validity of (9) in this particular case is a simple by-product of the discrete inequality (8), as it is stated in Remark 3.1(a) of [1]. The validity of (9) in the general case, i.e., when (10) is satisfied (and when g and its derivatives $g', g'', \ldots, g^{(n)}$ are merely assumed to be continuous in the open interval $(0,1)$), can be obtained from the above particular case. Indeed, let $\epsilon > 0$ be small, and consider the function $g_\epsilon(t) = g(\epsilon + (1-2\epsilon)t)$, $t \in [0,1]$. Then all functions

$$g_\epsilon^{(k)}(t) = (1-2\epsilon)^k g^{(k)}(\epsilon + (1-2\epsilon)t), \quad k = 0, 1, \ldots, n,$$

are continuous in $[0,1]$, so that (9) is applicable for g_ϵ, yielding

$$(-1)^n \left[\left(\int_0^1 g_\epsilon(x)dx \right)^2 - \sum_{k=0}^n \frac{(-1)^k}{k!(k+1)!} \left\{ \int_0^1 x^k(1-x)^k (g_\epsilon^{(k)}(x))^2 dx \right\} \right] \le 0. \quad (11)$$

Now, with the substitution $x = (t-\epsilon)/(1-2\epsilon)$ in the above integrals, (11) is equivalent to

$$(-1)^n \Bigg[\left(\int_\epsilon^{1-\epsilon} g(t)dt \right)^2 \\ -(1-2\epsilon) \sum_{k=0}^n \frac{(-1)^k}{k!(k+1)!} \left\{ \int_\epsilon^{1-\epsilon} (t-\epsilon)^k (1-\epsilon-t)^k (g^{(k)}(t))^2 dt \right\} \Bigg] \le 0. \quad (12)$$

Since $\int_0^1 (g(t))^2 dt < \infty$ (this condition is imposed by (10)), and therefore, $\int_0^1 |g(t)|dt < \infty$, it follows that $\lim_{\epsilon \searrow 0} \int_\epsilon^{1-\epsilon} g(t)dt = \int_0^1 g(t)dt$, because the functions $|g(t) I_{[\epsilon,1-\epsilon]}(t)|$, $\epsilon \in (0,1/2)$, are dominated by the integrable $|g(t)| I_{(0,1)}(t)$. Taking limits in (12) as $\epsilon \searrow 0$, observing that

$$(t-\epsilon)^k (1-\epsilon-t)^k (g^{(k)}(t))^2 I_{[\epsilon,1-\epsilon]}(t) \nearrow t^k(1-t)^k (g^{(k)}(t))^2 I_{(0,1)}(t), \quad \text{as } \epsilon \searrow 0,$$

and using monotone convergence, we obtain (9) for any g satisfying (10).

2. Discrete does not follow from continuous [Remark 3.1(b) of [1]]

The details needed for verifying Remark 3.1(b) of [1] are by far more complicated. One reason is that there are many possibilities for approximating a given discrete r.v. by a continuous one. In particular, if U_N is the discrete Uniform in $\{1,2,\ldots,N\}$, then the convolution $X_\epsilon = U_N + \epsilon X$ converges weakly to U_N, as $\epsilon \searrow 0$, for any (fixed) r.v. X, and X_ϵ is absolutely continuous whenever X is. Another more important reason is that given the N points $\{(j, g(j)),\ j = 1,2,\ldots,N\}$ in the plan, there are too many possibilities for fitting a function $\widetilde{g}(t)$, defined for $t \in [1,N]$, in such a way that the graph of $\widetilde{g}$ is passing through the given points. Clearly, for any $\widetilde{g}$ of this kind, one can apply the (old) variance bound given in [5] (see eq. (1.5) in the introduction of [1]), provided that $\widetilde{g}$ has n continuous derivatives in the smallest open interval $J(X_\epsilon)$ that contains the support of X_ϵ, and provided also that

$$\int_{J(X_\epsilon)} a_{k-1}^\epsilon(t) (\widetilde{g}^{(k)}(t))^2 < \infty, \quad k = 1,2,\ldots,n+1, \quad (13)$$

where $a_k^\epsilon(t)$ are the corresponding (for $X = X_\epsilon$) $a_k(t)$, given by (1.6) and (1.7) of [1] (see [5]). [Note that Theorem 2 of Johnson [3] (see eq. (1.9) in the introduction of [1]) is not applicable for X_ϵ, because X_ϵ lie outside the continuous Pearson family when ϵ is small, for any fixed absolutely continuous r.v. X.] Of course, $a_k^\epsilon(t)$ are well defined only if X_ϵ has a suitable number of moments, and this must be entered into the problem when the bound (1.5) of [1] is to be applied. Another (minor) constrain arises from the fact that $\mathrm{Var}\,\widetilde{g}(X_\epsilon)$ must be finite for some $\epsilon > 0$, and has to converge to $\mathrm{Var}\, g(U_N)$, as $\epsilon \searrow 0$. A final requirement is that the limit of the bound given in eq. (1.5) of [1] should exists, in order to be comparable with the corresponding discrete result given in Theorem 4.1 of [1]. Thus, we see that many technical details have to be considered, if one wants to obtain the discrete bound from the continuous one, even in the simplest case of a discrete uniform r.v. In the sequel we shall try to apply this procedure for U_N in its most general setup.

From now on, it is assumed that $N \geq 2$ is a given integer and $g(1), \ldots, g(N)$ are specified reals, defining a (fixed) function $g : \{1, 2, \ldots, N\} \longrightarrow \mathbb{R}$. The symbol $\widetilde{g}$ denotes any real function, defined at least in the interval $[1, N]$, such that $\widetilde{g}(1) = g(1), \ldots, \widetilde{g}(N) = g(N)$.

2.1. The limit of $a_k^\epsilon(t)$ does not depend on the convoluted absolutely continuous r.v. X

For any absolutely continuous r.v. Y with density f_Y and $\mathbb{E}|Y|^{k+1} < \infty$, we define the functions $a_k = a_k^Y$ as in (1.6) of [1] (see [5]), i.e.,

$$a_k^Y(t) = (-1)^k [\mathbb{E}(Y-t)^{k+1} \mathbb{E}[(Y-t)^k I(Y<t)] - \mathbb{E}(Y-t)^k \mathbb{E}[(Y-t)^{k+1} I(Y<t)]]. \tag{14}$$

Setting $L_k^Y(t) = \mathbb{E}[(t-Y)^k I(Y<t)]$ and $U_k^Y(t) = \mathbb{E}[(Y-t)^k I(Y>t)]$, it is easy to see that

$$\begin{aligned} a_k^Y(t) &= \mathbb{E}[(Y_2-Y_1)(t-Y_1)^k (Y_2-t)^k I(Y_1<t<Y_2)] &(15)\\ &= L_k^Y(t) U_{k+1}^Y(t) + L_{k+1}^Y(t) U_k^Y(t), &(16)\end{aligned}$$

where Y_1, Y_2 are independent copies of Y. These formulae show that a_k^Y is nonnegative.

Lemma 1 *Let X be any absolutely continuous r.v., stochastically independent of U_N, with $\mathbb{E}|X|^{k+1+\delta} < \infty$ for some $\delta > 0$. For $\epsilon > 0$ define $X_\epsilon = U_N + \epsilon X$ and $a_k^\epsilon = a_k^{X_\epsilon}$. Then, for any $t \in \mathbb{R} \setminus \{1, 2, \ldots, N\}$,*

$$a_k^\epsilon(t) \to a_k(t), \quad \text{as } \epsilon \searrow 0,$$

where

$$\begin{aligned} a_k(t) = {} & \mathbb{E}[(t-U_N)^{k+1} I(U_N<t)] \mathbb{E}[(U_N-t)^k I(U_N>t)] \\ & + \mathbb{E}[(t-U_N)^k I(U_N<t)] \mathbb{E}[(U_N-t)^{k+1} I(U_N>t)]. \end{aligned} \tag{17}$$

Proof: Define $U_k^\epsilon(t) = U_k^{X_\epsilon}(t)$, $L_k^\epsilon(t) = L_k^{X_\epsilon}(t)$, $g_t(x) = (t-x)^k I(x<t)$ and $h_t(x) = (x-t)^k I(x>t)$. Clearly the functions $g_t(x)$ and $h_t(x)$ are continuous at any point $x \neq t$ (note that only if $k=0$ they are not continuous at $x=t$), and thus, $g_t(X_\epsilon)$ converges weakly to $g_t(U_N)$, and $h_t(X_\epsilon)$ converges weakly to $h_t(U_N)$, because $t \notin \{1, 2, \ldots, N\}$. Observe that for $\delta' = \delta/k > 0$,

$$|h_t(X_\epsilon)|^{1+\delta'} = |h_t(X_\epsilon)|^{1+\delta/k} \leq |(X_\epsilon - t)|^{k+1} = |U_N + \epsilon X - t|^{k+1} \leq C_1 + C_2 |X|^{k+1},$$

where C_1 and C_2 are finite, independent of X and $\epsilon \in (0,1)$, and thus,

$$\sup_{0<\epsilon<1} \mathbb{E}|h_t(X_\epsilon)|^{1+\delta'} \leq C_1 + C_2 \mathbb{E}|X|^{k+1} < \infty.$$

Therefore, the r.v.'s $\{h_t(X_\epsilon),\ \epsilon\in(0,1)\}$ are uniformly integrable (as $\epsilon\searrow 0$), so that

$$\lim_{\epsilon\searrow 0} U_k^\epsilon(t)=\lim_{\epsilon\searrow 0}\mathbb{E}[h_t(X_\epsilon)]=\mathbb{E}[h_t(U_N)].$$

By the same arguments it follows that

$$\lim_{\epsilon\searrow 0} L_k^\epsilon(t)=\lim_{\epsilon\searrow 0}\mathbb{E}[g_t(X_\epsilon)]=\mathbb{E}[g_t(U_N)].$$

Finally, if we consider the functions $\widetilde{g}_t(x)=(t-x)^{k+1}I(x<t)$ and $\widetilde{h}_t(x)=(x-t)^{k+1}I(x>t)$, and if we apply the same arguments with $\delta'=\delta/(k+1)>0$, it follows that

$$U_{k+1}^\epsilon(t)=\mathbb{E}[\widetilde{h}_t(X_\epsilon)]\to\mathbb{E}[\widetilde{h}_t(U_N)] \text{ and } L_{k+1}^\epsilon(t)=\mathbb{E}[\widetilde{g}_t(X_\epsilon)]\to\mathbb{E}[\widetilde{g}_t(U_N)], \text{ as } \epsilon\searrow 0,$$

completing the proof. $\square$

2.2. A natural condition on $\widetilde{g}$ for obtaining the differential form of a variance bound for the discrete uniform

Lemma 1 implies that the function $a_k(t)$, i.e., the limit of $a_k^\epsilon(t)$ as $\epsilon\searrow 0$, is independent of X, and therefore, for convenience, we shall consider the particular case where $X=V$, with V uniformly distributed over the interval $(-1,1)$. In this case $X_\epsilon=U_N+\epsilon V$ has density

$$f_\epsilon(t)=\frac{1}{2N\epsilon}\sum_{j=1}^{N} I_{(j-\epsilon,j+\epsilon)}(t), \tag{18}$$

and the variance bound given in eq. (1.5) of the paper takes the form

$$(-1)^n[S_n^\epsilon(\widetilde{g})-\operatorname{Var}\widetilde{g}(X_\epsilon)]\ge 0, \quad\text{where } S_n^\epsilon(\widetilde{g})=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{(k-1)!k!}\int_{1-\epsilon}^{N+\epsilon} a_{k-1}^\epsilon(t)(\widetilde{g}^{(k)}(t))^2dt. \tag{19}$$

In formula (19) the functions $a_k^\epsilon(t)$ are given by (14), (15) or (16), for $Y=X_\epsilon$, i.e.,

$$a_k^\epsilon(t)=\int_{1-\epsilon}^{t}\int_{t}^{N+\epsilon}(y-x)(y-t)^k(t-x)^k f_\epsilon(x)f_\epsilon(y)dydx,\quad t\in(1-\epsilon,N+\epsilon), \tag{20}$$

where f_ϵ is given by (18).

Clearly the bound $S_n^\epsilon(\widetilde{g})$ in (19), and the corresponding variance inequality, are well-defined (and useful for our purpose) only if the function $\widetilde{g}$ satisfies the following condition:

Condition A. There exists an $\epsilon\in(0,1/2)$ such that the function $\widetilde{g}$ is defined at least for $t\in(1-\epsilon,N+\epsilon)$ in such a way that $\widetilde{g}(j)=g(j)$ for $j=1,2,\ldots,N$, and satisfies the following:
(i) $\mathbb{E}[\widetilde{g}(X_\epsilon)]^2<\infty$.
(ii) For $k=0,1,\ldots,n$, the function $\widetilde{g}^{(k)}(t)$, $t\in(1-\epsilon,N+\epsilon)$, is absolutely continuous with derivative $\widetilde{g}^{(k+1)}(t)$.
(iii) $\int_{1-\epsilon}^{N+\epsilon} a_{k-1}^\epsilon(t)(\widetilde{g}^{(k)}(t))^2dt<\infty$ for $k=1,2,\ldots,n+1$.

The class of functions $\widetilde{g}$ satisfying Condition A will be denoted by $\widetilde{C}_n$ or $\widetilde{C}_n(g)$.

Lemma 2 *If $\tilde{g} \in \tilde{C}_n(g)$ and $S_n^\epsilon(\tilde{g})$ is given by (19), then*

$$\lim_{\epsilon \searrow 0} S_n^\epsilon(\tilde{g}) = \tilde{S}_n(\tilde{g}) \in \mathbb{R}, \quad \textit{where} \quad \tilde{S}_n(\tilde{g}) = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!k!} \int_1^N a_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt, \quad (21)$$

and where $a_k(t)$ is given by (17). Moreover,

$$\lim_{\epsilon \searrow 0} \operatorname{Var} \tilde{g}(X_\epsilon) = \operatorname{Var} \tilde{g}(U_N) = \operatorname{Var} g(U_N). \quad (22)$$

Proof: Assume that (i), (ii) and (iii) of Condition A are satisfied for some $\epsilon = \epsilon_0 \in (0, 1/2)$, and choose $\epsilon \in (0, \epsilon_0/2)$. Observe that $a_{k-1}^{\epsilon_0}(t)$ is strictly positive and continuous in $(1-\epsilon_0, N+\epsilon_0)$, and therefore, there exist a constant m such that

$$0 < m \leq a_{k-1}^{\epsilon_0}(t), \quad \text{for all} \quad t \in [1-\epsilon_0/2, N+\epsilon_0/2].$$

This implies that

$$m \int_{1-\epsilon_0/2}^{N+\epsilon_0/2} (\tilde{g}^{(k)}(t))^2 dt \leq \int_{1-\epsilon_0/2}^{N+\epsilon_0/2} a_{k-1}^{\epsilon_0}(t)(\tilde{g}^{(k)}(t))^2 dt < \infty,$$

so that

$$\int_{1-\epsilon_0/2}^{N+\epsilon_0/2} (\tilde{g}^{(k)}(t))^2 dt < \infty. \quad (23)$$

Since $a_{k-1}^\epsilon(t)$ is uniformly bounded by a finite constant C (e.g., $C = (N+1)^{2k-1}$), it follows that

$$a_{k-1}^\epsilon(t)(\tilde{g}^{(k)}(t))^2 I_{[1-\epsilon,N+\epsilon]}(t) \leq C(\tilde{g}^{(k)}(t))^2 I_{[1-\epsilon,N+\epsilon]}(t) \leq C(\tilde{g}^{(k)}(t))^2 I_{(1-\epsilon_0/2,N+\epsilon_0/2)}(t),$$

and (23) shows that the function $C(\tilde{g}^{(k)}(t))^2 I_{(1-\epsilon_0/2,N+\epsilon_0/2)}(t)$ is integrable (and independent of ϵ). Also, from Lemma 1 we have that for almost all $t \in \mathbb{R}$,

$$\lim_{\epsilon \searrow 0} [a_{k-1}^\epsilon(t)(\tilde{g}^{(k)}(t))^2] = \begin{cases} a_{k-1}(t)(\tilde{g}^{(k)}(t))^2, & \text{if } t \in (1, N), \\ 0, & \text{if } t \notin (1, N), \end{cases}$$

where $a_{k-1}(t)$ is given by (17). Observe that $(1-\epsilon, N+\epsilon) \subset (1-\epsilon_0/2, N+\epsilon_0/2)$, and $a_{k-1}^\epsilon(t) = 0$ for $t \notin (1-\epsilon, N+\epsilon)$. Therefore, from dominated convergence we conclude that

$$\begin{aligned} \lim_{\epsilon \searrow 0} \int_{1-\epsilon}^{N+\epsilon} a_{k-1}^\epsilon(t)(\tilde{g}^{(k)}(t))^2 dt &= \lim_{\epsilon \searrow 0} \int_{1-\epsilon_0/2}^{N+\epsilon_0/2} a_{k-1}^\epsilon(t)(\tilde{g}^{(k)}(t))^2 dt \\ &= \int_{1-\epsilon_0/2}^{N+\epsilon_0/2} \lim_{\epsilon \searrow 0} [a_{k-1}^\epsilon(t)(\tilde{g}^{(k)}(t))^2] dt \\ &= \int_1^N a_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt. \end{aligned}$$

This proves (21). Assertion (22) follows immediately from the facts that $X_\epsilon \to U_N$ weakly, $\widetilde{g}(U_N) = g(U_N)$ w.p. 1, and that $\widetilde{g}$ is continuous in $(1-\epsilon_0, N+\epsilon_0)$, implying that it is uniformly continuous in $[1-\epsilon, N+\epsilon]$ for any $\epsilon \in (0, \epsilon_0)$. □

Obviously, as it was fairly expected, the limiting bound $\widetilde{S}_n(\widetilde{g})$ does not depend on the values $\widetilde{g}(t)$, $t \notin [1, N]$. Also, it does not depend on the values of its derivatives, $\widetilde{g}^{(k)}(t)$, for t outside the interval $(1, N)$. It is therefore natural to define a larger class of functions, C_n, as follows:

Condition B. We say that a real function $\widetilde{g}$, with $\widetilde{g}(j) = g(j)$, $j = 1, 2, \ldots, N$, belongs to $C_n = C_n(g)$, if it is defined in $[1, N]$, it is absolutely continuous with a.s. derivative $\widetilde{g}^{(1)}(t) = \widetilde{g}'(t)$, $t \in (1, N)$, and satisfies the following:
(i) For $k = 1, \ldots, n$, the function $\widetilde{g}^{(k)}(t)$, $t \in (1, N)$, is absolutely continuous with a.s. derivative $\widetilde{g}^{(k+1)}(t)$, $t \in (1, N)$.
(ii) $\int_1^N (\widetilde{g}^{(k)}(t))^2 dt < \infty$ for $k = 1, 2, \ldots, n+1$.

Let $\widetilde{g} \in \widetilde{C}_n(g)$, and consider its restriction h in the interval $[1, N]$. It becomes clear from the proof of Lemma 2 (see (22) and (23)) that $h \in C_n(g)$. Thus, $\widetilde{C}_n(g)$ is larger than $C_n(g)$. On the other hand, any $h \in C_n(g)$ can be translated to a function $\widetilde{g} \in \widetilde{C}_n(g)$, such that its restriction in $[1, N]$ coincides with the restriction of h in $[1, N]$. To see this, fix a small $\epsilon > 0$, and define $\widetilde{g}^{(n+1)}(t) = h^{(n+1)}(t) I_{(1,N)}(t)$. Next, fix a point $\theta \in (1, N)$ and define $\widetilde{g}^{(n)}, \widetilde{g}^{(n-1)}, \ldots, \widetilde{g}', \widetilde{g}$ by the recurrence

$$\widetilde{g}^{(n-k)}(t) = \begin{cases} -\displaystyle\int_t^\theta \widetilde{g}^{(n-k+1)}(u)du + h^{(n-k)}(\theta), & \text{if } 1-\epsilon < t \le \theta, \\ \displaystyle\int_\theta^t \widetilde{g}^{(n-k+1)}(u)du + h^{(n-k)}(\theta), & \text{if } \theta \le t < N+\epsilon, \end{cases} \qquad k = 0, 1, \ldots, n.$$

Then, the resulting $\widetilde{g} = \widetilde{g}^{(0)}$ belongs to $\widetilde{C}_n(g)$ and has the desired properties. Therefore, for our purposes, the classes $\widetilde{C}_n$ and C_n (i.e., Conditions A and B) are equivalent (in the sense that they produce the same differential bound).

Now, taking limits in (19), and in view of Lemma 2 and the preceding observations, we immediately obtain the following result:

Theorem 1 *Let $\widetilde{g}$ be any function defined in $[1, N]$, such that $\widetilde{g}(j) = g(j)$, $j = 1, 2, \ldots, N$, and $\widetilde{g}$ is absolutely continuous in $[1, N]$, with a.s. derivative $\widetilde{g}'(t) = \widetilde{g}^{(1)}(t)$, $t \in (1, N)$. Assume also that for $k = 1, \ldots, n$, the function $\widetilde{g}^{(k)}(t)$, $t \in (1, N)$, is absolutely continuous, with a.s. derivative $\widetilde{g}^{(k+1)}(t)$, $t \in (1, N)$. Also, suppose that*

$$\int_1^N (\widetilde{g}^{(k)}(t))^2 dt < \infty, \quad k = 1, 2, \ldots, n+1.$$

Then,

$$(-1)^n[\widetilde{S}_n(\widetilde{g}) - \mathrm{Var}\, g(U_N)] \ge 0, \quad \text{where } \widetilde{S}_n(\widetilde{g}) = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!k!} \int_1^N a_{k-1}(t)(\widetilde{g}^{(k)}(t))^2 dt, \tag{24}$$

and where $a_k(t)$ is given by (17).

Remark 1 (a) We tried to define the class C_n, under which the bound (24) holds true, in such a way that C_n remains as large as possible. The reason is that we want to compare the discrete bound with the largest possible class of differential bounds, as in (24). Of course, inequality (1.5) of [1] (i.e. (19)) has been proved for a more restrictive class of functions and r.v.'s than stated here (although, as the original Mohr and Noll inequality, obtained in [4], it can be slightly extended to a larger class). This fact, however, is in favor of the discrete bound, in the sense that we now consider more competitors than those that could be really obtained from the bound given in eq. (1.5) of [1].
(b) Although we tried to keep the class C_n as large as possible, in fact we did not manage to prove that it is completely independent of the r.v. X. The only detail that remained unproved is to verify that for any absolutely continuous X in a quite large class (as in Lemma 1),

$$\lim_{\epsilon \searrow 0} \int_{J(X_\epsilon)} a^{\epsilon}_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt = \int_1^N a_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt.$$

This holds when $X = V$, as Lemma 2 shows, but our derivation depends on (23), which again depends on the distribution of $X = V$. For example, (23) could not hold if V was chosen to be uniform in the interval $(0, 1)$. In this case, the nature of C_n could be slightly different, although it is intuitively clear that no simpler way exists to obtain meaningful bounds of the form (24). Moreover, it is worth to point out that if C_n could be extended in order to include as many functions $\tilde{g}$ as possible, then the least requirement on $\tilde{g}$ should be the condition

$$\int_1^N a_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt < \infty, \quad k = 1, 2, \ldots, n+1, \tag{25}$$

in place of Condition B(ii).

According to the preceding observations, in the sequel we shall consider functions $\tilde{g}$ from the largest possible class D_n, defined as follows:

Condition C. We say that a real function $\tilde{g}$, with $\tilde{g}(j) = g(j)$, $j = 1, 2, \ldots, N$, belongs to $D_n = D_n(g)$, if it is defined in $[1, N]$, it is absolutely continuous with a.s. derivative $\tilde{g}^{(1)}(t) = \tilde{g}'(t)$, $t \in (1, N)$, and satisfies the following:
(i) For $k = 1, \ldots, n$, the function $\tilde{g}^{(k)}(t)$, $t \in (1, N)$, is absolutely continuous with a.s. derivative $\tilde{g}^{(k+1)}(t)$, $t \in (1, N)$.
(ii) $\int_1^N a_{k-1}(t)(\tilde{g}^{(k)}(t))^2 dt < \infty$ for $k = 1, 2, \ldots, n+1$, where $a_k(t)$ is given by (17).

Clearly, D_n is strictly larger than C_n when $n \geq 1$, because $a_k(t) = O((t-1)^k)$, as $t \searrow 1$, and $a_k(t) = O((N-t)^k)$, as $t \nearrow N$. We do not know if the variance bound, given in (24), holds also for any $\tilde{g} \in D_n$ (it seems so, but we do not have a rigorous proof of it). However, independently if it holds true, or not, the comparison that will be done in the sequel is in favor of the discrete bound, due to the fact that D_n is larger than C_n.

2.3. Discrete versus continuous bound [cases $n=0$ and $n=1$]

The inequality suggested by Corollary 3.1(a) of [1] is equivalent to the variance bound

$$(-1)^n[S_n - \mathrm{Var}\, g(U_N)] \ge 0, \quad \text{where} \quad S_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k!(k+1)!} \frac{1}{N} \sum_{j=1}^{N-k} [j]_k (N-j)_k (\Delta^k g(j))^2. \tag{26}$$

In (26) the equality is attained if and only if the N points $\{(j, g(j)),\ j = 1, \ldots, N\}$ lie on a polynomial curve of degree at most $n+1$, so that $S_n = \mathrm{Var}\, g(U_N)$ whenever $n \ge N-2$.

On the other hand, the differential "bound" $\widetilde{S}_n(\widetilde{g})$, in (24), is meaningful for any $\widetilde{g} \in D_n(g)$, but it depends on the complicated functions $a_k(t)$, given by (17), i.e.,

$$a_k(t) = \frac{1}{N^2} \sum_{j_1 \le j_0} \sum_{j_2 > j_0} (j_2 - j_1)(j_2 - t)^k (t - j_1)^k, \quad j_0 < t < j_0 + 1, \quad j_0 = 1, 2, \ldots, N-1, \tag{27}$$

where the summation ranges over $j_1, j_2 \in \{1, 2, \ldots, N\}$ such that $j_1 \le j_0$ $(= [t])$ and $j_2 > j_0$. Our main purpose is to compare S_n, given in (26), with $\widetilde{S}_n(\widetilde{g})$, given in (24), for any function $\widetilde{g} \in D_n(g)$. To this end, we define

$$\Delta_n = \Delta_n(\widetilde{g}) = (-1)^n[\widetilde{S}_n(\widetilde{g}) - S_n], \quad \widetilde{g} \in D_n(g), \quad n = 0, 1, \ldots\ . \tag{28}$$

Clearly, if $n \ge N-2$, the N points $\{(j, g(j)),\ j = 1, 2, \ldots, N\}$ belong to a polynomial curve of degree at most $n+1$, namely, the Lagrange interpolation polynomial, $P(t)$, passing through the points, so that $P \in \widetilde{C}_n(g)$. Therefore, the inequality in (19) for $\widetilde{g} = P$ becomes an equality for any ϵ. Hence, working as in Theorem 1, it follows that $\widetilde{S}_n(\widetilde{g}) = \mathrm{Var}\, g(U_N)$ when $\widetilde{g} = P$, and the two bounds become equivalent. Thus, we have nothing to compare if $n \ge N-2$: In this case, the discrete bound S_n is better, in the trivial sense that S_n equals to $\mathrm{Var}\, g(U_N)$ while, of course, $\widetilde{S}_n(\widetilde{g})$ may take another value, if we make a wrong choice of $\widetilde{g} \in D_n(g)$. Thus, if $n \ge N-2$ then $\Delta_n(\widetilde{g}) \ge 0$ (assuming that for any $\widetilde{g} \in D_n(g)$, $\widetilde{S}_n(\widetilde{g})$ is indeed an upper/lower bound of the same kind as S_n, for $\mathrm{Var}\, g(U_N)$) and, furthermore,

$$\inf_{\widetilde{g} \in D_n(g)} \Delta_n(\widetilde{g}) = \min_{\widetilde{g} \in D_n(g)} \Delta_n(\widetilde{g}) = 0.$$

In the nontrivial case where $N \ge 3$ and $n \in \{0, 1, \ldots, N-3\}$, the comparison is rather difficult, except if $n = 0$. In this particular case it is easily seen that $a_0(t)$, in (27), is given by

$$a_0(t) = \frac{j(N-j)}{2N}, \quad t \in (j, j+1), \quad j = 1, 2, \ldots, N-1,$$

so that

$$\widetilde{S}_0(\widetilde{g}) = \frac{1}{2N} \sum_{j=1}^{N-1} j(N-j) \int_j^{j+1} (\widetilde{g}'(t))^2 dt. \tag{29}$$

Therefore, for any $\widetilde{g} \in D_0(g)$,

$$\Delta_0(\widetilde{g}) = \widetilde{S}_0(\widetilde{g}) - S_0 = \frac{1}{2N}\sum_{j=1}^{N-1} j(N-j)\left[\int_j^{j+1}(\widetilde{g}'(t))^2dt - \left(\int_j^{j+1}\widetilde{g}'(t)dt\right)^2\right]. \quad (30)$$

Eq. (30) shows that $\Delta_0(\widetilde{g}) \geq 0$, with equality if and only if $\widetilde{g}'(t)$ is a.s. constant in each interval $(j, j+1)$. Thus,

$$\inf_{\widetilde{g}\in D_0(g)} \Delta_0(\widetilde{g}) = \min_{\widetilde{g}\in D_0(g)} \Delta_0(\widetilde{g}) = 0, \quad (31)$$

and the minimum is attained only when $\widetilde{g}$ is the broken line passing through the $N \geq 3$ points.

For $n = 1$ (and $N \geq 4$), it can be seen after some algebra that

$$a_1(t) = \frac{j(N-j)}{2N}(t-j)(j+1-t) + \frac{[j]_2(N-j)_2}{6N}(t-j) + \frac{(j)_2[N-j]_2}{6N}(j+1-t),$$

when $t \in [j, j+1]$, $j = 1, 2, \ldots, N-1$. Moreover, it is easy to check that

$$\sum_{j=1}^{N-2}[j]_2(N-j)_2\int_j^{j+1}\int_{t_1}^{t_1+1}(\widetilde{g}''(t_2))^2dt_2dt_1$$
$$= \sum_{j=1}^{N-1}[j]_2(N-j)_2\int_j^{j+1}(t-j)(\widetilde{g}''(t))^2dt + \sum_{j=1}^{N-1}(j)_2[N-j]_2\int_j^{j+1}(j+1-t)(\widetilde{g}''(t))^2dt.$$

Using this formula and (29), we conclude that for any $\widetilde{g} \in D_1(g)$, $\widetilde{S}_1(\widetilde{g})$ of eq. (24) can be written as

$$\begin{aligned}\widetilde{S}_1(\widetilde{g}) \;&=\; \frac{1}{2N}\sum_{j=1}^{N-1} j(N-j)\left[\int_j^{j+1}(\widetilde{g}'(t))^2dt - \frac{1}{2}\int_j^{j+1}(t-j)(j+1-t)(\widetilde{g}''(t))^2dt\right]\\ &\quad -\frac{1}{12N}\sum_{j=1}^{N-2}[j]_2(N-j)_2\int_j^{j+1}\int_{t_1}^{t_1+1}(\widetilde{g}''(t_2))^2dt_2dt_1.\end{aligned}$$

Noting that for any $\widetilde{g} \in D_1(g)$, $\Delta g(j) = \int_j^{j+1}\widetilde{g}'(t)dt$, $j = 1, 2, \ldots, N-1$, and $\Delta^2 g(j) = \int_j^{j+1}\int_{t_1}^{t_1+1}\widetilde{g}''(t_2)dt_2dt_1$, $j = 1, 2, \ldots, N-2$ (see (1) and (2)), we conclude that

$$\Delta_1(\widetilde{g}) = S_1 - \widetilde{S}_1(\widetilde{g}) = \frac{1}{2N}\sum_{j=1}^{N-1} j(N-j)\theta_j(\widetilde{g}) + \frac{1}{12N}\sum_{j=1}^{N-2}[j]_2(N-j)_2\,\phi_j(\widetilde{g}), \quad (32)$$

where

$$\theta_j(\widetilde{g}) = \left(\int_j^{j+1}\widetilde{g}'(t)dt\right)^2 - \int_j^{j+1}(\widetilde{g}'(t))^2dt + \frac{1}{2}\int_j^{j+1}(t-j)(j+1-t)(\widetilde{g}''(t))^2dt,$$
$$\phi_j(\widetilde{g}) = \int_j^{j+1}\int_{t_1}^{t_1+1}(\widetilde{g}''(t_2))^2dt_2dt_1 - \left(\int_j^{j+1}\int_{t_1}^{t_1+1}(\widetilde{g}''(t_2))^2dt_2dt_1\right)^2.$$

It is now obvious that $\phi_j(\widetilde{g}) \ge 0$, while $\theta_j(\widetilde{g}) \ge 0$, as follows from the continuous Mohr and Noll inequality (see eq. (9)) applied to $\widetilde{g}'$ in the interval $(j, j+1)$. This verifies that

$$\inf_{\widetilde{g}\in D_1(g)} \Delta_1(\widetilde{g}) = \inf_{\widetilde{g}\in D_1(g)} [S_1 - \widetilde{S}_1(\widetilde{g})] \ge 0,$$

and thus, the discrete lower variance bound S_1 is always better than its continuous counterpart, $\widetilde{S}_1(\widetilde{g})$. More important is the fact that S_1 **cannot be recovered** by $\widetilde{S}_1(\widetilde{g})$, as $\widetilde{g}$ varies in $D_1(g)$, even in the limit, in the nontrivial case where (necessarily $N \ge 4$ and) the points $\{(j, g(j)),\ j = 1, 2, \ldots, N\}$ do not belong to a polynomial curve of degree at most 2 (otherwise, S_1 equals to $\mathrm{Var}\, g(U_N)$). In order to verify this fact it suffices, due to formula (32), to find a positive constant c_1, independent of $\widetilde{g} \in D_1(g)$, such that

$$\phi_j(\widetilde{g}) + \phi_{j+1}(\widetilde{g}) \ge c_1[\Delta^2 g(j+1) - \Delta^2 g(j)]^2, \quad j = 1, 2, \ldots, N-3. \tag{33}$$

This is indeed possible, and not so obvious (note that one can take $c_1 = 1/4$ in (33)), and it will be shown in Lemma 6, below (see also Remark 3), in a more general setup. It is now immediate from (32) and (33) (with $c_1 = 1/4$) that the following inequality holds:

$$\inf_{\widetilde{g}\in D_1(g)} \Delta_1(\widetilde{g}) \ge \frac{(N-1)(N-2)}{48N} \sum_{j=1}^{N-3} [\Delta^2 g(j+1) - \Delta^2 g(j)]^2.$$

Thus, the infimum is strictly positive, except in the trivial case where the N points belong to a quadratic.

2.3. Discrete versus continuous bound [the general case]

First, we find another description of the functions $a_k(t)$, defined in (17) or (27). To this end, let $U_1, U_2, \ldots$ be i.i.d. standard uniform r.v.'s, set $S_m = U_1 + \ldots + U_m$, and use the notation

$$a_k^m(t) = a_k^{S_m}(t), \tag{34}$$

(see eq.'s (14)–(16)). Then, we have the following.

Lemma 3 *Let $m \in \{1, 2, \ldots\}$ and $k \in \{0, 1, \ldots\}$. Then, for any $t \in [j_0, j_0+1]$, $j_0 = 0, 1, \ldots, m-1$, we have*

$$a_k^m(t) = \frac{(-1)^m k!(k+1)!}{(k+m)!(k+m+1)!} \times \sum_{j_1=0}^{j_0} \sum_{j_2=j_0+1}^{m} (-1)^{j_1+j_2} \binom{m}{j_1} \binom{m}{j_2} (j_2 - j_1)(t - j_1)^{k+m}(j_2 - t)^{k+m}, \tag{35}$$

while $a_k^m(t) = 0$ for $t \notin (0, m)$.

Proof: It is well-known that the density of S_m, say f_m, is given by

$$f_m(t) = \frac{1}{(m-1)!} \sum_{j_1=0}^{m} (-1)^{j_1} \binom{m}{j_1} (t - j_1)^{m-1} I(j_1 < t < m) \tag{36}$$

$$= \frac{1}{(m-1)!} \sum_{j_2=0}^{m} (-1)^{m-j_2} \binom{m}{j_2} (j_2 - t)^{m-1} I(0 < t < j_2). \tag{37}$$

From (15) we have

$$a_k^m(t) = \int_0^t \int_t^m (y-x)(t-x)^k (y-t)^k f_m(x) f_m(y) dy dx.$$

Substituting $f_m(x)$ from (36) and $f_m(y)$ from (37) in the above formula, we get (35). □

Lemma 4 *For any* $n \in \{0, 1, \ldots\}$, $N \in \{2, 3, \ldots\}$ *and* $t \in \mathbb{R} \setminus \{1, 2, \ldots, N\}$, *the functions* $a_n(t)$ (see (27)), $a_{n-k}^k(t)$, $k = 1, 2, \ldots, n$ (see (35)) *and* $f_{n+1}(t)$ (see (36), (37)) *are related through the following identity:*

$$\begin{aligned} N(n+1)(n+2)a_n(t) &= (n+1)\sum_{k=1}^{n} \binom{n}{k}\binom{n+2}{k+1} \sum_{j=1}^{N-k} [j]_k (N-j)_k a_{n-k}^k(t-j) \\ &+ \sum_{j=1}^{N-n-1} [j]_{n+1}(N-j)_{n+1} f_{n+1}(t-j), \end{aligned} \tag{38}$$

where empty sums should be treated as zero.

Proof: If $t \notin [1, N]$ then both sides of (38) vanish. Thus, we now assume that $t \in (j_0, j_0+1)$ for some fixed $j_0 \in \{1, 2, \ldots, N-1\}$. From (35) we have

$$\begin{aligned} a_{n-k}^k(t-j) &= \frac{(-1)^k (n-k)!(n+1-k)!}{n!(n+1)!} \\ &\times \sum_{j_1=j}^{j_0} \sum_{j_2=j_0+1}^{j+k} (-1)^{j_1+j_2} \binom{k}{j_1-j}\binom{k}{j_2-j} (j_2-j_1)(t-j_1)^n (j_2-t)^n, \end{aligned} \tag{39}$$

provided that $k \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, N-k\}$ are such that $j_0 - k + 1 \le j \le j_0$ (otherwise $a_{n-k}^k(t-j)$ vanishes). Similarly,

$$f_{n+1}(t-j) = \frac{(-1)^j}{n!} \sum_{j_1=j}^{j_0} (-1)^{j_1} \binom{n+1}{j_1-j} (t-j_1)^n, \tag{40}$$

provided that $j \in \{1, 2, \ldots, N-n-1\}$ is such that $j_0 - n \le j \le j_0$ (otherwise $f_{n+1}(t-j)$ vanishes). With the help of (39) and (40), (38) is equivalent to

$$\begin{aligned} &\frac{1}{N} \sum_{j_1 \le j_0} \sum_{j_2 > j_0} (j_2-j_1)(t-j_1)^n (j_2-t)^n \\ &= \sum_{j_1 \le j_0} \sum_{j_2 > j_0} (j_2-j_1)(t-j_1)^n (j_2-t)^n \sum_{k,j} \frac{(-1)^{k+j_1+j_2}}{k!(k+1)!} [j]_k (N-j)_k \binom{k}{j_1-j}\binom{k}{j_2-j} \\ &+ \frac{1}{(n+2)!} \sum_{j_1 \le j_0} (t-j_1)^n \sum_j (-1)^{j+j_1} [j]_{n+1} (N-j)_{n+1} \binom{n+1}{j_1-j}, \end{aligned} \tag{41}$$

where the summation $\sum_{j_1 \le j_0}$ ranges for $j_1 = 1, \ldots, j_0$, the summation $\sum_{j_2 > j_0}$ ranges for $j_2 = j_0+1, \ldots, N$, the summation $\sum_{k,j}$ ranges for all $k \in \{1, 2, \ldots, n\}$, $j \in$

$\{1,2,\ldots,N-k\}$, for which $j_2-k\le j\le j_1$ (if any), and the summation $\sum_j$ ranges for all $j\in\{1,2,\ldots,N-n-1\}$ for which $j_0-n\le j\le j_1$ (if any). Both sides of (41) present a polynomial in the interval (j_0,j_0+1), of degree $2n$. Therefore, the equality (41) can be deduced if we expand $(t-j_1)^n$ and $(j_2-t)^n$ according to binomial formula, on showing that the coefficients of t^i, $i=0,1,\ldots,2n$, in each side, are identical. □

Corollary 1 *Let $n\in\{0,1,\ldots\}$ and $N\in\{2,3,\ldots\}$ be fixed integers. Then, for any $h:[1,N]\longrightarrow\mathbb{R}$,*

$$\begin{aligned}&N(n+1)(n+2)\int_1^N a_n(t)h(t)dt\\&\quad=(n+1)\sum_{k=1}^{n}\binom{n}{k}\binom{n+2}{k+1}\sum_{j=1}^{N-k}[j]_k(N-j)_k\int_j^{j+k}a_{n-k}^k(t-j)h(t)dt\\&\qquad+\sum_{j=1}^{N-n-1}[j]_{n+1}(N-j)_{n+1}\int_j^{j+n+1}f_{n+1}(t-j)h(t)dt,\end{aligned}\tag{42}$$

provided that the integrals are finite.

We can now show the following general result.

Theorem 2 *Let $N\in\{3,4,\ldots\}$, $n\in\{0,1,\ldots,N-3\}$ and assume that $\tilde{g}\in D_n(g)$. Set $S_m=U_1+\ldots+U_m$, $m=1,2,\ldots$, where the U_i's are i.i.d. standard uniform r.v.'s, and define*

$$\phi_j(n,n+1)=\mathrm{Var}[\tilde{g}^{(n+1)}(j+S_{n+1})],\quad j=1,2,\ldots,N-n-1.\tag{43}$$

Also, for $k=1,2,\ldots,n$, $j=1,2,\ldots,N-k$, define

$$\phi_j(n,k)=(-1)^{n-k}\left[\sum_{s=0}^{n-k}\frac{(-1)^s}{s!(s+1)!}\int_j^{j+k}a_s^k(t-j)(\tilde{g}^{(k+s+1)}(t))^2dt-\mathrm{Var}[\tilde{g}^{(k)}(j+S_k)]\right],\tag{44}$$

where the functions $a_s^k(t)$, $t\in(0,k)$, corresponding to S_k, are given by (34) and (35). Then,

$$\Delta_n(\tilde{g})=(-1)^n[\tilde{S}_n(\tilde{g})-S_n]=\frac{1}{N}\sum_{k=1}^{n+1}\frac{1}{k!(k+1)!}\sum_{j=1}^{N-k}[j]_k(N-j)_k\,\phi_j(n,k),\tag{45}$$

where S_n is given by (26) and $\tilde{S}_n(\tilde{g})$ by (24).

Proof: We use induction on n. For $n=0$ the result holds for all $N\ge 3$ (see (30)), since $\phi_j(0,1)=\mathrm{Var}[\tilde{g}'(j+U_1)]=\int_j^{j+1}(\tilde{g}'(t))^2dt-\left(\int_j^{j+1}\tilde{g}'(t)dt\right)^2$. Eq. (45) is also true for $n=1$ and $N\ge 4$, as (32) shows. Indeed,

$$\begin{aligned}\phi_j(1,2)&=\mathrm{Var}[\tilde{g}''(j+U_1+U_2)]\\&=\int_0^1\int_0^1(\tilde{g}''(j+u_1+u_2))^2du_2du_1-\left(\int_0^1\int_0^1\tilde{g}''(j+u_1+u_2)du_2du_1\right)^2\\&=\int_j^{j+1}\int_{t_1}^{t_1+1}(\tilde{g}''(t_2))^2dt_2dt_1-\left(\int_j^{j+1}\int_{t_1}^{t_1+1}\tilde{g}''(t_2)dt_2dt_1\right)^2=\phi_j(\tilde{g}),\end{aligned}$$

where $\phi_j(\widetilde{g})$ is given in (32), and where we used the substitution $t_1 = j + u_1$ and $t_2 = j + u_1 + u_2$ in the above integrals. Also, by (44),

$$\phi_j(1,1) = \int_j^{j+1} a_0^1(t-j)(\widetilde{g}''(t))^2 dt - \mathrm{Var}[\widetilde{g}'(j+U_1)] = \theta_j(\widetilde{g})$$

(see (32)), because $a_0^1(t-j) = (t-j)(j+1-t)/2$ for $t \in (j, j+1)$. Therefore, (45) holds for $n = 0, 1$, and we now assume that it holds for $n-1$ ($n \geq 1$) and that $N - 3 \geq n$, and we shall prove that it is also true for n. By (24) and (26) we have

$$\Delta_n(\widetilde{g}) = -\Delta_{n-1}(\widetilde{g}) + \frac{1}{n!(n+1)!}\int_1^N a_n(t)(\widetilde{g}^{(n+1)}(t))^2 dt$$
$$-\frac{1}{N(n+1)!(n+2)!}\sum_{j=1}^{N-n-1}[j]_{n+1}(N-j)_{n+1}(\Delta^{n+1}g(j))^2.$$

Substituting $\Delta_{n-1}(\widetilde{g})$ from (45) (with $n-1$ in place of n), we see that (45) will be proved if it can be shown that

$$\frac{1}{n!(n+1)!}\int_1^N a_n(t)(\widetilde{g}^{(n+1)}(t))^2 dt$$
$$= \frac{1}{N}\sum_{k=1}^{n}\frac{1}{k!(k+1)!}\sum_{j=1}^{N-k}[j]_k(N-j)_k[\phi_j(n,k) + \phi_j(n-1,k)]$$
$$+\frac{1}{N(n+1)!(n+2)!}\sum_{j=1}^{N-n-1}[j]_{n+1}(N-j)_{n+1}[\phi_j(n,n+1) + (\Delta^{n+1}g(j))^2]. \quad (46)$$

Using the definitions (43) and (44) for $\phi_j(n,k)$, it is easily seen that for $k = 1, 2, \ldots, n$,

$$\phi_j(n,k) + \phi_j(n-1,k) = \frac{1}{(n-k)!(n+1-k)!}\int_j^{j+k} a_{n-k}^k(t-j)(\widetilde{g}^{(n+1)}(t))^2 dt. \quad (47)$$

Also, it follows from (1) and (2) (with $a = j$, $\delta = 1$, $k = n+1$), using the substitution $t_k = j + u_1 + \ldots + u_k$, $k = 1, 2, \ldots, n+1$, that

$$\begin{aligned}\Delta^{n+1}g(j) &= \int_j^{j+1}\int_{t_1}^{t_1+1}\cdots\int_{t_n}^{t_n+1}\widetilde{g}^{(n+1)}(t_{n+1})dt_{n+1}\ldots dt_2 dt_1\\ &= \int_0^1\int_0^1\cdots\int_0^1 \widetilde{g}^{(n+1)}(j+u_1+u_2+\ldots+u_{n+1})du_{n+1}\ldots du_2 du_1\\ &= \mathbb{E}[\widetilde{g}^{(n+1)}(j+S_{n+1})], \end{aligned} \quad (48)$$

and thus,

$$\phi_j(n,n+1) + (\Delta^{n+1}g(j))^2 = \mathbb{E}[\widetilde{g}^{(n+1)}(j+S_{n+1})]^2 = \int_j^{j+n+1} f_{n+1}(t-j)(\widetilde{g}^{(n+1)}(t))^2 dt, \quad (49)$$

where f_{n+1} is the density of S_{n+1}.

Now observe that, in view of (47) and (49), (42) with $h = (\widetilde{g}^{(n+1)})^2$ leads to (46), completing the proof. □

Remark 2 The quantities $\phi_j(n,k)$, $k=1,\ldots,n+1$, appearing in (45), are nonnegative. This is obvious for $\phi_j(n,n+1)$ (see (43)), while for the other ϕ_j's, defined by (44), this fact follows from the inequality (1.5) of [1], applied to $X=j+S_k$, for $g=\widetilde{g}^{(k)}$, and with $n-k$ in place of n (see also [5]). Therefore, for all $n\le N-3$, $n=0,1,\ldots$,

$$\inf_{\widetilde{g}\in D_n(g)}(-1)^n[\widetilde{S}_n(\widetilde{g})-S_n]\ge 0, \tag{50}$$

showing that S_n of (26) is always a better bound than $\widetilde{S}_n(\widetilde{g})$ of (24), for all nontrivial cases, and for any $\widetilde{g}$ passing through the points, and satisfying natural regularity assumptions (see Condition C). Furthermore, another conclusion is that $\widetilde{S}_n(\widetilde{g})$ of (24) is, indeed, a variance bound for $\mathrm{Var}\,g(U_N)$, for all $\widetilde{g}\in D_n(g)$. In particular, it is indeed true that

$$\widetilde{S}_{2n+1}(\widetilde{g})\le S_{2n+1}\le \mathrm{Var}\,g(U_N)\le S_{2n}\le \widetilde{S}_{2n}(\widetilde{g}),$$

where the first inequality holds whenever $2n+1\le N-3$ and $\widetilde{g}\in D_{2n+1}(g)$, while the last one holds whenever $2n\le N-3$ and $\widetilde{g}\in D_{2n}(g)$.

2.4. The discrete bound is strictly better than the continuous one

The last detail that will be proved here is that equality is not possible in (50), except in trivial cases. To this end, we first prove a useful Lemma.

Lemma 5 *Let f_k, $k\ge 2$, be the density of $S_k=U_1+\ldots+U_k$* (see (36) and (37)), *and define*

$$A_k=\int_0^k \frac{f_k^2(t)}{f_k(t)+f_k(t-1)}dt. \tag{51}$$

Then,

$$\frac{1}{2}<A_k<1. \tag{52}$$

Proof: First observe that

$$A_k=\int_0^{k+1} \frac{f_k^2(t)}{f_k(t)+f_k(t-1)}dt,$$

and the denominator is strictly positive in $(0,k+1)$. Also define

$$B_k=\int_0^{k+1} \frac{f_k(t)f_k(t-1)}{f_k(t)+f_k(t-1)}dt=\int_1^{k} \frac{f_k(t)f_k(t-1)}{f_k(t)+f_k(t-1)}dt$$

and

$$C_k=\int_0^{k+1} \frac{f_k^2(t-1)}{f_k(t)+f_k(t-1)}dt=\int_1^{k+1} \frac{f_k^2(t-1)}{f_k(t)+f_k(t-1)}dt=\int_0^{k} \frac{f_k^2(t)}{f_k(t)+f_k(t+1)}dt.$$

Since $k \geq 2$ and the integrands are strictly positive in $[1, k]$, and nonnegative in $(0, k+1)$, it is obvious that $A_k > 0$, $B_k > 0$ and $C_k > 0$. Observe that $f_k(t) = f_k(k-t)$. Hence, with the substitution $t = k - u$ in the last integral defining C_k, we get

$$C_k = \int_0^k \frac{f_k^2(k-u)}{f_k(k-u) + f_k(k-(u-1))} du = \int_0^k \frac{f_k^2(u)}{f_k(u) + f_k(u-1)} du = A_k.$$

Therefore,

$$\begin{aligned} 2A_k + 2B_k = A_k + 2B_k + C_k &= \int_0^{k+1} \frac{f_k^2(t) + 2f_k(t)f_k(t-1) + f_k^2(t-1)}{f_k(t) + f_k(t-1)} dt \\ &= \int_0^{k+1} (f_k(t) + f_k(t-1)) dt = 2, \end{aligned}$$

because $f_k(t)$ and $f_k(t-1)$ are probability densities for $t \in (0, k+1)$. This shows that $B_k = 1 - A_k$, and since $B_k > 0$, it follows that $A_k < 1$. Similarly,

$$\begin{aligned} 2A_k - 2B_k = A_k - 2B_k + C_k &= \int_0^{k+1} \frac{f_k^2(t) - 2f_k(t)f_k(t-1) + f_k^2(t-1)}{f_k(t) + f_k(t-1)} dt \\ &= \int_0^{k+1} \frac{(f_k(t) - f_k(t-1))^2}{f_k(t) + f_k(t-1)} dt > 0, \end{aligned}$$

i.e., $2A_k - 2(1 - A_k) > 0$, which shows (52). $\Box$

Remark 3 It is easy to see that

$$A_2 = 5/6, \tag{53}$$

while for $k \geq 3$ the exact calculation of A_k is not that easy.

Lemma 6 *Let $n \in \{1, 2, \ldots\}$ and $N - 3 \geq n$. Then, there exists a constant $c_n > 0$ such that for any $\widetilde{g} \in D_n(g)$ and any $j = 1, 2, \ldots, N - n - 2$,*

$$\phi_j(n, n+1) + \phi_{j+1}(n, n+1) \geq c_n [\Delta^{n+1} g(j+1) - \Delta^{n+1} g(j)]^2, \tag{54}$$

where $\phi_j(n, n+1)$ is given by (43). The constant c_n in (54) can be taken as

$$c_n = \frac{1 - A_{n+1}}{2A_{n+1} - 1} > 0, \tag{55}$$

where $A_{n+1} \in (1/2, 1)$ is given by (51). [In particular, $c_1 = 1/4$.]

Proof: For $t \in (j, j + n + 2)$ we define

$$u_1(t) = f_{n+1}(t - j) + f_{n+1}(t - j - 1), \quad u_2(t) = f_{n+1}(t - j - 1) - f_{n+1}(t - j),$$

where f_{n+1} is the density of $S_{n+1} = U_1 + \ldots + U_{n+1}$. Also, for $\widetilde{g} \in D_n(g)$ we set (see (48))

$$\begin{aligned} A &= \Delta^{n+1} g(j) = \mathbb{E}[\widetilde{g}^{(n+1)}(j + S_{n+1})] = \int_j^{j+n+2} f_{n+1}(t - j) \widetilde{g}^{(n+1)}(t) dt, \\ B &= \Delta^{n+1} g(j+1) = \mathbb{E}[\widetilde{g}^{(n+1)}(j + 1 + S_{n+1})] = \int_j^{j+n+2} f_{n+1}(t - j - 1) \widetilde{g}^{(n+1)}(t) dt, \end{aligned}$$

so that

$$A+B=\int_j^{j+n+2} u_1(t)\widetilde{g}^{(n+1)}(t)dt, \quad B-A=\int_j^{j+n+2} u_2(t)\widetilde{g}^{(n+1)}(t)dt.$$

It is clear that $u_1(t)>0$ for $t\in(j,j+n+2)$, and obviously,

$$\int_j^{j+n+2}\left[\widetilde{g}^{(n+1)}(t)-\frac{1}{2}(A+B)-\frac{1}{2(2A_{n+1}-1)}(B-A)\frac{u_2(t)}{u_1(t)}\right]^2 u_1(t)dt\geq 0, \qquad (56)$$

where A_{n+1} is given by (51). Expanding the square in (56) and taking into account the fact that (see the proof of Lemma 5)

$$\int_j^{j+n+2} u_1(t)dt=2, \quad \int_j^{j+n+2}\frac{u_2^2(t)}{u_1(t)}dt=2(2A_{n+1}-1), \quad \int_j^{j+n+2} u_2(t)dt=0,$$

we obtain the inequality

$$\int_j^{j+n+2} u_1(t)(\widetilde{g}^{(n+1)}(t))^2dt-\frac{1}{2}(A+B)^2-\frac{1}{2(2A_{n+1}-1)}(B-A)^2\geq 0,$$

that is,

$$\int_j^{j+n+2} u_1(t)(\widetilde{g}^{(n+1)}(t))^2dt-\frac{1}{2}(A+B)^2-\frac{1}{2}(B-A)^2\geq\frac{1-A_{n+1}}{2A_{n+1}-1}(B-A)^2. \qquad (57)$$

Now, using $(A+B)^2/2+(B-A)^2/2=A^2+B^2$, and observing that

$$\begin{aligned}\int_j^{j+n+2} u_1(t)(\widetilde{g}^{(n+1)}(t))^2dt-A^2-B^2 &= \mathrm{Var}[\widetilde{g}^{(n+1)}(j+S_{n+1})] \\ &\quad+\mathrm{Var}[\widetilde{g}^{(n+1)}(j+1+S_{n+1})] \\ &= \phi_j(n,n+1)+\phi_{j+1}(n,n+1),\end{aligned}$$

we see that (57) is equivalent to (54), completing the proof. □

The final conclusion of the present analysis is included the following Theorem.

Theorem 3 *Let $n\in\{1,2,\ldots\}$ and $N\geq n+3$. Then, there is a constant $\theta(n,N)>0$, such that for any $\widetilde{g}\in D_n(g)$, the following inequality holds:*

$$(-1)^n[\widetilde{S}_n(\widetilde{g})-S_n]\geq\theta(n,N)\sum_{j=1}^{N-n-2}[\Delta^{n+1}g(j+1)-\Delta^{n+1}g(j)]^2. \qquad (58)$$

Moreover, $\theta(n,N)$ in (58) can be taken as

$$\theta(n,N)=\frac{1}{2N(n+2)}\binom{N-1}{n+1}\frac{1-A_{n+1}}{2A_{n+1}-1}, \qquad (59)$$

where $A_{n+1}\in(1/2,1)$ is given by (51).

Proof: Keeping the last term of the sum in (45) we have

$$(-1)^n[\widetilde{S}_n(\widetilde{g}) - S_n] \geq \frac{1}{N(n+1)!(n+2)!} \sum_{j=1}^{N-n-1} [j]_{n+1}(N-j)_{n+1}\ \phi_j(n, n+1),$$

and since

$$\min_{j\in\{1,2,\ldots,N-n-1\}} [j]_{n+1}(N-j)_{n+1} = [1]_{n+1}(N-1)_{n+1} = [(n+1)!]^2 \binom{N-1}{n+1},$$

it follows that

$$(-1)^n[\widetilde{S}_n(\widetilde{g}) - S_n] \geq \frac{1}{N(n+2)} \binom{N-1}{n+1} \sum_{j=1}^{N-n-1} \phi_j(n, n+1).$$

Obviously,

$$\sum_{j=1}^{N-n-1} \phi_j(n, n+1) \geq \frac{1}{2} \sum_{j=1}^{N-n-2} [\phi_j(n, n+1) + \phi_{j+1}(n, n+1)],$$

and Lemma 6 (see (54)) completes the proof. □

Final Conclusion: *In cases where the sum in the* RHS *of* (58) *is strictly positive, it is* **impossible** *to recover the discrete bound* (26) *by the continuous one* (24)*, even in the limit. Thus, unless* $n = 0$ *or the* N *points* $\{(j, g(j)),\ j = 1, 2, \ldots, N\}$ *belong to a polynomial curve of degree at most* $n + 1$*, the discrete bound is* **strictly better**.

References

[1] Afendras, G., Papadatos, N. and Papathanasiou, V. (2007). The discrete Mohr and Noll Inequality with applications to variance bounds. *Sankhyā* (accepted for publication).

[2] Afendras, G., Papadatos, N. and Papathanasiou, V. (2005). The discrete Mohr and Noll Inequality with applications to variance bounds. Technical Report TR\12\2005, Dept. of Mathematics and Statistics, University of Cyprus.
(http://www.mas.ucy.ac.cy/technicalreports/ListofTechnicalReports2005.htm)

[3] Johnson, R. W. (1993). A note on variance bounds for a function of a Pearson variate. *Statist. Decisions* **11** 273–278.

[4] Mohr, E. and Noll, W. (1952). Eine Bemerkung zur Schwarzschen Ungleichheit (in German). *Math. Nachr.* **7** 55–59.

[5] Papathanasiou, V. (1988). Variance bounds by a generalization of the Cauchy-Schwarz inequality. *Statist. Probab. Lett.* **7** 29–33.