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DISTANCE IN VARIATION BETWEEN TWO ARBITRARY DISTRIBUTIONS VIA THE ASSOCIATED w-FUNCTIONS

Для расстояния по полной вариации между двумя произвольными вероятностными мерами получены вариационные неравенства в терминах w-функций, соответствующих этим мерам. Эти результаты распространяются на расстояние между распределением суммы зависимых величин и произвольным распределением. Приведено несколько примеров.

Ключевые слова и фразы: расстояние по полной вариации, w-функция.

1. Introduction. Cacoullos, Papathanasiou, and Utev [1] established upper bounds for the distance in variation between an arbitrary distribution and the standard normal distribution in terms of the Fisher information and the w-function (see (2.1)). Moreover, they obtained another proof of the central limit theorem. Also, Papathanasiou and Utev [2] extended these results in the discrete case to approximate an arbitrary discrete distribution by the Poisson one.

Recently Papadatos and Papathanasiou [3] obtained variational inequalities for two arbitrary probability measures in terms of a Fisher-type information and gave applications to extreme-value theory.

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In the present paper bounds are obtained for the total variation distance between two arbitrary distributions F, G

$$\rho(F,G) = \sup_A \left|F(A) - G(A)\right|, \qquad A \text{ is a Borel set,}$$

via the corresponding w-functions. The results are extended for the distance between a distribution of a sum of dependent random variables (r. v.) and an arbitrary distribution. The results are obtained by using some identities for the covariance (see [4], [5]) and choosing a special function $l(x)$ (see (2.3)). The results are illustrated by some examples, given in Section 5.

2. Distance in variation and the w-function. This section deals with the evaluation of the distance $\rho(F,G)$ between F and G in terms of their associated w-functions (see [4]); if X is a continuous r. v. with density f and interval support I, the corresponding w-function is defined by

$$\sigma^2 w(x)\, f(x) = \int_{-\infty}^{x} (\mu - t)\, f(t)\, dt, \tag{2.1}$$

provided that $\mu = \mathrm{E}[X]$ and $\sigma^2 = \mathrm{Var}[X]$ exist.

The function $w(x)$ determines uniquely the density $f(x)$ and thus it characterizes the corresponding distribution, for example, $w(x) \equiv 1$ if and only if X is normal (see [4]).

Now consider two arbitrary random variables X and Y with interval supports (a,b), (a',b'), $-\infty \leqslant a' \leqslant a < b \leqslant b' \leqslant +\infty$, distribution functions (d. f.) F, G and densities f, g, respectively.

We need the following two lemmas.

Lemma 2.1 ([4]). *Under the preceding conditions, for every absolutely continuous function h with*

$$\mathrm{E}\left|(X-\mu)\, h(X)\right| < \infty, \quad \mathrm{E}\left|w(X)\, h'(X)\right| < \infty,$$

we have the identity

$$\mathrm{Cov}\left[X,\, h(X)\right] = \sigma^2 \mathrm{E}\left[w(X)\, h'(X)\right]. \tag{2.2}$$

Define now a function $l(x)$ by

$$l(x,A) = \frac{1}{g(x) w_Y(x)} \int_{a'}^{x} \Big(I_A(u) - G(A)\Big)\, g(u)\, du, \tag{2.3}$$

(cf. [6]), where A is a Borel set, $x \in (a',b')$, $w_Y(x)$ is the w-function associated with Y and $\mathrm{E}[Y] = \mu_Y$, $\mathrm{Var}[Y] = \sigma_Y^2$.

Lemma 2.2. *Suppose at least one of the following conditions hold:*

(i) $a' > -\infty$ *and* $b' < +\infty$;

(ii) $a' > -\infty$, $b' = +\infty$ *and* $\displaystyle\lim_{x\to+\infty} \frac{1 - G(x)}{x\, g(x)}$ *exists;*

(iii) $a' = -\infty$, $b' < +\infty$ *and* $\displaystyle\lim_{x\to-\infty} \left(-\frac{G(x)}{x\, g(x)}\right)$ *exists;*

(iv) $a' = -\infty$, $b' = +\infty$ *and* $\displaystyle\lim_{x\to-\infty} \left(-\frac{G(x)}{x\, g(x)}\right), \lim_{x\to+\infty} \frac{1 - G(x)}{x\, g(x)}$ *exist.*

Then there exist constants $c_1 < \infty$ and $c_2' < \infty$ depending only on G such that

$$\sup_A \sup_x \left|l(x,A)\right| \leqslant c_2', \tag{2.4}$$

$$\sup_A \sup_x \left|w_Y(x)\, l'(x,A)\right| \leqslant c_1. \tag{2.5}$$

P r o o f. We only consider case (iii). The remaining cases are similar. Suppose that $a' = -\infty$, $b' < +\infty$. The existence of

$$\lim_{x \to -\infty} -\frac{G(x)}{xg(x)} = s$$

guarantees that the limit s belongs to the interval $[0, 1]$.

Indeed, s is also the limit[1)] of $H(x)$ as $x \to -\infty$, where

$$H(x) = \frac{\int_{-\infty}^{x} G(y)\, dy}{\int_{-\infty}^{x} (-y\, g(y))\, dy}.$$

Integrating by parts the numerator, we have

$$H(x) = 1 - \frac{(-x\, G(x))}{\int_{-\infty}^{x} (-yg(y))\, dy} \in [0, 1],$$

since the functions $(-xG(x))$ and $(-xg(x))$ are non-negative for $x < 0$. This implies that $s \in [0, 1]$.

By definition (2.3), we also have

$$l(x, A) = -\frac{1}{w_Y(x)\, g(x)} \int_x^{b'} \Big(I_A(u) - G(A)\Big)\, g(u)\, du.$$

Hence,

$$\Big|l(x, A)\Big| \leqslant \frac{1}{w_Y(x)\, g(x)} \min\Big\{G(x),\, 1 - G(x)\Big\}, \qquad a' < x < b'. \tag{2.6}$$

We observe that

$$\lim_{x \to -\infty} \frac{\min\{G(x),\, 1 - G(x)\}}{w_Y(x)\, g(x)} = 0$$

and similarly the limit is finite if $x \to b'$.

Therefore, the function $(w_Y(x)\, g(x))^{-1} \min\{G(x),\, 1 - G(x)\}$ is continuous and has finite limits at both end-points a', b'. Hence, there exists a constant $c_2' < \infty$ such that

$$\sup_x \frac{\min\{G(x),\, 1 - G(x)\}}{w_Y(x)\, g(x)} = c_2'$$

and by (2.6) we conclude (2.4).

We also have

$$w_Y(x)\, l'(x, A) = I_A(x) - G(A) + \frac{x - \mu_Y}{\sigma_Y^2}\, l(x, A), \qquad a' < x < b'. \tag{2.7}$$

Thus,

$$\Big|w_Y(x)\, l'(x, A)\Big| \leqslant 1 + \frac{1}{\sigma_Y^2} \Big\{c_2' |\mu_Y| + \Big|x\, l(x, A)\Big|\Big\}.$$

By (2.6) we obtain

$$\Big|xl(x, A)\Big| \leqslant \frac{|x|}{w_Y(x)\, g(x)} \min\Big\{G(x),\, 1 - G(x)\Big\}.$$

[1)] Thanks are due to the anonymous referee for this remark.

The relation

$$\lim_{x \to -\infty} \frac{|x|}{w_Y(x)\, g(x)} \min\Big\{G(x),\, 1 - G(x)\Big\} \leqslant \sigma_Y^2$$

is valid and similarly the limit is finite as $x \to b'$.

In all cases the continuous function $(|x|/(w_Y(x)\, g(x))) \min\{G(x),\, 1 - G(x)\}$ has finite limits at both end-points and hence it is bounded by a constant c_3.

Setting

$$c_1 = 1 + \frac{1}{\sigma_Y^2} \Big\{c_2'|\mu_Y| + c_3\Big\},$$

we obtain the desired result (2.5).

We give now the main result as stated in the following theorem.

Theorem 2.1. *Given two arbitrary distribution functions F and G satisfying the conditions of Lemmas 2.1 and 2.2, there exist constants $c_1 < \infty$ and $c_2 < \infty$ depending only on G such that*

$$\rho(F, G) \leqslant c_1 \mathrm{E} \left| 1 - \frac{\sigma_X^2 w_X(X)}{\sigma_Y^2 w_Y(X)} \right| + c_2 |\mu_X - \mu_Y|, \tag{2.8}$$

where $\mu_X = \mathrm{E}[X]$, $\sigma_X^2 = \mathrm{Var}[X]$, and $w_X(x)$ is the w-function associated with X.

P r o o f. Taking expectations in (2.7); we obtain

$$F(A) - G(A) = \mathrm{E}\left[w_Y(X)\, l'(X, A)\right] - \frac{1}{\sigma_Y^2} \mathrm{E}\left[(X - \mu_Y)\, l(X, A)\right]. \tag{2.9}$$

Using Lemma 2.1, we have

$$\begin{aligned} \mathrm{E}\left[(X - \mu_Y)\, l(X, A)\right] &= \mathrm{Cov}\,\Big(X, l(X, A)\Big) + (\mu_X - \mu_Y)\, \mathrm{E}\Big[l(X, A)\Big] \\ &= \sigma_X^2 \mathrm{E}\left[w_X(X)\, l'(X, A)\right] + (\mu_X - \mu_Y)\, \mathrm{E}\Big[l(X, A)\Big]. \end{aligned}$$

This combined with (2.9) gives

$$\begin{aligned} \Big|F(A) - G(A)\Big| &\leqslant \mathrm{E}\left\{ \Big|w_Y(X)\, l'(X, A)\Big| \left|1 - \frac{\sigma_X^2 w_X(X)}{\sigma_Y^2 w_Y(X)}\right| \right\} \\ &\quad + \frac{|\mu_X - \mu_Y|}{\sigma_Y^2} \mathrm{E}\Big|l(X, A)\Big|. \end{aligned} \tag{2.10}$$

Taking in (2.10) the sup with respect to X and A and applying Lemma 2.2 complete the proof of the theorem with $c_2 = c_2'/\sigma_Y^2$.

Theorem 2.1 also holds in all cases, where the functions l and wl' are bounded.

Corollary 2.1. *If X and Y have the same means and variances, then*

$$\rho(F, G) \leqslant c_1 \mathrm{E} \left| 1 - \frac{w_X(X)}{w_Y(X)} \right|. \tag{2.11}$$

Corollary 2.2. *If Y is normal and both X and Y have mean 0 and variance 1, then*

$$\rho(F, \Phi) \leqslant 2\, \mathrm{E} \Big| 1 - w_X(X) \Big| \tag{2.12}$$

(*see* [1]).

P r o o f. $\Phi(x)$ satisfies (iv) of Lemma 2.2; moreover, the constant $c_1 = 2$ and (2.12) follows from Corollary 2.1.

3. Discrete case. Here we obtain similar results for the discrete case. Consider two arbitrary discrete random variables X and Y with probability functions (p. f.) p, q, d. f. P, Q, means μ_X, μ_Y and variances $\sigma_X^2 < \infty$, $\sigma_Y^2 < \infty$, respectively. Suppose also that X and Y take values in integer intervals, $\{0, 1, \dots, b\}$ and $\{0, 1, \dots, b'\}$, respectively, where $0 < b \leqslant b' \leqslant +\infty$. The w-function is defined here for each point in the support of X by

$$w_X(x) = \frac{1}{\sigma_X^2 p(x)} \sum_{k=0}^{x} (\mu_X - k)\, p(k), \qquad x = 0, 1, \dots, b. \tag{3.1}$$

The following lemmas hold.

Lemma 3.1 ([4]). *For each real-valued function $h(x)$ with*

$$\mathrm{E}\left|w_X(X)\,\Delta h(X)\right| < \infty, \quad \mathrm{E}\left|(X - \mu_X)\, h(X)\right| < \infty,$$

we have

$$\mathrm{Cov}\left[X,\, h(X)\right] = \sigma_X^2\, \mathrm{E}\left[w_X(X)\,\Delta h(X)\right]. \tag{3.2}$$

We define the function l by

$$l(x, A) = \frac{1}{w_Y(x-l)\, q(x-l)} \sum_{k=0}^{x-1} \Big(I_A(k) - Q(A)\Big)\, q(k) \tag{3.3}$$

(we set $l(x, A) = 0$ for $x \neq 1, 2, \dots, b'$).

For the function l we give the following

Lemma 3.2. *Let one of the following statements hold:*

(i) $b' < \infty$;

(ii) $b' = +\infty$ *and* $\lim_{x\to+\infty} \sup \dfrac{1 - Q(x)}{w_Y(x) q(x)} < +\infty$. *Then there exist constants* $c_1' < \infty$ *and* $c_2' < \infty$ *(depending only on Q) such that*

$$\sup_A \sup_x \left|l(x, A)\right| \leqslant c_2', \tag{3.4}$$

$$\sup_A \sup_x \left|\Delta\, l(x, A)\right| \leqslant c_1'. \tag{3.5}$$

P r o o f. When (i) holds relation (3.4) is obvious. If $b' = +\infty$, we conclude (3.4) by the boundedness of $(1 - Q(x))/(w_Y(x)\, q(x))$, since

$$\left|l(x, A)\right| \leqslant \frac{\min\{Q(x-1)\, 1 - Q(x-1)\}}{w_Y(x-1)\, q(x-1)}.$$

Relation (3.5) is consequence of (3.4), in view of

$$\left|\Delta\, l(x, A)\right| \leqslant \left|l(x+1, A)\right| + \left|l(x, A)\right|.$$

The main result is stated as

Theorem 3.1. *Let Q be as in Lemma 3.2. Then there exist constants $c_1 < \infty$ and $c_2 < \infty$ such that*

$$\rho(P, Q) \leqslant c_1\, \mathrm{E}\left|\sigma_X^2 w_X(X) - \sigma_Y^2 w_Y(X)\right| + c_2|\mu_X - \mu_Y|, \tag{3.6}$$

where w_Y is the w-function of q.

P r o o f. Relation (3.3) yields

$$I_A(x) - Q(A) = w_Y(X)\, \Delta l(x, A) - \frac{x - \mu_Y}{\sigma_Y^2}\, l(x, A).$$

Taking expectations, we obtain

$$P(A) - Q(A) = \mathrm{E}\left[w_Y(X)\,\Delta l(X, A)\right] - \frac{1}{\sigma_Y^2}\,\mathrm{E}\left[(X - \mu_Y)\,l(X, A)\right].$$

Applying Lemma 3.1 for $h(x) = l(x, A)$ and Lemma 3.2 for $c_1 = c_1'/\sigma_Y^2$, $c_2 = c_2'/\sigma_Y^2$ completes the proof of Theorem 3.1.

If X and Y have the same means and variances, then we have the following corollaries.

Corollary 3.1. *Under the above conditions,*

$$\rho(P, Q) \leqslant c\mathrm{E}\left|w_X(X) - w_Y(X)\right|. \tag{3.7}$$

Corollary 3.2. *If Q is the Poisson d.f., then*

$$\rho(P, Q) \leqslant c\mathrm{E}\left|w_X(X) - 1\right| \tag{3.8}$$

(*see* [2]).

4. Distance for the sum of dependent components. Let $\mathbf{X}$ be a continuous n-dimensional random vector with density $f(\mathbf{x})$ in the n-rectangle I^n: $a_i \leqslant x_i < b_i$, $-\infty \leqslant a_i < b_i \leqslant \infty$, $i = 1, 2, \ldots, n$, and dispersion matrix $D(\mathbf{X}) = (\sigma_{ij}) = \Sigma > 0$. Consider the linear function

$$q^i(\mathbf{X}) = \sum_{j=1}^{n} \sigma_{ij}^* X_j, \qquad i = 1, \ldots, n, \tag{4.1}$$

where $\Sigma^{-1} = (\sigma_{ij}^*) = \Sigma^*$, that is, in matrix form

$$q(\mathbf{x}) = \Sigma^* \mathbf{x}.$$

Cacoullos and Papathanasiou [5] defined the functions $w^i(\mathbf{x})$ for every point $\mathbf{x}$ in the support of f by the relations

$$w^i(\mathbf{x})\,f(\mathbf{x}) = \int_{a_i}^{x_i} \left(\mu^i - q^i(u, t_i, v)\right) f(u, t_i\, v)\,dt_i, \tag{4.2}$$

where $u = (x_1, \ldots, x_{i-1})$, $v = (x_{i+1}, \ldots, x_n)$ and $\mu^i = \mathrm{E}\left[q^i(\mathbf{X})\right]$. Furthermore, assuming that the functions $w^i(\mathbf{x})\,f(x) \to 0$ monotonically as $\mathbf{x}$ approaches any boundary point of I^n along the coordinate axes, they established the identity

$$\mathrm{Cov}\left[q^i(\mathbf{X}), h(\mathbf{X})\right] = \mathrm{E}\left[w^i(\mathbf{X})\,h_i(\mathbf{X})\right], \qquad i = 1, \ldots, n, \tag{4.3}$$

for every differentiable h, provided that

$$\mathrm{E}\left|w^i h_i\right| \leqslant \infty, \quad \mathrm{E}\left|(q^i - \mu^i)\,h\right| < \infty,$$

where $h_i = \frac{\partial}{\partial x_i} h(\mathbf{x})$. Here, for our purposes, we modify identity (4.3) and we give the following

Lemma 4.1. *Under the above regularity conditions,*

$$\mathrm{Cov}\left[\sum_{i=1}^{n} X_i, h(\mathbf{X})\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\,\mathrm{E}\left[w^i(\mathbf{X})\,h_i(\mathbf{X})\right]. \tag{4.4}$$

P r o o f. Identity (4.4) is an immediate consequence of (4.3), since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\,\mathrm{E}\left[w^i(\mathrm{X})\,h_i(\mathrm{X})\right]=\sum_{i=1}^{n}\mathrm{Cov}\left[q^i(\mathrm{X}),\,h(\mathrm{X})\right]\sum_{j=1}^{n}\sigma_{ij}$$
$$=\sum_{k=1}^{n}\mathrm{Cov}\left[X_k,\,h(\mathrm{X})\right]\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma^{*}_{ik}\sigma_{ij}=\mathrm{Cov}\left[\sum_{i=1}^{n}X_i,\,h(\mathrm{X})\right].$$

The main result is given in the next

Theorem 4.1. *Let Y be a univariate r.v. with d.f. G as in Theorem 2.1, and X a random vector as in Lemma 4.1 with $\sum_{i=1}^n a_i \geqslant a'$, $\sum_{i=1}^n b_i \leqslant b'$. Then there exist constants $c_1<\infty$ and $c_2<\infty$ such that*

$$\rho(F,\,G)\leqslant c_1\mathrm{E}\left|1-\frac{\sum_{i=1}^n\sum_{j=1}^n\sigma_{ij}w^i(\mathrm{X})}{\sigma_Y^2\,w_Y(X)}\right|+c_2|\mu_X-\mu_Y|, \tag{4.5}$$

where F is the d.f. of $X=\sum_{i=1}^n X_i$, $\mu_X=\mathrm{E}[X]=\sum_{i=1}^n\mu_i$, $\mu_Y=\mathrm{E}[Y]$, $\sigma_Y^2=\mathrm{Var}[Y]$.

P r o o f. Identity (4.4) for $h(\mathrm{x})=l(x,A)$ becomes ($x=\sum_{i=1}^n x_i$)

$$\mathrm{Cov}\left[X,\,l(X,A)\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\,\mathrm{E}\left[w^i(\mathrm{X})\,l'(X,A)\right].$$

Using the same arguments as in Theorem 2.1, we obtain the required result (4.5).

Corollary 4.1. *If G is normal with $\sigma_Y^2=1$ and $\mu_X=\mu_Y$, then Theorem 4.1 yields*

$$\rho(F,\,G)\leqslant c_1\mathrm{E}\left|1-\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}w^i(\mathrm{X})\right|. \tag{4.6}$$

We extend the foregoing results to the discrete case. Let X be an n-dimensional discrete random vector with probability function $p(\mathrm{x})$.

The discrete analogue of identity (4.3) (see [5]) is

$$\mathrm{Cov}\left[q^i(\mathrm{X}),\,h(\mathrm{X})\right]=\mathrm{E}\left[w^i(\mathrm{X})\,\Delta_i h(\mathrm{X})\right],\qquad i=1,\ldots,n, \tag{4.7}$$

where Δ_i denotes the ith partial difference operator:

$$\Delta_i h(\mathrm{x})=h(x_1,\ldots,x_i+1,\ldots,x_n)-h(x_1,\ldots,x_i,\ldots,x_n)$$

and the $w^i(\mathrm{x})$ are defined by

$$w^i(\mathrm{x})\,p(\mathrm{x})=\sum_{k_i=0}^{x_i}\left(\mu^i-q^i(u,\,k_i,\,v)\right)p(u,\,k_i,\,v),\qquad i=1,\ldots,n. \tag{4.8}$$

We formulate the discrete analogue of Lemma 4.1 and Theorem 4.1. The proofs, being easily established, are omitted.

Lemma 4.2. *Let X be an n-dimensional random vector as above and let $h(\mathrm{x})$ be an arbitrary real-valued function such that*

$$\mathrm{E}\left|w^i(\mathrm{X})\,\Delta_i h(\mathrm{X})\right|<\infty,\quad \mathrm{E}\left|\left(\mu^i-q^i(\mathrm{X})\,h(\mathrm{X})\right|<\infty,$$

then

$$\mathrm{Cov}\left[\sum_{i=1}^{n}X_i,\,h(\mathrm{X})\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\,\mathrm{E}\left[w^i(\mathrm{X})\,\Delta_i h(\mathrm{X})\right]. \tag{4.9}$$

Theorem 4.2. *Let Q be a univariate d.f. as in Theorem 3.1 and let $b' \geqslant \sum_{i=1}^{n} b_i$. If P is the d.f. of $X = \sum_{i=1}^{n} X_i$, then there exist constants $c_1 < \infty$ and $c_2 < \infty$ such that*

$$\rho(P, Q) \leqslant c_1 \mathrm{E}\left|\sigma_Y^2 w_Y(X) - \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} w^i(\mathrm{X})\right| + c_2|\mu_X - \mu_Y|, \tag{4.10}$$

where $\mu_X = \mathrm{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mu_i$, w_Y is the w-function corresponding to the Q, $\mu_Y = \mathrm{E}[Y]$, $\sigma_Y^2 = \mathrm{Var}[Y]$.

If $\mu_X = \mu_Y$ and Q is Poisson, then (4.10) *gives*

$$\rho(P, Q) \leqslant c_1 \mathrm{E}\left|\sigma_Y^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} w^i(\mathrm{X})\right|. \tag{4.11}$$

5. Applications. In this section we illustrate the previous results with some applications.

Omey and Rachev [7] and Omey [8] established uniform rates of convergences in extreme-value theory. Example 5.1 concerns the asymptotic behavior of the minimum of independent iniformly distributed r. v.'s.

E x a m p l e 5.1. Let $X_1, \ldots, X_n$ be i. i. d. r. v.'s uniformly distributed on the interval $(0,1)$ and $X = nX_{1:n}$, where $X_{1:n} = \min\{X_1, \ldots, X_n\}$. Then

$$\mu_X = \frac{n}{n+1}, \quad \sigma_X^2 = \frac{n^3}{(n+1)^2(n+2)},$$
$$w_X(x) = \frac{(n+1)(n+2)}{n^2}\, x\left(1 - \frac{x}{n}\right), \qquad 0 < x < n.$$

For $G(x) = 1 - e^{-x}$, $x > 0$, we have $a' = 0$, $b' = +\infty$ and $\lim_{x\to+\infty}(1-G(x))/(xg(x)) = 0$. Hence, applying Theorem 2.1 with $w_Y(x) = x$, $\mu_Y = 1$, $\sigma_Y^2 = 1$ we obtain

$$\rho(F, G) \leqslant \frac{1}{n+1}\left(\frac{2n+1}{n+1}\, c_1 + c_2\right). \tag{5.1}$$

Thus, $\mathbf{P}\{nX_{1:n} \leqslant x\}$ tends to $1 - e^{-x}$ in the sense of the total variation distance and the rate of convergence is at least $O(1/n)$.

E x a m p l e 5.2. If X has the t-distribution with $n > 3$ degrees of freedom, then $\mu_X = 0$, $\sigma_X^2 = n/(n-2)$ and $w_X(x) = (n-2)/(n-1)(1 + x^2/n)$. Applying Corollary 2.2, we have

$$\rho(F, \Phi) \leqslant \frac{4}{n-2} \longrightarrow 0 \tag{5.2}$$

and the rate of convergence is at least $O(1/n)$.

E x a m p l e 5.3. If $X = nZ$, where Z is $F_{n,m}$ (F-distribution with n and m degrees of freedom) and $G = \chi_n^2$ (chi-square with n degrees of freedom), then Theorem 2.1 gives

$$\rho(F, G) \leqslant \frac{1}{m-2}\left[2nc_2 + c_1\left(2 + \frac{mn}{m-2}\right)\right], \tag{5.3}$$

which guarantees that $nF_{n,m}$ tends to χ_n^2 as $m \to \infty$ and the rate of convergence is at least $O(1/m)$.

E x a m p l e 5.4. Suppose that X_n is beta $B(a_n, b_n)$ with parameters $a_n > 0$, $b_n > 0$. If $X = m_n X_n$ and the sequence $\{m_n > 0\}$ is properly chosen so that

$$m_n \to \infty, \quad \mathrm{E}(X) \to \mu > 0 \quad \text{and} \quad \mathrm{Var}(X) \to \sigma^2 > 0,$$

then X tends in the sense of the total variation distance to the gamma distribution $G(\mu^2/\sigma^2, \sigma^2/\mu)$. Indeed, Theorem 2.1 gives

$$\rho(F, G) \leqslant c_1 \left| 1 - \frac{\mu}{\sigma^2} \frac{m_n}{a_n + b_n} \right| + c_1 \frac{\mu}{\sigma^2} \frac{m_n a_n}{(a_n + b_n)^2} + c_2 \left| \frac{m_n a_n}{a_n + b_n} - \mu \right|. \tag{5.4}$$

Finally we give an application of Theorem 4.2. Let $\mathbf{X}$ be a random vector with probability function

$$p(\mathbf{x}) = \frac{\binom{\Lambda}{x}\binom{M}{n-x}}{\binom{\Lambda+M}{n}\binom{n}{x}},$$

where $x = \sum_{i=1}^n x_i$, $\mathbf{x} \in \{0,1\}^n$, $n \leqslant \min\{\Lambda, M\}$, and let Q be binomial with parameters n, p. After some calculations we have

$$w^i(\mathbf{x}) = \frac{(\Lambda + M)(\Lambda + M - 1)}{\Lambda M(\Lambda + M - n)} (\Lambda - x)(1 - x_i), \qquad i = 1, \ldots, n,$$

$$\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} w^i(\mathbf{x}) = \frac{(\Lambda - x)(n - x)}{\Lambda + M}, \qquad \mu_X = \frac{n\Lambda}{\Lambda + M}.$$

If P is the d.f. of $X = \sum_{i=1}^n X_i$, then Theorem 4.2 gives

$$\rho(P, Q) \leqslant c_1 \mathrm{E} \left| (n - X) \left(\frac{\Lambda - X}{\Lambda + M} \right) - p \right| + c_2 n \left| \frac{\Lambda}{\Lambda + M} - p \right|. \tag{5.5}$$

Hence,

$$\rho(P, G) \leqslant n\, c_1 (1 - p) \max \left\{ \left| \frac{\Lambda}{\Lambda + M} - p \right|, \left| \frac{\Lambda - n}{\Lambda + M} - p \right| \right\} + n\, c_2 \left| \frac{\Lambda}{\Lambda + M} - p \right|. \tag{5.6}$$

Letting $\Lambda \to \infty$, $M \to \infty$, $\Lambda/(\Lambda + M) \longrightarrow p$, we have

$$\rho(P, Q) \to 0.$$

We are led to the same result applying Theorem 3.1 directly.

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