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Orthogonal polynomials in the cumulative Ord family and its application to variance bounds

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ABSTRACT

This article presents and reviews several basic properties of the Cumulative Ord family of distributions; this family contains all the commonly used discrete distributions. A complete classification of the Ord family of probability mass functions is related to the orthogonality of the corresponding Rodrigues polynomials. Also, for any random variable X of this family and for any suitable function g in $L^2(\mathbb{R}, X)$, the article provides useful relationships between the Fourier coefficients of g (with respect to the orthonormal polynomial system associated to X) and the Fourier coefficients of the forward difference of g (with respect to another system of polynomials, orthonormal with respect to another distribution of the system). Finally, using these properties, a class of bounds for the variance of $g(X)$ is obtained, in terms of the forward differences of g. These bounds unify and improve several existing results.

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1. Indroduction

Ord [1] introduced the discrete analogue of Pearson's system. Ord's family contains all integer-valued random variables (rvs) whose probability mass function (pmf), p, satisfies

$$\frac{\Delta p(j-1)}{p(j)} = \frac{a-j}{(a+b_0)+(b_1-1)j+b_2 j(j-1)}. \qquad (1)$$

Here, Δ is the forward difference operator and $p(j)$ is the pmf of the discrete rv X and j takes values in an integer interval. In the sequel, the term 'discrete rv' is customized to mean 'integer-valued rv'. Equation (1) is the discrete analogue of Pearson's differential equation. Ord classified these distributions according to the values of the parameters a, b_0, b_1 and b_2; see [2, Table 2.1, p.87].

The present work is concerned with the *Cumulative Ord Family* of discrete distributions, defined as follows.

Definition 1.1 (Cumulative Ord Family): Let X be a discrete rv with finite mean μ and pmf $p(j) = \mathbb{P}(X = j)$, $j \in \mathbb{Z}$. We say that X belongs to the Cumulative Ord family (or that p belongs to

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the Cumulative Ord system) if there exists a quadratic $q(j) = \delta j^2 + \beta j + \gamma$ such that

$$\sum_{k \le j} (\mu - k)p(k) = q(j)p(j) \quad \text{for all } j \in \mathbb{Z}. \tag{2}$$

If Equation (2) is satisfied, we write $X \sim \mathrm{CO}(\mu; q)$ or $p \sim \mathrm{CO}(\mu; q)$, or more explicitly, X or $p \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$.

Let $X \sim \mathrm{CO}(\mu; q)$. Afendras et al. [3] studied the orthogonal polynomials generated by a Rodrigues-type formula (see Theorem 7.3 below) and based on these polynomials, they prove Stein-type covariance identities (see [3, Equation(2.7), p.512]). First-order covariance identities, $k=1$, of that kind have been studied by Sudheesh and Luisa [4] for the Ord, Katz as well as modified power series families of distributions. Afendras et al. [3], using Bessel's inequality (for $m \in \mathbb{N} = \{0, 1, \ldots\}$ such that $\mathbb{E}|X|^{2m} < \infty$ and $\mathbb{E}q^{[m]}(X)|\Delta^m g(X)| < \infty$), showed that

$$\mathsf{Var}\, g(X) \ge \sum_{k=1}^{m} \frac{\mathbb{E}^2 q^{[k]}(X)\Delta^k g(X)}{k!\Pi_\delta^{[k]}(k-1)\mathbb{E}q^{[k]}(X)}; \tag{3}$$

the equality holds iff g coincides with a polynomial of degree at most m in the support of X. For $q^{[k]}$, $\Pi_\delta^{[k]}$ and Δ^k, see Notations 2.1 below. Also, for $n \in \mathbb{N}$ such that $\mathbb{E}|X|^{2n} < \infty$ and $\mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2 < \infty$, Afendras et al. [5], by applying a discrete Mohr and Noll inequality, established some Poincaré-type bounds for the variance of $g(X)$, of the form

$$(-1)^n\big[\mathsf{Var}\, g(X) - S_n\big] \ge 0, \quad \text{where } S_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{\mathbb{E}q^{[k]}(X)\left[\Delta^k g(X)\right]^2}{k!\Pi_\delta^{[k]}(0)}; \tag{4}$$

the equality holds iff g is identified with a polynomial of degree at most n in the support of X.

We first present a simple example for illustrating the improvement achieved by the results of the present article. Let $X \sim \mathrm{Poisson}(\lambda)$. For $m = n = 1$, Equations (3) and (4) produce the double inequality

$$\lambda\mathbb{E}^2\Delta g(X) \le \mathsf{Var}\, g(X) \le \lambda\mathbb{E}[\Delta g(X)]^2, \tag{5}$$

where both equalities hold iff g is a linear polynomial. Applying the results of the present paper (see Theorem 9.1), we get the strengthened inequality

$$\mathsf{Var}\, g(X) \le \frac{\lambda}{2}\mathbb{E}^2\Delta g(X) + \frac{\lambda}{2}\mathbb{E}[\Delta g(X)]^2, \tag{6}$$

in which the equality holds iff g is a polynomial of degree at most two. It is clear that the upper bound in (6) improves the upper bound in Equation (5) and, in fact, it is strictly better, unless g is linear.

The rest of this paper is organized as follows. In Section 2, we present some elementary properties of the Cumulative Ord (CO) family of distributions. In Section 3, we give an algorithm (Algorithm 1) which checks if a pair $(\mu; q)$ is admissible, according to Definition 3.2. Also, we provide a detailed classification of the CO family. It turns out that, up to an (integer-valued) location transformation and/or multiplication by -1, there are six different types of pmfs, described in Table 1, while Section 4 offers a comparison between Ord's Discrete Student distributions and that ones presented in this article. Section 5 presents the symmetric distributions of the CO family of distributions; moreover, in this section, we define the noncentrality parameter as well as the degrees of freedom of a discrete Student distribution. In Section 6, we show that for any $p \sim \mathrm{CO}(\mu; q)$, the pmf $p_i \propto q^{[i]}p$ belongs to the CO system, under appropriate moment conditions. Also, using a known covariance identity, we obtain close-form expressions for $\mathsf{Var}(X_i)$ (where $X_i \sim p_i$) and for $\mathbb{E}q^{[i]}(X)$. Recurrence relations for the

Table 1. Probabilities of the cumulative Ord family.

Type	Notation	$p(j)$	Support S	$q(j)$	Parameters	Mean μ	Classification rule
1. Poisson-type	$X \sim P(\lambda)$	$\mathrm{e}^{-\lambda}\frac{\lambda^j}{j!}$	$\mathbb{N}$	λ	$\lambda > 0$	λ	$\delta = \beta = 0$
2. Binomial-type	$X \sim \mathrm{Bin}(N, p)$	$\binom{N}{j}p^j(1-p)^{N-j}$	$0, 1, \ldots, N$	$p(N-j)$	$N=1,2,\ldots$ $0<p<1$	Np	$\delta = 0, \beta = -p \in (-1, 0)$
3. Negative Binomial-type	$X \sim \mathrm{NB}(r, p)$	$\frac{[r]_j}{j!}p^r(1-p)^j$	$\mathbb{N}$	$\frac{1-p}{p}(r+j)$	$r>0$ $0<p<1$	$\frac{r(1-p)}{p}$	$\delta = 0, \beta = \frac{1-p}{p} > 0$
4a. Negative Hypergeometric-type	$X \sim \mathrm{NHgeo}(N; r, s)$	$\binom{N}{j}\frac{(-r)_j(-s)_{N-j}}{(-r-s)_N}$	$0, 1, \ldots, N$	$\frac{(r+j)(N-j)}{r+s}$	$N=1,2,\ldots$ $r,s>0$	$\frac{Nr}{r+s}$	$\delta = \frac{-1}{r+s} < 0$
4b. Hypergeometric-type	$X \sim \mathrm{Hgeo}(N; r, s)$	$\binom{N}{j}\frac{(r)_j(s)_{N-j}}{(-r-s)_N}$	$0, 1, \ldots, N$	$\frac{(r-j)(N-j)}{r+s}$	$N=1,2,\ldots$ $r,s>N-1$	$\frac{Nr}{r+s}$	$\delta = \frac{1}{r+s} > 0$, finite S
5. Discrete F-type[a]	$X \sim \mathrm{d}{-}F(\rho; r, s)$	$\frac{\Gamma(\rho-r)\Gamma(\rho-s)}{\Gamma(\rho-r-s)\Gamma(\rho)}\frac{[r]_j[s]_j}{j![\rho]_j}$	$\mathbb{N}$	$\frac{(r+j)(s+j)}{\rho-r-s-1}$	$(r,s)\in\mathcal{C}_2$ $\rho>\max\{0,r+s+1\}$	$\frac{rs}{\rho-r-s-1}$	$\delta = \frac{1}{\rho-r-s-1} > 0$, one-side infinite S
6. Discrete Student-type[b,c]	$X \sim \mathrm{d}{-}t(\boldsymbol{z}, \boldsymbol{w})$	$C\frac{[z_1]_j[z_2]_j}{[w_1+1]_j[w_2+1]_j}$	$\mathbb{Z}$	$\frac{(z_1+j)(z_2+j)}{w_1+w_2-z_1-z_2}$	$\boldsymbol{z},\boldsymbol{w}\in\tilde{\mathcal{C}}_2$: $w_1+w_2>z_1+z_2$	$\frac{z_1z_2-w_1w_2}{w_1+w_2-z_1-z_2}$	$\delta = \frac{1}{w_1+w_2-z_1-z_2} > 0$, two-side infinite S

[a] $\mathcal{C}_2 \doteq \{(z, \bar{z}) : z \in \mathbb{C} \smallsetminus \mathbb{R}\} \cup (0, \infty)^2 \cup \{\bigcup_{n=0}^{\infty}(-n-1, -n)^2\}$.

[b] We were not able to find a closed formula for the normalizing constant C of this pmf.

[c] $\tilde{\mathcal{C}}_2 \doteq \{(z, \bar{z}) : z \in \mathbb{C} \smallsetminus \mathbb{R}\} \cup \{\bigcup_{n\in\mathbb{Z}}(n, n+1)^2\}$.

factorial moments are also given. In Section 7, we study the Rodrigues-type orthogonal polynomials of a rv that belongs to the CO Family. The main result of this section is that the forward differences of orthogonal polynomials of a pmf within the CO system are also orthogonal polynomials corresponding to another pmf of the system; see Lemma 7.5 and Theorem 7.6. In Section 8, we present expressions for the Fourier coefficients of a function g, with respect to the corresponding orthonormal polynomials. One of the important facts is that when the polynomials are dense in $L^2(\mathbb{R}, X)$, the expectation $\mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2$ can be expressed as a series (finite or infinite) in terms of the Fourier coefficients of g. This is utilized in Section 9 where, upon applying the series expansion for $\mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2$, we present a wide class of (upper/lower) bounds for $\mathsf{Var}\, g(X)$.

2. Preliminaries

In the present section, we investigate the basic properties of the CO family. In the sequel, we shall make use of the following notation.

Notation 2.1: (a) $S \equiv S(X) = \{j \in \mathbb{Z} : \mathbb{P}(X = j) > 0\}$ will denote the support of a discrete rv X. Also, we define $S_\circ \doteq S \setminus \{\text{the lower endpoint of } S\}$ and $S^\circ \doteq S \setminus \{\text{the upper endpoint of } S\}$. Of course, if S does not have a finite lower (upper) endpoint, then $S_\circ = S$ ($S^\circ = S$);

(b) For each real function f and $k = 0, 1, \ldots$, we define $f^{[k]}(x) = f(x) \cdots f(x + k - 1)$ and $f^{[-k]}(x) = 1/[f(x-1) \cdots f(x-k)]$ (provided that $f(x-1) \cdots f(x-k) \neq 0$), with $f^{[0]}(x) = 1$;

(c) For $z \in \mathbb{C}$ and $k = 0, 1, \ldots$, we define $[z]_k = z(z+1) \cdots (z+k-1)$ and $[z]_{-k} = [(z-1) \cdots (z-k)]^{-1}$ (provided that $z \neq 1, \ldots, k$), with $[z]_0 = 1$;

(d) For $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, we define $(z)_k = (-1)^k[-z]_k$, provided that the quantity $[-z]_k$ is well-defined. That is, for $k = 0, 1, \ldots$, we have $(z)_k = z(z-1) \cdots (z-k+1)$ and $(z)_{-k} = [(z+1) \cdots (z+k)]^{-1}$ (provided that $z \neq -1, \ldots, -k$), with $(z)_0 = 1$;

(e) $\Delta^k = \Delta(\Delta^{k-1})$ with $\Delta^0 = I$, the kth order forward difference operator;

(f) For each $\delta \in \mathbb{R}$ and $k \in \mathbb{N}$, $\Pi_\delta^{[k]}(m) \doteq \prod_{j=m}^{k+m-1}(1 - j\delta)$, $m \in \mathbb{N}$, with $\Pi_\delta^{[0]}(m) = 1$;

(g) Let $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2) \in \mathbb{C}^2$. We denote $\boldsymbol{z} \mathrel{=\!\!=} \boldsymbol{w}$ if $(z_1, z_2) = (w_1, w_2)$ or $(z_1, z_2) = (w_2, w_1)$.

Remark 2.2: For every non-degenerate discrete rv X with finite mean μ and pmf p, the function $f(j) = \sum_{k \le j}(\mu - k)p(k), j \in \mathbb{Z}$, is non-negative, unimodal (increases for $j \le \lfloor \mu \rfloor$, the integer part of μ, and then decreases) and takes its maximum value at the point $j = \lfloor \mu \rfloor$ (if $\mu \in \mathbb{Z}$, the maximum value is attained at the points $\mu - 1$ and μ).

Definition 2.3: A subset S of $\mathbb{Z}$ is called an *integer chain* if for every $j_1, j_2 \in S$ with $j_1 \le j_2$, we have $[j_1, j_2] \cap \mathbb{Z} \subseteq S$.

Remark 2.4: An integer chain S of $\mathbb{Z}$ can be written as $\{\alpha, \ldots, \omega\}$, where $\alpha \in \mathbb{Z} \cup \{-\infty\}$ and $\omega \in \mathbb{Z} \cup \{\infty\}$, in the sense that $\{\alpha, \ldots, \infty\} = \{\alpha, \alpha + 1, \ldots\}$ when $\alpha > -\infty$, $\{-\infty, \ldots, \omega\} = \{\ldots, \omega - 1, \omega\}$ when $\omega < \infty$ and $\{-\infty, \ldots, \infty\} = \mathbb{Z}$.

From Definition 1.1, we can prove the following lemma.

Lemma 2.5: *Let* $X \sim \mathrm{CO}(\mu; q)$. *Then:*

(a) *The rv* X *is supported on an integer chain, denoted by* $S = \{\alpha, \ldots, \omega\}$, *where* $\alpha \in \mathbb{Z} \cup \{-\infty\}$ *and* $\omega \in \mathbb{Z} \cup \{\infty\}$ *with* $\alpha \le \omega$. *Thus,* $S_\circ = \{\alpha + 1, \ldots, \omega\}$ *and* $S^\circ = \{\alpha, \ldots, \omega - 1\}$. *Note that if* $\alpha = \omega$ *then* $S_\circ = S^\circ = \varnothing$;

(b) $q(j) > 0$ *for all* $j \in S^\circ$;

(c) *If* $\omega < \infty$, *then* $q(\omega) = 0$. *If* $\alpha = 0$, *then* $\mu = q(0) = \gamma$ *and, in general, if* $\alpha > -\infty$, *then* $q(\alpha) = \mu - \alpha$;
(d) *For every* $r \in \mathbb{Z}$, *the rv* $Y = X + r$ *follows* $\mathrm{CO}(\mu_Y = \mu + r; q_Y(j) = q(j - r))$;
(e) *The rv* $W = -X$ *follows* $\mathrm{CO}(\mu_W = -\mu; q_W(j) = q(-j) - j - \mu)$;
(f) $\underline{q}(j) > 0$ *for all* $j \in S_\circ$, *where* $\underline{q}(j) \doteq q(j) + j - \mu$;
(g) $p(j) = r(j-1)p(j-1)$ *for all* $j \in S_\circ$, *where* $r(j) \doteq q(j)/\underline{q}(j+1), j \in S^\circ$.

Proof: (a), (b) and (c) are obvious by Equation (2) and Remark 2.2.

(d) The rv Y has mean $\mu_Y = \mu + r$ and support $S(Y) = r + S = \{r + j : j \in S\}$ (integer chain). Observe that $\sum_{k \le j}(\mu_Y - k)\mathbb{P}(Y = k) = \sum_{k \le j}[\mu - (k - r)]\mathbb{P}(X = k - r) = \sum_{s \le j - r}(\mu - s)\mathbb{P}(X = s) = q(j - r)\mathbb{P}(X = j - r) = q(j - r)\mathbb{P}(Y = j)$.

(e) The rv W has mean $\mu_W = -\mu$ and support $S(W) = -S = \{-j : j \in S\}$. Now, write $\sum_{k \le j}(\mu_W - k)\mathbb{P}(W = k) = \sum_{k \le j}(-\mu - k)\mathbb{P}(X = -k) = -\sum_{k \le j}[\mu - (-k)]\mathbb{P}(X = -k) = -\sum_{s \ge -j}(\mu - s)\mathbb{P}(X = s) = (-\mu - j)\mathbb{P}(X = -j) - \sum_{s > -j}(\mu - s)\mathbb{P}(X = s)$ and observe that $\sum_{s \le -j}(\mu - s)\mathbb{P}(X = s) + \sum_{s > -j}(\mu - s)\mathbb{P}(X = s) = 0$. Thus, $\sum_{k \le j}(\mu_W - k)\mathbb{P}(W = k) = (-\mu - j)\mathbb{P}(X = -j) + \sum_{s \le -j}(\mu - s)\mathbb{P}(X = s) = (-\mu - j)\mathbb{P}(X = -j) + q(-j)\mathbb{P}(X = -j) = [q(-j) - j - \mu]\mathbb{P}(W = j)$.

(f) It is obvious that $j \in S_\circ(X) \Leftrightarrow -j \in S^\circ(-X)$. Thus, from (b) and (e), we see that for every $j \in S_\circ$, $q_{-X}(-j) > 0$, i.e. $q(j) + j - \mu > 0$.

(g) From (f), it follows that $r(j)$ is well-defined. For each $j \in S_\circ$, (2) gives $q(j)p(j) = \sum_{k \le j}(\mu - k)p(k) = q(j-1)p(j-1) + (\mu - j)p(j)$, that is, $\underline{q}(j)p(j) = q(j-1)p(j-1)$. ■

Now, we present a useful lemma concerning the existence of moments (see [5, p. 176]).

Lemma 2.6: *Let* $X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$. *If* $S(X)$ *is finite or* $\delta \le 0$, *then* X *has finite moments of any order. Furthermore, if* $S(X)$ *is infinite and* $\delta > 0$, *then* X *has finite moments of any order* $\theta \in [0, 1 + 1/\delta)$, *while* $\mathbb{E}|X|^{1+1/\delta} = \infty$.

Remark 2.7: We can find a rv $X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$ with $\delta > 0$ and finite support S (with cardinality $|S| \ge 3$). However, the inequality $\delta < [2(|S| - 2)]^{-1}$ should be necessarily satisfied in this case; see Section 3.3.1.

Lemma 2.8: *Suppose a discrete rv* X *is supported on an infinite integer chain* $S(X)$, *has finite mean* μ, *and satisfies the relation* $\sum_{k \le j}(c - k)p(k) = q(j)p(j)$, $j \in S(X)$, *where* c *is a constant and* q *is a polynomial of degree at most two. If* $S(X)$ *is upper (resp., lower) unbounded, then* $q(j)p(j) \to 0$ *as* $j \to \infty$ *(resp.,* $j \to -\infty$*).*

Proof: If q is a linear polynomial, then the result is obvious because $\mathbb{E}|X| < \infty$. We shall examine only the case where $q(j) = \delta j^2 + \beta j + \gamma$ with $\delta \ne 0$. If $S(X)$ is upper unbounded, the quantity $\sum_{k \in \mathbb{Z}}(c - k)p(k) = \lim_{j \to \infty}\sum_{k \le j}(c - k)p(k)$ is well-defined, since $\mathbb{E}|X| < \infty$. Thus, $\lim_{j \to \infty} p(j)q(j) = C \in \mathbb{R}$. Observe that $\lim_{j \to \infty} j^2 p(j) = \lim_{j \to \infty}[j^2/q(j)]q(j)p(j) = C/\delta \ge 0$. Assuming $C > 0$, we can find an integer $j_0 > 0$ such that $jp(j) > C/(2\delta j)$ for all $j \ge j_0$. Thus, $\mathbb{E}|X| \ge \sum_{j \ge j_0} jp(j) \ge \sum_{j \ge j_0} C/(2\delta j) = \infty$, a contradiction. For the lower unbounded case, we can use analogous arguments. ■

The result of Lemma 2.8 applies to all rvs of the CO family whose support is infinite. For this family, the results of the above lemma can be generalized; see Proposition 6.5.

Remark 2.9: In Lemma 2.8, if the support does not have a finite upper bound, then the constant c is necessarily the mean μ of X, i.e. the rv X belongs to the CO family. However, if the support has a finite upper bound, it may happen that $c \neq \mu$. For example, let $X \sim \text{Poisson}(\lambda) = \text{CO}(\mu = \lambda;\ q(j) = \lambda)$. Then, from Lemma 2.5(e), the rv $W = -X \sim \text{CO}(\mu_W = -\lambda; q_W(j) = -j)$, i.e. $\sum_{k=-\infty}^{j}(\mu_W - k)p_W(k) = q_W(j)p_W(j)$ for all $j \in \mathbb{Z}$. Now, consider the truncated $\tilde{W}$ at the point zero $[p_{\tilde{W}}(j) = \vartheta p_W(j),\ j = -1, -2, \ldots]$. Then, for each $j \in S(\tilde{W}) = \{-1, -2, \ldots\}$, we have $\sum_{k=-\infty}^{j}(\mu_W - k)p_{\tilde{W}}(k) = \vartheta \sum_{k=-\infty}^{j}(\mu_W - k)p_W(k) = \vartheta q_W(j)p_W(j) = (-j)p_{\tilde{W}}(j)$ and $\mathbb{E}\tilde{W} \neq \mu_W$ (note that this $\tilde{W}$ does not belong to the CO family).

Now we compare the CO system, i.e. the pmfs satisfying Equation (2), with the ordinary Ord system, i.e. the pmfs satisfying the Ord's difference Equation (1).

Proposition 2.10: *Assume that a discrete rv X has pmf p and finite mean. Then,* (a) *and* (b) *are equivalent, where*

(a) $X \sim \text{CO}(\mu; q)$;
(b) (i) *The support S of X is an integer chain, $S = \{\alpha, \ldots, \omega\}$ with $\alpha \leq \omega, \alpha \in \mathbb{Z} \cup \{-\infty\}, \omega \in \mathbb{Z} \cup \{\infty\}$;*
 (ii) *There exist polynomials p_1 (of degree at most one) and q (of degree at most two) such that $[\Delta p(j-1)]q(j-1) = p_1(j)p(j)$, for all $j \in S$;*
 (iii) *For the above polynomials, there exists a constant μ such that $p_1(j) + \Delta q(j-1) = \mu - j$, for all $j \in S$. If $\omega < \infty$, then it is further required that $\mu = \mathbb{E}X$.*

Proof: It is obvious that (a) implies (b). We now prove that (b) implies (a). Note that we do not assume that μ is the mean of X. Combining (i) and (iii), we have, after some algebra, that $p(j)(\mu - j) = \Delta[p(j-1)q(j-1)]$. Fix an integer i with $i \leq j$. Then, $\sum_{k=i}^{j} p(k)(\mu - k) = p(j)q(j) - p(i-1)q(i-1)$. If $\alpha > -\infty$, we choose $i = \alpha$ and since $p(\alpha - 1) = 0$, we have $\sum_{k \leq j} p(k)(\mu - k) = \sum_{k=\alpha}^{j} p(k)(\mu - k) = p(j)q(j)$, for all $j \in S$. For $\alpha = -\infty$, since $\mathbb{E}|X| < \infty$, the quantity $\sum_{k \leq j}(\mu - k)p(k) = \lim_{i \to -\infty} \sum_{k=i}^{j}(\mu - k)p(k)$ is well-defined. Thus, $\lim_{i \to -\infty} p(i-1)q(i-1) = C \in \mathbb{R}$. If q is of degree at most one, then it is obvious that $C = 0$. If $q(j) = \delta j^2 + \beta j + \gamma$, $\delta \neq 0$, we use the same arguments as in the proof of Lemma 2.8 and we conclude that $C = 0$. Thus, in any case, $\sum_{k \leq j}(\mu - k)p(k) = q(j)p(j)$, $j \in S$. Lemma 2.8 and Remark 2.9 then complete the proof. ■

Remark 2.11: All assumptions of Proposition 2.10(b) are necessary for a rv to lie in the CO family: It is obvious that (ii) is necessary for all $j \in S$. Regarding the assumption (i), consider the rv X with pmf $p(0) = p(2) = 0.5$ and observe that (ii) and (iii) are fulfilled for $p_1(j) = 2 - j$, $q(j) = 1 - j$ and $\mu = 1 = \mathbb{E}X$. But, this rv does not belong to the CO family. Regarding the assumption (iii), consider the truncated Poisson $X \sim p(j) \propto \lambda^j/j!, j = 0, 1, \ldots, N$. Observe that (i)–(iii) are satisfied for $p_1(j) = \lambda - j$, $q(j) = \lambda$ and $\mu = \lambda$. However, since $\lambda \neq \mathbb{E}X$, this rv does not belong to the CO family; this is so because S has a finite upper endpoint.

3. A complete classification of the cumulative Ord family

In this section, we classify the distributions of the CO family. The classification is based on the mean μ and the parameters of the quadratic q. The most important role is played by the parameter δ, the coefficient of the square power of q.

The natural question is to ask whether the mean μ and the quadratic q, together, characterize the distribution. The answer is given in the following proposition.

Proposition 3.1: *Suppose the rv X follows the* $\mathrm{CO}(\mu; q)$ *distribution. Then:*

(a) *The support $S(X)$ is uniquely determined by μ and q;*
(b) *The distribution is characterized by the pair $(\mu; q)$.*

Proof: (a) First, we consider the special case when $\mu \in \mathbb{Z}$ and $q(\mu) = 0$. From Remark 2.2, it follows easily that the rv takes the value μ with probability 1. Otherwise, we define $N \equiv N(\mu; q) = \{\text{the first } j \in \mathbb{Z} \cap [\mu, \infty) \text{ such that } q(N) \le 0\}$. If $N = \infty$, then the support $S(X)$ does not have a finite upper endpoint, otherwise the value N is the upper endpoint of $S(X)$ and, then, $q(N) = 0$ (otherwise the pair $(\mu; q)$ could not satisfy the relation (2)). Regarding the lower endpoint: The rv $-X$ follows $\mathrm{CO}(\mu_{-X}; q_{-X})$ with $S(-X) = -S(X)$, $\mu_{-X} = -\mu_X$ and $q_{-X}(j) = q(-j) - j - \mu$. As before, we can determine $N' \equiv N(\mu_{-X}; q_{-X})$. If $N' = \infty$, then the support $S(-X)$ does not have a finite upper endpoint, i.e. the support $S(X)$ does not have a finite lower endpoint. Otherwise, the value N' is the upper endpoint of $S(-X)$, i.e. $-N'$ is the lower endpoint of $S(X)$.

(b) Consider the pmf p of X and its support, $S(X)$, which is determined by the pair $(\mu; q)$. Now, let $\tilde{p} \sim \mathrm{CO}(\mu; q)$. Consider the function r of Lemma 2.5(g) and fix $j_0 \in S(X)$. For every $k \in \mathbb{Z}$ with $j_0 + k \in S(X)$, it follows that $p(j_0 + k) = r^{[k]}(j_0)p(j_0)$ and $\tilde{p}(j_0 + k) = r^{[k]}(j_0)\tilde{p}(j_0)$. Consequently, $\tilde{p} \propto p$, and so $\tilde{p} = p$ because p and $\tilde{p}$ are pmfs. ■

Definition 3.2: Let $\mu \in \mathbb{R}$ and $q(j) = \delta j^2 + \beta j + \gamma$. We say that the pair $(\mu; q)$ is *admissible* if there exists a pmf p in the CO system such that $p \sim \mathrm{CO}(\mu; q)$.

Now, the natural question is 'How one can check the admissibility of a given pair $(\mu; q)$?' Also, if a pair is admissible, how can the corresponding pmf be obtained by this pair? The answer is given in Algorithm 1.

Algorithm 1 Admissibility of the pair $(\mu; q)$.

1: Consider the polynomial $\underline{q}(j)$ of Lemma 2.5(f) and define

$$\alpha \doteq \sup\{j \in (-\infty, \mu] \cap \mathbb{Z} : \underline{q}(j) = 0\}, \quad \omega \doteq \inf\{j \in [\mu, \infty) \cap \mathbb{Z} : q(j) = 0\},$$

noting that $\sup\{\varnothing\} = -\infty$ and $\inf\{\varnothing\} = \infty$. Next, define $S = [\alpha, \omega] \cap \mathbb{Z}$. The pair $(\mu; q)$ is admissible if and only if $q(j) > 0$ for all $j \in S^\circ$ and $\underline{q}(j) > 0$ for all $j \in S_\circ$. If $(\mu; q)$ is admissible, go to Step 2, end otherwise.

2: Let p be the corresponding pmf to the pair $(\mu; q)$ in the CO system. By application of Lemma 2.5(g), we get[1]:

$$\text{if } \alpha > -\infty, \qquad p(\alpha + i) \propto r^{[i]}(\alpha), \qquad i = 0, 1, \ldots, \omega - \alpha; \tag{7a}$$

$$\text{if } \alpha = -\infty \text{ and } \omega < \infty, \qquad p(\omega - i) \propto r^{[-i]}(\omega), \qquad i = 0, 1, \ldots; \tag{7b}$$

$$\text{if } \alpha = -\infty \text{ and } \omega = \infty, \qquad p(j) \propto r^{[j]}(0), \qquad j \in \mathbb{Z}. \tag{7c}$$

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[1] For the pmfs in (7), we observe the following. Since Lemma 2.5(g) holds, as in the analysis presented in [5, Lemma 4.1, pp. 176–178], one can see that $\sum_{j\in S} |j|p(j) < \infty$ (so the mean of X is finite), and hence, $\sum_{j\in S} p(j) < \infty$. By construction, these pmfs satisfy the relation $[\Delta p(j-1)]q(j-1) = [-\Delta q(j-1) - j + \mu]p(j), j \in S$. As in the proof of Proposition 2.10, we get $\sum_{k\le j}(\mu - k)p(k) = q(j)p(j)$ for all $j \in S$. If $\omega = \infty$, then $q(j)p(j) \to 0$ as $j \to \infty$; see Lemma 2.8; thus, $\sum_{k\in S}(\mu - k)p(k) = 0$. If $\omega < \infty$, then $q(\omega) = 0$, see Step 1 of Algorithm 1, and so $\sum_{k\in S}(\mu - k)p(k) = \sum_{k\le\omega}(\mu - k)p(k) = q(\omega)p(\omega) = 0$. In both cases, $\omega < \infty$ and $\omega = \infty$, μ is the mean value.

Next, we present a detailed classification of the CO system.

3.1. The case $\delta = 0$

We have to further distinguish between the cases $\beta = 0$ and $\beta \neq 0$.

3.1.1. The subcase $\beta = 0$ (Poisson-type distributions)

Then, $q(j) = \gamma > 0$, $j \in S$. The support S does not have a finite upper endpoint, but it must have a finite lower endpoint (because the quadratic $q_{-X}(j) = \gamma - j - \mu$ of $-X$, see Lemma 2.5(e), takes negative values for large values of j). Without lost of generality, we assume $S = \mathbb{N}$. Since $\alpha = 0$, $\mu = \gamma$. Observe that the Poisson distribution with parameter $\lambda = \gamma$ follows $\mathrm{CO}(\gamma; 0, 0, \gamma)$. Using Proposition 3.1, we have $X \sim \mathrm{Poisson}(\gamma)$.

3.1.2. The subcase $\beta \neq 0$

We have the following sub-subcases.

3.1.2.1. The sub-subcase $\beta > 0$ (Negative Binomial-type distributions). The support S does not have a finite upper endpoint, but it has a lower one. Again, we may assume that $S = \mathbb{N}$ (of course $\mu > 0$). Since $\alpha = 0$, $q(j) = \beta j + \mu$. Consider the Negative Binomial distribution with parameters $r = \beta/\mu > 0$ and $p = 1/(1 + \beta) \in (0, 1)$, i.e.

$$p(j) = \frac{[r]_j}{j!} p^r (1 - p)^j, \quad j = 0, 1, \ldots,$$

which follows $\mathrm{CO}(\mu; 0, \beta, \mu)$. From Proposition 3.1, $X \sim \mathrm{NB}(r = \beta/\mu, p = 1/(1 + \beta))$.

3.1.2.2. The sub-subcase $-1 < \beta < 0$ (Binomial-type distributions). The support S has a finite upper endpoint. Also, $q_{-X}(j) = -(1 + \beta)j + \gamma - \mu$ where, $-(1 + \beta) < 0$. Thus, S has a finite lower endpoint. Assume that $S = \{0, 1, \ldots, N\}$ and $0 < \mu < N$. Since $q(0) = \mu$, $q(N) = 0$ and q is a linear polynomial, we get $q(j) = \mu(N - j)/N$. Consider the Binomial distribution with parameters N and $p = \mu/N$, i.e.

$$p(j) = \binom{N}{j} p^j (1 - p)^{N-j}, \quad j = 0, 1, \ldots, N,$$

which follows $\mathrm{CO}(p; 0, -p, Np)$. From Proposition 3.1, we see that $X \sim \mathrm{Bin}(N, p = \mu/N)$.

3.1.2.3. The sub-subcase $\beta = -1$ (Poisson-type distributions). Here, $q_{-X}(j) = \gamma - \mu$ (constant). Thus, this sub-subcase is the negative of the case 3.1.1.

3.1.2.4. The sub-subcase $\beta < -1$ (Negative Binomial-type distributions). In this case, $q_{-X}(j) = -(1 + \beta)j + \gamma - \mu$, $-(1 + \beta) > 0$. This case is the negative of the case 3.1.2.1.

3.2. The case $\delta < 0$ (Negative Hypergeometric-type distributions)

It is obvious that S is finite. Without loss of generality, assume that $S = \{0, 1, \ldots, N\}$ with $0 < \mu < N$. Since $q(N) = 0$ and $q(0) = \mu$, it follows that $q(j) = \delta[\mu/(N\delta) - j](N - j)$. Consider now the Negative Hypergeometric distribution with parameters $N \in \{1, 2, 3, \ldots\}$, $r = -\mu/(N\delta) > 0$ and $s = (\mu - N)/(N\delta) > 0$, i.e. with pmf

$$p(j) = \binom{N}{j} \frac{(-r)_j (-s)_{N-j}}{(-r - s)_N}, \quad j = 0, 1, \ldots, N.$$

This follows $\mathrm{CO}(Nr/(r + s); -1/(r + s), (N - r)/(r + s), Nr/(r + s))$. From Proposition 3.1, $X \sim \mathrm{NHgeo}(N; r, s)$.

3.3. The case $\delta > 0$

We study the following subcases, relating to the support.

3.3.1. Finite S (Hypergeometric-type distributions)

Set $S=\{0,1,\ldots,N\}$ and $0<\mu<N$. As in Section 3.2 ($\delta<0$), it follows that $q(j)=\delta[\mu/(N\delta)-j](N-j)$. From Lemma 2.5(b), we get $\mu/(N\delta)-(N-1)>0$, or equivalently $\delta<\mu/[N(N-1)]$. Now, since $\underline{q}(j)=j[\delta j+(1-\mu/N-N\delta)]$, from Lemma 2.5(f), we have that $\delta+(1-\mu/N-N\delta)>0$, or equivalently $\delta<(N-\mu)/[N(N-1)]$. So,

$$0<\delta<\min\{\mu,N-\mu\}/[N(N-1)].$$

Note that $\min\{\mu,N-\mu\}/[N(N-1)]\le 1/[2(N-1)]$. Consider the Hypergeometric distribution with parameters $N\in\{2,3,\ldots\}$, $r=\mu/(N\delta)>N-1$ and $s=(N-\mu)/(N\delta)>N-1$, with pmf

$$p(j)=\binom{N}{j}\frac{(r)_j(s)_{N-j}}{(r+s)_N},\quad j=0,1,\ldots,N.$$

Thus, $p\sim \mathrm{CO}(Nr/(r+s);1/(r+s),-(r+N)/(r+s),Nr/(r+s))$. From Proposition 3.1, $X\sim \mathrm{Hgeo}(N;r,s)$.

3.3.2. One-side infinite S (Generalized Inverse Polya or Discrete F-type distributions)

First, we give an example. Let $q(j)=j^2+1$ and $\mu=1$. It follows that $\underline{q}(j)=j(j+1)$. Step 1 of Algorithm 1 gives $\alpha=0$ and $\omega=\infty$, namely, $S=\mathbb{N}$, and the pair $(\mu;q)=(1;j^2+1)$ is admissible. Using (7a), we find the pmf $p\sim\mathrm{CO}(1;1,0,1)$ which is $p(j)=[\pi/\sinh(\pi)][\prod_{k=0}^{j-1}(k^2+1)]/[j!(j+1)!]$, $j\in\mathbb{N}$; this pmf can be written as

$$p(j)=\frac{\pi}{\sinh(\pi)}\frac{[\imath]_j[-\imath]_j}{j!(j+1)!},\quad j=0,1,\ldots,\tag{8}$$

where $\imath$ is the complex unity.

Let us consider the general case when $S=\mathbb{N}$ and $\mu>0$. Since $\alpha=0$, the quadratic q is of the form $q(j)=\delta j^2+\beta j+\mu$. Write $q(j)=\delta(j+z_1)(j+z_2)$, where $-z_1,-z_2$ are the complex roots of q. Since $q(j)>0$ for every $j\in S$, we get $(z_1,z_2)\in\mathcal{C}_2\subset\mathbb{C}^2$, where

$$\mathcal{C}_2\doteq\{(z,\bar{z}):z\in\mathbb{C}\setminus\mathbb{R}\}\cup(0,\infty)^2\cup\left\{\bigcup_{n=0}^{\infty}(-n-1,-n)^2\right\}.$$

Observe that $\underline{q}(j)=\delta j(j+\rho)$, where $\rho=(\delta+\beta+1)/\delta$. It is also required that $\underline{q}(j)>0$ for all $j\in\mathbb{N}^*=\{1,2,\ldots\}$, that is, $\rho>-1$, or equivalently, $\beta>-\delta-1$. Under the above restrictions, the pair $(\mu;q)$ is admissible. Step 2 of the algorithm then yields

$$p(j)=\frac{\Gamma(\rho-z_1)\Gamma(\rho-z_2)}{\Gamma(\rho)\Gamma(\rho')}\frac{[z_1]_j[z_2]_j}{j![\rho]_j},\quad j=0,1,\ldots,\tag{9}$$

where $\rho'=1+1/\delta$. The substitution $z_{1,2}\mapsto\pm\imath$ and $\rho\mapsto 2$ in Equation (9) yields Equation (8).

3.3.3. Two-side infinite S (Discrete Student-type distributions)

First, we give an example. Let $q(j)=j^2+1$ and $\mu=0$. It follows that $\underline{q}(j)=j^2+j+1$. Applying Algorithm 1, Step 1 gives $\alpha=-\infty$ and $\omega=\infty$, namely, $S=\mathbb{Z}$, and the pair $(\mu;q)=(0;j^2+1)$ is admissible. Applying Equation (7c), the pmf of $\mathrm{CO}(0;1,0,1)$ distribution is

$$p(j)\propto\frac{[\imath]_j[-\imath]_j}{\left[3/2+\imath\sqrt{3}/2\right]_j\left[3/2-\imath\sqrt{3}/2\right]_j},\quad j\in\mathbb{Z}.\tag{10}$$

Note that the above choice of $(\mu;q)$ forces S to be the entire $\mathbb{Z}$.

For the general case, let $\mu \in \mathbb{R}$, $q(j) = \delta(z_1 + j)(z_2 + j)$ and $\underline{q}(j) = \delta(w_1 + j)(w_2 + j)$, where $-z_1$, $-z_2$ and $-w_1$, $-w_2$ are the complex roots of q and $\underline{q}$, respectively. Writing $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$, since $q(j) > 0$ and $\underline{q}(j) > 0$ for all $j \in \mathbb{Z}$, it follows that $\boldsymbol{z}, \boldsymbol{w} \in \tilde{\mathcal{C}}_2$, where

$$\tilde{\mathcal{C}}_2 \doteq \{(z, \bar{z}) : z \in \mathbb{C} \smallsetminus \mathbb{R}\} \cup \left\{ \bigcup_{n \in \mathbb{Z}} (n, n+1)^2 \right\}.$$

Note that the pair (w_1, w_2) is a function of $(\mu; q)$, i.e. a function of the values μ, δ, z_1 and z_2. The pair $(\mu; q)$ is admissible; see Step 1 of Algorithm 1. From Step 2, we obtain a formula for the pmf as follows (cf. [6]):

$$p(j) \propto \frac{[z_1]_j [z_2]_j}{[w_1 + 1]_j [w_2 + 1]_j}, \quad j \in \mathbb{Z}. \tag{11}$$

Substituting $z_{1,2} \mapsto \pm\imath$ and $w_{1,2} \mapsto 1/2 \pm \imath\sqrt{3}/2$ in (11), we obtain (10).

All the above possibilities (Sections 3.1–3.3) are summarized in Table 1 .

Remark 3.3: It is easy to check that if the cardinality $|S|$ of the support equals 2, then different types lead to identical distributions (since every such rv is Bernoulli).

4. A comparison with Ord's discrete student distributions

Here, we offer a comparison between Ord's Discrete Student distributions and the Discrete Student-type distributions that are presented in this article.

Ord [6] defined the discrete student distribution as one with pmf

$$p(j) \propto \prod_{r=0}^{k} \frac{1}{(j + r + a)^2 + b^2}, \quad j \in \mathbb{Z}, \tag{12}$$

where $k \in \mathbb{N}$, $a \in [0, 1]$ and $0 < b^2 < \infty$ are the parameters of the distribution.

We are interested in answering the following questions: (a) Does p in Equation (12) belong to the CO system? (b) If yes, what is the corresponding pmf in the Table 1? (c) Does Equation (12) describe the set of Discrete Student-type distributions?

Before our analysis, we state the following relations that arise by the definition of $[z]_j$; see Notations 2.1(c). Let $z \in \mathbb{C}$, $r \in \mathbb{N}^*$ and $j \in \mathbb{Z}$. Then, one can easily check that the following identities hold:

$$[z]_{-j} = (-1)^j / [-z + 1]_j \quad \text{and} \quad [z]_{r+j} = [z]_j [z + j]_r = [z]_r [z + r]_j, \tag{13}$$

provided that the quantities that appear are well-defined.

Now, set the complex numbers $z_{1,2} = a \pm \imath b$, $w_{1,2} = a + k \pm \imath b$ and then $\boldsymbol{z}_{k,a,b} = (z_1, z_2)$ and $\boldsymbol{w}_{k,a,b} = (w_1, w_2)$. Obviously, $\boldsymbol{z}_{k,a,b}, \boldsymbol{w}_{k,a,b} \in \tilde{\mathcal{C}}_2$ and $w_1 + w_2 - z_1 - z_2 = 2k \in 2\mathbb{N}$. We observe that $\prod_{r=0}^{k} [(j + r + a)^2 + b^2] = [z_1 + j]_{k+1} [z_2 + j]_{k+1}$. An application of Equation (13) shows that $\prod_{r=1}^{k} [(j + r + a)^2 + b^2] = [z_1]_{k+1} [z_2]_{k+1} [w_1 + 1]_j [w_2 + 1]_j / ([z_1]_j [z_2]_j)$. Since the quantity $[z_1]_{k+1} [z_2]_{k+1}$ is positive and independent of j, the pmf in Equation (12) takes the form

$$p(j) \propto \frac{[z_1]_j [z_2]_j}{[w_1 + 1]_j [w_2 + 1]_j}, \quad j \in \mathbb{Z}.$$

If $k = 0$, the pmf p does not belong to the CO system; this is an expected result because p does not have expected value due to the divergence of the harmonic series. In this case, p is the pmf of a discrete Cauchy distribution. If $k \in \mathbb{N}^*$, the pmf p belongs to the CO system. In conclusion, let us denote the distribution of p in Equation (12) by $\mathrm{d}^{\mathrm{ORD}}{-}t(k,a,b)$; then, for each $k \in \mathbb{N}$ and a,b as above, the pmf $p \sim \mathrm{d}^{\mathrm{ORD}}{-}t(k,a,b)$ belongs to the CO system iff $k \in \mathbb{N}^*$; in particular, $\mathrm{d}^{\mathrm{ORD}}{-}t(k,a,b) \equiv \mathrm{d}{-}t(\boldsymbol{z}_{k,a,b}, \boldsymbol{w}_{k,a,b})$.

Let $k \in \mathbb{N}^*$, $0 \le a \le 1$ and $b^2 > 0$, and let us consider $p \sim \mathrm{d}^{\mathrm{ORD}}{-}t(k,a,b)$. Then, from the above analysis, it follows that $p \sim \mathrm{d}{-}t(\boldsymbol{z}_{k,a,b}, \boldsymbol{w}_{k,a,b})$. Observe that $\boldsymbol{z}_{k,a,b}, \boldsymbol{w}_{k,a,b} \in \{(z,\bar{z}) : z \in \mathbb{C} \setminus \mathbb{R}\} \subsetneq \tilde{\mathcal{C}}_2$ with $w_1 + w_2 - z_1 - z_2 \in 2\mathbb{N}^*$. In view of Table 1, it is obvious that the class of the discrete student-type distributions of the CO system is strictly bigger than Ord's class of the Discrete Student distributions which have finite mean value. Consequently, Equation (12) cannot describe the whole of Discrete Student-type distributions of the CO system. Of course, we must note that the Ord's class of Discrete Student distributions contains discrete Cauchy distributions (for $k = 0$); in contrast to Ord's system, any pmf of the CO system has finite mean value.

Finally, it is worth noting the relationship between the finite moment-order and the parameter k of $p \sim \mathrm{d}^{\mathrm{ORD}}{-}t(k,a,b)$. Consider the case $k = 0$. Then, $p(j) \propto [(j+a)^2 + b^2]^{-1}$ and it is obvious that p has finite moment of order θ iff $0 \le \theta < 1$. Suppose now that $k \in \mathbb{N}^*$. Based on the previous analysis and Table 1, $p \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$ with $\delta = 1/(2k) > 0$. Since $1 + 1/\delta = 2k + 1$, Lemma 2.6 shows that p has finite moment of order θ iff $0 \le \theta < 2k+1$. Observe that the rule 'p has finite moment of order θ iff $0 \le \theta < 2k+1$' holds for every $k \in \mathbb{N}$.

5. The symmetric pmfs of the CO system

In this section, we are interested in characterizing the symmetric pmfs of the CO system. In investigating this aspect, we state and prove the following theorem. First, observe that if X is a symmetric integer-valued rv with finite mean value, then the expected value of X is an integer or half-integer number (the set of the half-integer numbers is denoted by $\mathbb{Z} + 1/2$).

Theorem 5.1: *Let $p \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$. The pmf p is symmetric, around its mean value μ, iff $\mu \in \frac{1}{2}\mathbb{Z} \doteq \mathbb{Z} \cup \{\mathbb{Z} + 1/2\}$ and $4\delta\mu + 2\beta = -1$.*

Proof: Suppose $X \sim p$. We prove separately the cases $\mu \in \mathbb{Z}$ and $\mu \in \mathbb{Z} + 1/2$.

Let $\mu \in \mathbb{Z}$ and let us consider the rv $Y = X - \mu$. Then, the rv X, and so the pmf p, is symmetric iff $Y \overset{\mathrm{d}}{=} -Y$. Using Lemma 2.5(d),(e), it follows that $Y \sim \mathrm{CO}(\mu_Y; q_Y)$, where $\mu_Y = 0$ and $q_Y(j) = \delta j^2 + (2\delta\mu + \beta)j + (\delta\mu^2 + \beta\mu + \gamma)$, and $-Y \sim \mathrm{CO}(\mu_{-Y}; q_{-Y})$, where $\mu_{-Y} = 0$ and $q_{-Y}(j) = \delta j^2 - (2\delta\mu + \beta + 1)j + (\delta\mu^2 + \beta\mu + \gamma)$. Applying Proposition 3.1(b), $Y \overset{\mathrm{d}}{=} -Y$ iff $4\delta\mu + 2\beta = -1$.

Let $\mu \in \mathbb{Z} + 1/2$, say $\mu = \lfloor\mu\rfloor + 1/2$. Consider the rvs $Y_1 = X - \lfloor\mu\rfloor$ and $Y_2 = -Y_1 + 1$. Then, the rv X, and so the pmf p, is symmetric iff $Y_1 \overset{\mathrm{d}}{=} Y_2$. Again, from Lemma 2.5(d),(e), we get $Y_1 \sim \mathrm{CO}(\mu_1; q_1)$, where $\mu_1 = 1/2$ and $q_1(j) = \delta j^2 + (2\delta\lfloor\mu\rfloor + \beta)j + (\delta\lfloor\mu\rfloor^2 + \beta\lfloor\mu\rfloor + \gamma)$, and $Y_2 \sim \mathrm{CO}(\mu_2; q_2)$, where $\mu_2 = 1/2$ and $q_2(j) = \delta j^2 - [2\delta(\lfloor\mu\rfloor + 1) + \beta + 1]j + [\delta\lfloor\mu\rfloor^2 + \delta(2\lfloor\mu\rfloor + 1) + \beta(\lfloor\mu\rfloor + 1) + \gamma + 1/2]$. An application of Proposition 3.1(b) implies that $Y_1 \overset{\mathrm{d}}{=} Y_2$ iff $4\delta\mu + 2\beta = -1$. ∎

Now, we are interested in finding the types of the CO system that contain symmetric pmfs. If $X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$, there exist $s \in \{-1, 1\}$ and $r \in \mathbb{Z}$ such that the pmf of $Y = sX + r$ belongs to Table 1. It is obvious that X is a symmetric rv iff Y is symmetric. Under this observation and using Theorem 5.1 and Table 1, we have the following list:

- The *Poisson-type distributions* do not contain symmetric pmfs, due to non-symmetric support. Alternatively, since $\delta = \beta = 0$, we have that $4\delta\mu + 2\beta = 0 \ne -1$;
- The *Binomial-type distributions* contain symmetric pmfs. Since $\delta = 0$ and $\beta = -p$, $4\delta\mu + 2\beta = -1$ is equivalent with $p = 1/2$ which implies $\mu = N/2 \in \frac{1}{2}\mathbb{Z}$;

- The *Negative Binomial-type* of distributions does not contain symmetric pmfs, due to non-symmetric support. Alternatively, $\delta = 0$, $\beta = (1-p)/p$ and so $4\delta\mu + 2\beta = 2(1-p)/p > 0$;
- The *Negative Hypergeometric-type distributions* contain symmetric pmfs. If p is a symmetric pmf, its support $S = 0, 1, \ldots, N$ must be symmetric around μ; consequently, $\mu = rN/(r+s) = N/2$, equivalently, $r = s$. Conversely, for $r = s$, we have that $\mu = N/2$, $\delta = -1/(2r)$, $\beta = (N-r)/(2r)$ and so $4\delta\mu + 2\beta = -1$;
- The *Hypergeometric-type distributions* contain symmetric pmfs. Using the same arguments as in the Negative Hypergeometric-type distributions, a pmf in this subsystem is symmetric iff $r = s$;
- The *discrete F-type distributions* do not contain symmetric pmfs, due to non-symmetric support. Alternatively, setting $\theta = \rho - r - s - 1 > 0$, we have that $\mu = rs/\theta$, $\delta = 1/\theta$ and $\beta = (r+s)/\theta$. Observe that $rs > 0$ and $r + s \in \mathbb{R}$ because $(r, s) \in \mathcal{C}_2$. The relation $4\delta\mu + 2\beta = -1$ implies that $\theta^2 + 2(r+s)\theta + 4rs = 0$, which has discriminant $\Delta = 4(s-r)^2$. If r and s are complex conjugate numbers, $\theta \in \mathbb{C} \setminus \mathbb{R}$, a contradiction; therefore, $(r, s) \in (0, \infty)^2 \cup \{\bigcup_{n=0}^{\infty}(-n-1, -n)^2\}$. Solving the equation $\theta^2 + 2(r+s)\theta + 4rs = 0$, we get $\theta = -2r$ or $-2s$. Observe that $\mu = -s/2 > 0$ (or $\mu = -r/2 > 0$) belongs in $\frac{1}{2}\mathbb{Z}$. Hence, s (or r) is a negative integer, a contradiction;
- The *discrete student-type distributions* contain symmetric pmfs. Consider the vectors $\boldsymbol{z} = (-1/2, -1/2)$, $\boldsymbol{w} = (1/2, 1/2) \in \widetilde{\mathcal{C}}_2$. Then, $w_1 + w_2 - z_1 - z_2 = 2$, $\mu = 0$, $\beta = -1/2$ and so $4\delta\mu + 2\beta = -1$. Using (11), we find that the corresponding symmetric pmf is $p(0) = (2\pi^2 - 55/3)^{-1}$, $p(j) = (6\pi^2 - 165)^{-1}$ when $j = \pm 1$, and $p(j) = (2\pi^2 - 55/3)^{-1}(j^2 - 1/4)^{-2}$ for $j = \pm 2, \pm 3, \ldots$.

  Now, we determine the class of the symmetric discrete-t rvs. Suppose $X \sim \text{d}{-}t(\boldsymbol{z}, \boldsymbol{w})$, where $\boldsymbol{z}, \boldsymbol{w} \in \tilde{\mathcal{C}}_2$ with $\delta^{-1} = w_1 + w_2 - z_1 - z_2 > 0$. Then, $X \sim \text{CO}(\mu_X; q_X)$ for an admissible pair $(\mu_X; q_X)$, and the rv $Y = X - \lfloor \mu_X \rfloor$ follows $\text{CO}(\mu_Y; q_Y)$, see Lemma 2.5(d), and has mean value $\mu_Y \in [0, 1)$. Since X is a symmetric rv iff Y is symmetric, it is sufficient to find the generator class of the symmetric discrete-t distributions for which the mean value is 0 or 1/2. We distinguish the cases $\mu = 0$ and $\mu = 1/2$.

  - Case $\mu = 0$. In view of Table 1, $\mu_X = \delta(z_1 z_2 - w_1 w_2) = 0$, $q_X(j) = \delta(z_1 + j)(z_2 + j)$ and $p_X(j) \propto [z_1]_j[z_2]_j/([w_1+1]_j[w_2+1]_j)$, $j \in \mathbb{Z}$. The rv $Y = -X$ has pmf $p_Y(j) = p_X(-j) \propto [z_1]_{-j}[z_2]_{-j}/([w_1+1]_{-j}[w_2+1]_{-j})$, $j \in \mathbb{Z}$. Applying (13), $p_Y(j) \propto [-w_1]_j[-w_2]_j/([-z_1+1]_j[-z_2+1]_j)$, $j \in \mathbb{Z}$. Lemma 2.5(d) gives that Y follows $\text{CO}(\mu_Y; q_Y)$; moreover, from Table 1, $\mu_Y = \delta(w_1 w_2 - z_1 z_2) = 0$ and $q_Y(j) = \delta(-w_1 + j)(-w_2 + j)$. Obviously, X is a symmetric rv iff $X \stackrel{\text{d}}{=} Y$; using Proposition 3.1(b), X is a symmetric rv iff $\boldsymbol{w} = -\boldsymbol{z}$ and $z_1 + z_2 < 0$ (since $\delta > 0$). The symmetric discrete-t rvs with mean value zero is the set

  $$\mathcal{S}_{\text{d}-t}^{\{0\}} \doteq \left\{\text{d}{-}t(\boldsymbol{z}, -\boldsymbol{z}) \colon \boldsymbol{z} \in \tilde{\mathcal{C}}_2 \text{ with } z_1 + z_2 < 0\right\};$$

  - Case $\mu = 1/2$. Again from Table 1 we have that $\mu_X = \delta(z_1 z_2 - w_1 w_2) = 1/2$ and $q_X(j)$, $p_X(j)$ as in the case $\mu = 0$. The rv $Y = -X+1$ has pmf $p_Y(j) = p_X(-j+1) \propto [z_1]_{-j+1}[z_2]_{-j+1}/([w_1+1]_{-j+1}[w_2+1]_{-j+1})$, $j \in \mathbb{Z}$. Applying (13), $p_Y(j) \propto [-w_1-1]_j[-w_2-1]_j([-z_1]_j[-z_2]_j)$, $j \in \mathbb{Z}$. Lemma 2.5(d),(e) give that Y follows $\text{CO}(\mu_Y; q_Y)$. By construction, $\mu_Y = \mu_X = 1/2$; furthermore, Table 1 implies $q_Y(j) = \delta(-w_1 - 1 + j)(-w_2 - 1 + j)$. Since X is a symmetric rv iff $X \stackrel{\text{d}}{=} Y$, Proposition 3.1(b) gives that X is a symmetric rv iff $\boldsymbol{w} = -\boldsymbol{z} - \mathbf{1}$ and $z_1 + z_2 < -1$ (since $\delta > 0$). The symmetric discrete-t rvs with mean value half is

  $$\mathcal{S}_{\text{d}-t}^{\{1/2\}} \doteq \left\{\text{d}{-}t(\boldsymbol{z}, -\boldsymbol{z} - \mathbf{1}) \colon \boldsymbol{z} \in \tilde{\mathcal{C}}_2 \text{ with } z_1 + z_2 < -1\right\}.$$

  The symmetric discrete-t rvs with mean value zero or half is the generator-set of the symmetric discrete-t rvs,

  $$\mathcal{S}_{\text{d}-t}^{\{0,1/2\}} = \mathcal{S}_{\text{d}-t}^{\{0\}} \cup \mathcal{S}_{\text{d}-t}^{\{1/2\}}.$$

The set of the symmetric discrete-t rvs is

$$\mathcal{S}_{\text{d}-t} \doteq \mathcal{S}_{\text{d}-t}^{\{0,1/2\}} + \mathbb{Z} = \left\{X + r \colon X \in \mathcal{S}_{\text{d}-t}^{\{0,1/2\}},\ r \in \mathbb{Z}\right\}.$$

Finally, we define the noncentrality parameter as well as the degrees of freedom of a discrete-t distribution. In view of Theorem 5.1 and the fact that the t_ν distribution has finite absolute moments of order θ for each $0 < \theta < \nu$ while its νth absolute moment is infinity, cf. Lemma 2.6, we give the following definition.

Definition 5.2: Let $X \sim \text{d}-t(\boldsymbol{z}, \boldsymbol{w})$ with $\boldsymbol{z}, \boldsymbol{w} \in \tilde{\mathcal{C}}_2$ and $\delta^{-1} = w_1 + w_2 - z_1 - z_2 > 0$, and let us consider the parameters $\mu = \delta(z_1 z_2 - w_1 w_2)$ and $\beta = \delta(z_1 + z_2)$. Then, the *noncentrality parameter* of X is defined by

$$\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2), \quad \text{where } \varepsilon_1 = \min_{x \in \frac{1}{2}\mathbb{Z}} |x - \mu| \quad \text{and} \quad \varepsilon_2 = |4\delta\mu + 2\beta + 1|,$$

and the *degrees of freedom* of X are defined as $\text{df} = 1 + 1/\delta$.

6. Moment relations in the cumulative Ord family

This section presents some properties about the moments of a rv of the CO family.

For a discrete rv $X \sim \text{CO}(\mu; q) \equiv \text{CO}(\mu; \delta, \beta, \gamma)$, the following covariance identity holds

$$\mathsf{Cov}[X, g(X)] = \mathbb{E}q(X)\Delta g(X), \tag{14}$$

provided that $\mathbb{E}q(X)|\Delta g(X)| < \infty$; see [7]. Setting $g(x) = x$, we get

$$\sigma^2 \doteq \mathsf{Var}X = \mathbb{E}q(X),$$

provided $\mathbb{E}X^2 < \infty$. Writing $q(X) = \delta(X - \mu)^2 + q'(\mu)(X - \mu) + q(\mu)$ and taking expectations, we have

$$\sigma^2 = q(\mu)/(1 - \delta), \tag{15}$$

noting that from Lemma 2.6 and Remarks 2.7 and 3.3, the denominator $1 - \delta$ is positive.

Now, we prove a lemma concerning the pmf $p^*(j) \propto q(j)p(j)$.

Lemma 6.1: *Suppose a non-constant rv $X \sim \text{CO}(\mu; q)$ and $\mathbb{E}|X|^3 < \infty$. Let X^* be the rv with pmf $p^* \propto qp$. Then, X^* is supported on the set $S(X^*) \doteq S^\circ(X)$ (i.e. $\alpha^* = \alpha$, $\omega^* = \omega - 1$) and $X^* \sim \text{CO}(\mu^*; q^*)$, where $\mu^* = (\mu + \beta + \delta)/(1 - 2\delta)$ and $q^*(j) = q(j + 1)/(1 - 2\delta)$.*

Proof: Lemma 2.6 proves that $1 - 2\delta > 0$ because $\mathbb{E}|X|^3 < \infty$; from this and Lemma 2.5(b),(c), it follows that the function p^* is non-negative on $\mathbb{Z}$ and takes strictly positive values on the set $S(X^*) = S^\circ(X)$. Using $\Delta[h_1(k)h_2(k)] = h_1(k)\Delta h_2(k) + h_2(k+1)\Delta h_1(k)$, we have $\Delta[q(k)q(k-1)p(k-1)] = q(k)\Delta[q(k-1)p(k-1)] + q(k)p(k)\Delta q(k) = q(k)(\mu - k)p(k) + q(k)p(k)(2\delta k + \delta + \beta) = [\mu + \delta + \beta - (1 - 2\delta)k]q(k)p(k)$. Thus, $\sum_{k \le j}(\mu^* - k)p^*(k) = \{\sum_{k \le j} \Delta[q(k)q(k-1)p(k-1)]\}/[(1 - 2\delta)\mathbb{E}q(X)]$. Since $\mathbb{E}|X|^3 < \infty$, using the same arguments as in the proof of Proposition 2.10, we obtain that $\sum_{k \le j} \Delta[q(k)q(k-1)p(k-1)] = q(j+1)q(j)p(j)$. So, $\sum_{k \le j}(\mu^* - k)p^*(k) = q^*(j)p^*(j)$. It remains to show that the value μ^* is the mean of X^*. Of course, $\mathbb{E}|X^*| < \infty$ because $\mathbb{E}|X|^3 < \infty$. If $S(X^*)$ has a finite upper endpoint $\omega^* = \omega - 1$, then $q^*(\omega^*) = q(\omega)/(1 - 2\delta) = 0$, since ω is the upper endpoint of $S(X)$. If $\omega^* = \omega = \infty$, we use the same arguments as in the proof of Proposition 2.10. For both cases $\omega < \infty$ and $\omega = \infty$, $\sum_{k \in S(X^*)}(\mu^* - k)p^*(k) = 0$, i.e. $\mathbb{E}X^* = \mu^*$. ∎

The quadratic q takes non-negative values on the support of X. Therefore, we can create new pmfs by defining $p_i \propto q^{[i]}p$. But, if the support of X is finite, then for each i greater than or equal to the cardinality of $S(X)$, the function p_i vanishes identically on $\mathbb{Z}$. Thus, it is useful to define the quantity $M = M(X)$ as follows:

$$M = M(X) \doteq |S(X)| - 1 = \omega - \alpha \in \{0, 1, \ldots\} \cup \{\infty\}.$$

Proposition 6.2: *Let $X \sim \mathrm{CO}(\mu; q) = \mathrm{CO}(\mu; \delta, \beta, \gamma)$ with pmf p and $\mathbb{E}|X|^{2n+1} < \infty$ for some $n \in \{0, 1, \ldots, M(X)\}$. For all $i = 0, 1, \ldots, n$, we consider the rvs X_i with pmfs $p_i \propto q^{[i]}p$ [note that $X_i = X^*_{i-1}$, $i = 1, \ldots, n$, where $X_0 = X$], and we define*

$$\mu_i = \frac{\delta i^2 + \beta i + \mu}{1 - 2i\delta}, \quad q_i(j) = \frac{q(j + i)}{1 - 2i\delta},$$
$$\delta_i = \frac{\delta}{1 - 2i\delta}, \quad \psi(i) = \frac{q\left([-\delta i^2 + (\beta + 1)i + \mu]/[1 - 2i\delta]\right)}{1 - (2i + 1)\delta}.$$

Then:

(a) *The rv X_i is supported on the set $S_i \doteq S(X_i) = \{\alpha_i, \ldots, \omega_i\} = \{\alpha, \ldots, \omega - i\}$;*
(b) *$X_i \sim \mathrm{CO}(\mu_i; q_i)$;*
(c) *$\mathrm{Var}X_i = \psi(i)$ (for $i = n$, it is additionally required that $\mathbb{E}|X|^{2n+2} < \infty$);*
(d) *$A_i = A_i(\mu; q) \doteq \mathbb{E}q^{[i]}(X) = \psi^{[i]}(0)\Pi^{[i]}_{2\delta}(0)$;*
(e) *The descending factorial moments of X, $\mu_{(r)} = \mathbb{E}(X)_r$, and the ascending factorial moments of X, $\mu_{[r]} = \mathbb{E}[X]_r$, satisfy the following second-order recurrence relations:*

$$(1 - r\delta)\mu_{(r+1)} = \{\mu + r[\beta - 1 + (2r - 1)\delta]\}\mu_{(r)} + r\{\gamma + (r - 1)[\beta + (r - 1)\delta]\}\mu_{(r-1)},$$
$$(1 - r\delta)\mu_{[r+1]} = \{\mu + r[\beta + 2 - (2r - 1)\delta]\}\mu_{[r]} + r\{\gamma - \mu - (r - 1)[\beta + 1 - (r - 1)\delta]\}\mu_{[r-1]},$$

with initial conditions $\mu_{(0)} = \mu_{[0]} = 1$ and $\mu_{(1)} = \mu_{[1]} = \mu$, for all $r = 1, \ldots, 2n$;
(f) *The factorial moments of X, $\mu_{(r)}$ and $\mu_{[r]}$, satisfy the following recurrence relations:*

$$(1 - r\delta)\mu_{[r+1]} = [\mu + r(\beta - r\delta + 1)]\mu_{[r]} + \gamma r! \sum_{k=0}^{r-1} \frac{\mu_{[k]}}{k!},$$
$$(1 - r\delta)\mu_{(r+1)} = (\delta r^2 + \beta r + \mu)\mu_{(r)} + (\gamma - \mu)r! \sum_{k=0}^{r-1} (-1)^{r+k+1}\frac{\mu_{(k)}}{k!}.$$

Proof: (a) Observe that $1 - 2i\delta > 0$ for all $i = 0, 1, \ldots, n$ (if $\delta \le 0$, it is obvious; if $\delta > 0$, the case of infinity support follows by Lemma 2.6 while the case of finite support by Remark 2.7). Therefore, Lemma 2.5(b),(c) show that $q^{[i]}p$ is supported on S_i.

(b) The proof will be done by induction on i. For $i = 1$, the result follows from Lemma 6.1. Assuming that it holds for $i - 1 \in \{0, 1, \ldots, n - 1\}$, we will prove that it is true for i. By assumption, $X_{i-1} \sim \mathrm{CO}(\mu_{i-1}; q_{i-1})$. From $\mathbb{E}|X|^{2n+1} < \infty$, it follows that $\mathbb{E}|X_{i-1}|^3 < \infty$. As in Lemma 6.1, we consider the rv $X^*_{i-1} \sim \mathrm{CO}(\mu^*_{i-1}; q^*_{i-1})$ with $\mu^*_{i-1} = (\mu_{i-1} + \beta_{i-1} + \delta_{i-1})/(1 - 2\delta_{i-1})$ and $q^*_{i-1} = q_{i-1}(j + 1)/(1 - 2\delta_{i-1})$. Hence, after some algebra, we get $\mu^*_{i-1} = \mu_i$ and $q^*_{i-1} = q_i$. Finally, observe that $p^*_{i-1} \propto q_{i-1}p_{i-1}$ and $q_{i-1}p_{i-1} \propto q^{[i]}p$; so, $p^*_{i-1} \propto q^{[i]}p$. By definition, $p_i \propto q^{[i]}p$. Hence, we conclude that $p_i = p^*_{i-1}$ because p_i, p^*_{i-1} are pmfs with support S_i.

(c) It is immediate from (b) and Equation (15).

(d) Using (c), an application of Equation (15) gives $(1-2j\delta)\psi(j) = (1-2j\delta)\mathbb{E}q_j(X_j) = \mathbb{E}q(X_j + j) = \mathbb{E}q^{[j+1]}(X)/\mathbb{E}q^{[j]}(X) = A_{j+1}/A_j$, $j = 0, 1, \ldots, n-1$, where $A_0 = 1$. By multiplying these relations for $j = 0, 1, \ldots, k-1$, the result follows.

(e) Write $\mathbb{E}(X)_{r+1} = (\mu - r)\mathbb{E}(X)_r + \mathbb{E}(X-\mu)(X)_r = (\mu - r)\mathbb{E}(X)_r + \mathsf{Cov}[X, (X)_r]$. Using the covariance identity (14) and since $\Delta(j)_r = r(j)_{r-1}$, it follows that $\mathsf{Cov}[X, (X)_r] = r\mathbb{E}q(X)(X)_{r-1}$. Moreover, $q(x)(x)_{r-1} = \delta(x)_{r+1} + [\beta + (2r-1)\delta](x)_r + \{\gamma + (r-1)[\beta + (r-1)\delta]\}(x)_{r-1}$. Thus, $\mathbb{E}q(X)(X)_{r-1} = \delta\mu_{(r+1)} + [\beta + (2r-1)\delta]\mu_{(r)} + \{\gamma + (r-1)[\beta + (r-1)\delta]\}\mu_{(r-1)}$. Upon combining the above relations, the result follows.

For the second relation, we consider the rv $Y = -X \sim \mathrm{CO}(-\mu; \delta, -\beta - 1, \gamma - \mu)$; see Lemma 2.5(e). Observe that $\mu^Y_{(r)} = (-1)^r\mu_{[r]}$. An application of the first relation, with some algebra, shows the result.

(f) Using the same arguments as in (e), we write $\mu_{[r+1]} = (\mu + r)\mu_{[r]} + \mathsf{Cov}(X, [X]_r)$. Utilizing (14) and the fact that $\Delta[X]_r = r[X+1]_{r-1}$, we get $\mathsf{Cov}(X, [X]_r) = r\mathbb{E}q(X)[X+1]_{r-1}$. Write $q(x)[x+1]_{r-1} = \delta[x]_{r+1} + (\beta - r\delta)[x]_r + \gamma\sum_{k=0}^{r-1}(r-1)_{r-1-k}[x]_k$. Then, $\mu_{[r+1]} = (\mu + r)\mu_{[r]} + r[\delta\mu_{[r+1]} + (\beta - r\delta)\mu_{[r]} + \gamma(r-1)!\sum_{k=0}^{r-1}\mu_{[k]}/k!]$. Finally, the proof of the recurrence relation of $\mu_{(r)}$s is similar to that of (e). ■

Now, suppose the rv X belongs to the CO family and its support has lower endpoint $\alpha = 0$. Then, $\gamma = \mu$ (see Lemma 2.5(c)) and so the second recurrence relation of Proposition 6.2(f) takes the form $(1 - r\delta)\mu_{(r+1)} = q(r)\mu_{(r)}$. Under this observation, the following corollary follows immediately.

Corollary 6.3: *Let $X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$. If the support of X has lower endpoint $\alpha = 0$, then for each positive integer k such that $\mathbb{E}|X|^k < \infty$, the kth descending factorial moment of X is $\mu_{(k)} = q^{[k]}(0)/\Pi_\delta^{[k]}(0) = \prod_{j=0}^{k-1}[q(j)/(1-j\delta)]$.*

We apply Corollary 6.3 to the distributions of the types 1–5 that are presented in Table 1.

Application 6.4: 1. Poisson Distribution: If $X \sim P(\lambda)$ with $\lambda > 0$, then $q(j)/(1-j\delta) = \lambda$ and so $\mu_{(k)} = \lambda^k$ for all $k = 0, 1, \ldots$;

2. Binomial distribution: If $X \sim \mathrm{Bin}(N, p)$ with $N = 1, 2, \ldots$ and $0 < p < 1$, then $q(j)/(1 - j\delta) = p(N-j)$ and so $\mu_{(k)} = p^k(N)_k$ for all $k = 0, 1, \ldots$;

3. Negative Binomial distribution: If $X \sim \mathrm{NB}(r, p)$, $r > 0$ and $0 < p < 1$, then $q(j)/(1-j\delta) = [(1-p)/p](r+j)$ and so $\mu_{(k)} = [(1-p)/p]^k[r]_k$ for all $k = 0, 1, \ldots$;

4a. Negative Hypergeometric distribution: If $X \sim \mathrm{NHgeo}(N; r, s)$ with $N = 1, 2, \ldots$ and $r,s > 0$, then $q(j)/(1-j\delta) = (r+j)(N-j)/(r+s+j)$ and so $\mu_{(k)} = [r]_k(N)_k/[r+s]_k$ for all $k = 0, 1, \ldots$;

4b. Hypergeometric distribution: If $X \sim \mathrm{NHgeo}(N; r, s)$ with $N = 1, 2, \ldots$ and $r,s > N-1$, then $q(j)/(1-j\delta) = (r-j)(N-j)/(r+s-j)$ and so $\mu_{(k)} = (r)_k(N)_k/(r+s)_k$ for all $k = 0, 1, \ldots$;

5. Discrete F-type distribution: If $X \sim \mathrm{d}{-}F(\rho; r, s)$ with $(r, s) \in \mathcal{C}_2$ and $\rho > \max\{0, r+s+1\}$, then $q(j)/(1-j\delta) = (r+j)(s+j)/(\rho - r - s - 1 - j)$ and so $\mu_{(k)} = [r]_k[s]_k/(\rho - r - s - 1)_k$ for all $k = 0, 1, \ldots$ such that $k < \rho - r - s$.

Next, we generalize the results of Lemma 2.8 in CO family.

Proposition 6.5: *Let $X \sim \mathrm{CO}(\mu; q) = \mathrm{CO}(\mu; \delta, \beta, \gamma)$ and assume that it has an upper (resp. lower) unbounded support and $\mathbb{E}|X|^{2i-1} < \infty$ for some $i \in \{1, 2, \ldots\}$. Then, $j^{2i}p(j) \to 0$ as $j \to \infty$ (resp. $j \to -\infty$).*

Proof: Note that $\delta \geq 0$ since the support is finite if $\delta < 0$. For the case $\delta = 0$, the result is obvious since X has finite moments of any order; see Lemma 2.6. When $\delta > 0$, then, as in Proposition 6.2, consider the rv $X_{2i-2} \sim \mathrm{CO}(\mu_{2i-2}; q_{2i-2})$. From Lemma 2.8, it follows that $q_{2i-2}(j)p_{2i-2}(j) \to 0$ as $j \to \infty$ (resp. $j \to -\infty$) and since $\lim_{j\to\pm\infty} q_{2i-2}(j)p_{2i-2}(j) \propto \lim_{j\to\pm\infty} j^{2i}p(j)$, the proof is complete. ■

7. Orthogonal polynomials in the cumulative Ord family

In this section, we present results for the orthogonal polynomials of a probability measure of the CO family. These polynomials are obtained by a *discrete Rodrigues-type formula*.

First, we present a brief review. Hildebrandt [8, Chapter IV, pp.419–439] studied the nonzero solutions $u(j)$ of the Pearson difference equation,

$$\Delta u(j) = \frac{N(j)}{D(j)} u(j), \tag{16}$$

where the numerator N is a polynomial of degree at most one and the denominator D is a polynomial of degree at most two. He showed that the functions $Q_n(j)$, produced by the Rodrigues-type formula

$$Q_n(j) = \frac{\Delta^n \left[D^{[n]}(j-n)u(j)\right]}{u(j)}, \tag{17}$$

are polynomials of degree at most n; see [8, p. 425]. Note that Hildebrandt makes use of the descending power notation, $D^{(n)}(j-1) \doteq D(j-1)D(j-2)\cdots D(j-n) = D^{[n]}(j-n)$. He farther established several properties of these polynomials. In the sequel of this section, when we say that a function is the solution of a difference equation, we will always mean a pmf solution.

In Hildebrandt's results, the orthogonality of the produced polynomials was not an issue. However, these polynomials are orthogonal only when we make a correct choice of the set on which we seek a solution, and provided that we used the correct writing of the ratio of the polynomials N and D in Equation (17). Next, we present some examples to illustrate this issue.

Here, we note that Equations (1) and (16) are equivalent, excluding the case $\Delta p(j) = 0$. Specifically, $\Delta p(j)/p(j) = N(j)/D(j)$ is equivalent with $\Delta p(j-1)/p(j) = N(j-1)/[D(j-1) + N(j-1)]$.

Example 7.1: (a) Consider the difference equation $\Delta p(j)/p(j) = (\lambda - j - 1)/(j + 1)$, where λ is a positive constant. This difference equation is of the form Equation (1) and (16). Of course, in order to solve a difference equation, we must specify the support set on which we seek the solution. If this set is $\mathbb{N}$, then the solution is $\mathrm{e}^{-\lambda}\lambda^j/j!$, $j = 0, 1, \ldots$ (Poisson distribution with parameter λ). If the set is $\{0, 1, \ldots, N\}$, then the solution is $C\lambda^j/j!$, $j = 0, 1, \ldots, N$ (truncated Poisson distribution with parameter λ). The polynomials obtained by Equation (17) are the Charlier polynomials which are orthogonal with respect to the Poisson pmf, but not with respect to the truncated Poisson pmf.

(b) Consider the pmf of the geometric distribution with parameter $p \in (0, 1)$, i.e. $p(j) = p(1-p)^j$, $j = 0, 1, \ldots$. This pmf satisfies the difference equation $\Delta p(j)/p(j) = -p$, which can be rewritten in the form Equation (16) in many ways. Specifically, $\Delta p(j)/p(j) = -p(bj + a)/(bj + a)$, where $bj+a$ is a constant (when $b=0$), or a linear polynomial without roots on $\mathbb{N}$. For any choice of a and b, Hildebrandt's results are valid. However, the polynomials in Equation (17) are orthogonal with respect to the geometric pmf only when we make the choice $a = b \neq 0$ (Meixner polynomials).

(c) Now, consider the difference equation $\Delta p(j)/p(j) = 0$ supported on an integer chain. Of course, if the support is infinite, then it has no pmf solutions; thus, we consider a finite integer chain, and without loss of generality take $S = \{1, 2, \ldots, N\}$. The solution is $p(j) = 1/N, j = 1, 2, \ldots, N$, i.e. X is uniformly distributed on the support. The equation can be rewritten in the form (16) in many ways, i.e. $N(j) = 0$ and $D(j) = cj^2 + bj + a$ any quadratic polynomial without roots on

$\{1, 2, \ldots, N-1\}$. Again, the polynomials in Equation (17) are orthogonal with respect to pmf p only when one makes the correct choice $D(j) \propto j(N-j)$ (Hahn polynomials).

It is true that the denominator in Equation (1), under suitable conditions, generates orthogonal polynomials with respect to the pmf solution of this equation; see Proposition 2.10 and also the next theorem.

Remark 7.2: In view of Example 7.1, we observe the following. The Rodrigues-type formula (17) is a mechanism for producing polynomials, that may have some elegant properties regarding their coefficients. On the other hand, the specific cases of Example 1 clearly indicate that the relation (1) (or the equivalent relation (16)) ignores the information about the production of the Rodrigues-orthogonal polynomials, while the relation (2) provides the whole of the information that is needed.

Independently of Hildebrandt's results, Afendras et al. [3] studied the orthogonality of the Rodrigues polynomials in the CO family:

Theorem 7.3 ([3, Lemma 2.3, Theorems 2.1 and 2.2])**:** *Let* $X \sim \mathrm{CO}(\mu; q) = \mathrm{CO}(\mu; \delta, \beta, \gamma)$. *For each* $k = 0, 1, 2, \ldots,$ *define the functions* $P_k(j), j \in S$, *by the Rodrigues-type formula*

$$P_k(j) = \frac{(-1)^k}{p(j)} \Delta^k \left[q^{[k]}(j-k)p(j-k) \right] = \frac{1}{p(j)} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} q^{[k]}(j-i)p(j-i). \tag{18}$$

Then:

(a) *Each* P_k *is a polynomial of degree at most k, with*

$$\mathrm{lead}(P_k) = \Pi_\delta^{[k]}(k-1) \doteq c_k(\delta) \tag{19}$$

[in the sense that the function $P_k(j)$, $j \in S$, *is the restriction of a real polynomial* $G_k(x) = \sum_{i=0}^{k} c(k,i)x^i$, $x \in \mathbb{R}$, *of degree at most k, such that* $c(k,k) = \mathrm{lead}(P_k)$*];*

(b) *Provided that* $\mathbb{E}X^{2n} < \infty$ *for some* $n \geq 1$, *the polynomials* P_k, $k = 0, 1, \ldots, n$, *satisfy the orthogonality condition*

$$\mathbb{E}P_k(X)P_m(X) = \delta_{k,m}c_k(\delta)\mathbb{E}q^{[k]}(X) = \delta_{k,m}c_k(\delta)A_k, \quad k, m = 0, 1, \ldots, n, \tag{20}$$

where $\delta_{k,m}$ *is Kronecker's delta;*

(c) *Provided that* $k \geq 1$ *and* $\mathbb{E}|X|^{2k-1} < \infty$, *the following 'Rodrigues inversion formula' holds:*

$$q^{[k]}(j)p(j) = \frac{1}{(k-1)!} \sum_{i>j} (i-j-1)_{k-1} P_k(i)p(i). \tag{21}$$

Remark 7.4: (a) In (18) when $k > M$, we have $\mathbb{E}P_k^2(X) = 0$, since the polynomial $q^{[k]}$ vanishes identically on S. Thus, in the sequel, we study the polynomials P_k only when $k \leq M$.

(b) Provided that $\mathbb{E}X^{2k} < \infty$ and $k \leq M$, the quantities $1 - j\delta$, $j = 0, 1, \ldots, 2k-2$, are strictly positive. If $\delta \leq 0$, this is obvious. If $\delta > 0$ and S is infinite, this follows from Lemma 2.6; when S is finite, it follows from Remark 2.7. Thus, the quantity $\Pi_\delta^{[k]}(k-1)$ is strictly positive. Also, since the polynomial $q^{[k]}$ is non-negative on S and $\mathbb{P}[q^{[k]}(X) > 0] > 0$, it follows that $0 < \mathbb{E}q^{[k]}(X) < \infty$.

For a non-negative integer n such that $n \le M$ and $\mathbb{E}X^{2n} < \infty$, Remark 7.4(b) shows that we can define the standardized Rodrigues polynomials,

$$\phi_k(j) = [k! c_k(\delta) A_k]^{-1/2} P_k(j), \quad k = 0, 1, \ldots, n. \tag{22}$$

The set $\{\phi_k\}_{k=0}^n \subset L^2(\mathbb{R}, X)$ is an orthonormal basis for all polynomials with degree at most n. Moreover, Equation (19) shows that the leading coefficient is given by

$$\text{lead}(\phi_k) \doteq d_k(\mu; q) = [c_k(\delta)/(k! A_k)]^{1/2} > 0, \quad k = 0, 1, \ldots, n. \tag{23}$$

Let X be any rv of the CO family with $\mathbb{E}|X|^{2n} < \infty$, where n is less than the cardinality of the support of X. It is well known that we can always construct an orthonormal set of real polynomials up to order n. This construction is based on the first $2n$ moments of X and is a by-product of the Gram–Schmidt orthonormalization process, applied to the linearly independent system $\{1, x, x^2, \ldots, x^n\} \subset L^2(\mathbb{R}, X)$. The orthonormal polynomials are then uniquely defined, apart from the fact that we can multiply each polynomial by ± 1. It follows that the standardized Rodrigues polynomials ϕ_k of Equation (22) are the unique orthonormal polynomials that can be defined for a pmf $p \sim \text{CO}(\mu; \delta, \beta, \gamma)$, provided that $\text{lead}(\phi_k) > 0$. Therefore, it is useful to express the L^2-norm of each P_k in terms of the parameters δ, β, γ and μ. This result is given by Equation (20) and Proposition 6.2(d).

Consider the rvs X_i with pmfs p_i as defined in Proposition 6.2. From (18), the corresponding Rodrigues polynomials are given by

$$P_{k,i}(j) = \frac{(-1)^k}{p_i(j)} \Delta^k \left[q_i^{[k]}(j-k) p_i(j-k) \right]. \tag{24}$$

Thus, the standardized Rodrigues polynomials, orthonormal with respect to the pmf of X_i, are given by

$$\phi_{k,i}(j) = \left[k! c_k(\delta_i) A_k(\mu_i; q_i) \right]^{-1/2} P_{k,i}(j). \tag{25}$$

Note that for $i = 1$, the rv X_1 is denoted by X^* ($p_1 \equiv p^*$ etc.). Therefore, we may denote the polynomial $P_{k,1}$ by P_k^* and the standardized polynomial $\phi_{k,1}$ by ϕ_k^*. An important observation is that the forward difference of ϕ_k is scalar multiple of ϕ_{k-1}^*. Specifically, we have the following lemma.

Lemma 7.5: *If $X \sim \text{CO}(\mu; q) = \text{CO}(\mu; \delta, \beta, \gamma)$ and $\mathbb{E}X^{2n} < \infty$ for some $1 \le n \le M$, then the polynomials ϕ_k of* (22) *and $\phi_{k,1} \equiv \phi_k^*$ of Equation* (25) *are related through*

$$\Delta\phi_k(j) = v_{k-1}\phi_{k-1}^*(j), \quad k = 1, 2, \ldots, n, \quad \text{where } v_{k-1} = v_{k-1}(\mu; q) \doteq \{k[1 - (k-1)\delta]/A_1\}^{1/2}. \tag{26}$$

Proof: First, we show that for $1 \le m < k \le n$, $\mathbb{E}\Delta\phi_k(X^*)\Delta\phi_m(X^*) = 0$. We have

$$[\Delta\phi_k(j)\Delta\phi_m(j)]q(j)p(j) = \Delta\left\{\phi_k(j)[\Delta\phi_m(j-1)]q(j-1)p(j-1)\right\} - \phi_k(j)\text{pol}_m(j)p(j), \tag{27}$$

where $\text{pol}_m(j) \doteq [\Delta^2\phi_m(j-1)]q(j) + [\Delta\phi_m(j-1)](\mu - j)$ is a polynomial with $\deg(\text{pol}_m) \le m$. Summing (27) for all $j \in \{\alpha, \ldots, \omega\}$, we observe the following: The lhs of the sum is $\mathbb{E}[\Delta\phi_k(X^*)\Delta\phi_m(X^*)]\mathbb{E}q(X)$. The first part of the rhs of the sum is $\phi_k(j)[\Delta\phi_m(j-1)]q(j-1)p(j-1)|_\alpha^{\omega+1} = 0$ (for finite α and ω, this follows from $p(\alpha-1) = q(\omega) = 0$; for infinite α and ω, it follows from Proposition 6.5). The second part of the rhs of the sum is $\mathbb{E}\phi_k(X)\text{pol}_m(X) = 0$, because ϕ_k is orthogonal to any polynomial of degree less than k. From the moment conditions, it is obvious that $\mathbb{E}[\Delta\phi_k(X^*)]^2 < \infty$. Thus, it suffices to show that $\mathbb{E}[\Delta\phi_k(X^*)]^2 > 0$. The polynomial $\Delta\phi_k(x)$, $x \in \mathbb{R}$, is not identically zero, since $\text{lead}(\Delta\phi_k) = k\,\text{lead}(\phi_k) > 0$, and can not vanish identically on the support of X^*, since $\deg(\Delta\phi_k) = k - 1$ is less than the cardinality of the support of X^*. Finally, since $\deg(\Delta\phi_k) = \deg(\phi^*_{k-1}) = k-1$, $k = 1, \ldots, n$, the uniqueness of the orthogonal polynomial system implies that there exist constants $\upsilon_k \neq 0$ such that $\Delta\phi_k = \upsilon_{k-1}\phi^*_{k-1}$. Equating the leading coefficients, we obtain $\text{lead}(\Delta\phi_k) = \upsilon_{k-1}\text{lead}(\phi^*_{k-1})$, that is, $\upsilon_{k-1} = \text{lead}(\Delta\phi_k)/\text{lead}(\phi^*_{k-1}) = k\,\text{lead}(\phi_k)/\text{lead}(\phi^*_{k-1}) = k\{[(k-1)!c_k(\delta)A_{k-1}(\mu^*;q^*)]/[k!c_{k-1}(\delta^*)A_k]\}^{1/2}$; see Equation (23). Moreover, one can easily see that $c_k(\delta) = [1-(k-1)\delta](1-2\delta)^{k-1}c_{k-1}(\delta^*)$ and $A_k = (1-2\delta)^{k-1}A_1 A_{k-1}(\mu^*;q^*)$. Thus, $\upsilon_{k-1} = \{k[1-(k-1)\delta]/A_1\}^{1/2}$. ■

Applying now Lemma 7.5, inductively it is easy to verify the following result.

Theorem 7.6: *Let $X \sim \text{CO}(\mu;q) = \text{CO}(\mu;\delta,\beta,\gamma)$ and assume that $\mathbb{E}X^{2n} < \infty$ for some integer n with $1 \leq n \leq M$. Then,*

$$\Delta^m\phi_k(j) = \upsilon^{(m)}_{k-m}\phi_{k-m,m}(j), \quad m = 0,1,\ldots,n, \;\; k = m, m+1, \ldots, n,$$

$$\text{with } \upsilon^{(m)}_{k-m} = \upsilon^{(m)}_{k-m}(\mu;q) \doteq \left\{k!\Pi^{[m]}_\delta(k-1)/[(k-m)!A_m]\right\}^{1/2}, \tag{28}$$

where the polynomials ϕ_k, $\phi_{k-m,m}$ are as given in Equations (22) *and* (25), *respectively.*

Proof: The proof follows by induction on m. For $m=0$, the result is obvious, noting that $\phi_{k,0} = \phi_k$ and $\nu^{(0)}_{k,0} = 1$. For $m=1$, the result follows by Lemma 7.5, since $\phi_{k,1} = \phi^*_k$ and $\nu^{(1)}_{k,1} = \nu_k$. Assuming that it is true for $m-1 \in \{0,1,\ldots,n-1\}$, we will show that it holds for m. By the assumption of induction, $\Delta^{m-1}\phi_k(j) = \upsilon^{(m-1)}_{k-m+1}\phi_{k-m+1,m-1}(j)$, and $\upsilon^{(m-1)}_{k-m+1} = \{k!\Pi^{[m-1]}_\delta(k-1)/[(k-m+1)!A_{m-1}]\}^{1/2}$. Applying Lemma 7.5 for $X_{m-1} \sim \text{CO}(\mu_{m-1};q_{m-1})$, $\Delta^m\phi_k(j) = \Delta[\Delta^{m-1}\phi_k(j)] = \upsilon^{(m-1)}_{k-m+1}\Delta[\phi_{k-m+1,m-1}(j)] = \upsilon^{(m-1)}_{k-m+1}\upsilon_{k-m}(\mu_{m-1};q_{m-1})\phi^*_{k-m,m-1}(j) = \upsilon^{(m-1)}_{k-m+1}\upsilon_{k-m}(\mu_{m-1};q_{m-1})\,\phi_{k-m,m}(j)$, where $\upsilon^{(m-1)}_{k-m+1}\upsilon_{k-m}(\mu_{m-1};q_{m-1}) = (\{k!\Pi^{[m-1]}_\delta(k-1)/[(k-m+1)!A_{m-1}]\}\{(k-m+1)[1-(k-m)\delta_{m-1}]/[A_1(\mu_{m-1};q_{m-1})]\})^{1/2}$; see (26). Finally, it is easily shown that $A_1(\mu_{m-1};q_{m-1}) = A_m/\{[1-2(m-1)\delta]A_{m-1}\}$ and $1-(k-m)\delta_{m-1} = [1-(k-2m-2)\delta]/[1-2(m-1)\delta]$. Thus, $\upsilon^{(m-1)}_{k-m+1}\upsilon_{k-m}(\mu_{m-1};q_{m-1}) = \upsilon^{(m)}_{k-m}$, completing the proof. ■

8. L^2 completeness and expansions

We now study the Fourier coefficients of a function regarding its expansion in the L^2 Hilbert space. First, we present the following basic result.

Theorem 8.1 ([3, Theorem 2.2])**:** *Suppose $X \sim \text{CO}(\mu;q)$ and that $\mathbb{E}X^{2k} < \infty$ for some $k \geq 1$. If g is a function defined on S with $\mathbb{E}q^{[k]}(X)|\Delta^k g(X)| < \infty$, then $\mathbb{E}|P_k(X)g(X)| < \infty$ and the following covariance identity holds:*

$$\mathbb{E}P_k(X)g(X) = \mathbb{E}q^{[k]}(X)\Delta^k g(X). \tag{29}$$

Note that if the support S has a finite upper endpoint, $\omega < \infty$, then $\Delta^k g(j), j \in S$, may depend on some values $\{g(j), j \notin S\}$; however, only the values $\{j : j \in S, j \leq \omega - k\}$ are relevant in the covariance identity. This is so because for $j > \omega - k$, the ascending power $q^{[k]}(j)$ includes the factor $q(\omega) = 0$. Thus, assuming any values for $g(j)$ when j lies in the set $\{\omega+1, \omega+2, \ldots\}$, e.g. $g(j) = 0, j = \omega+1, \omega+2, \ldots$, will not affect the covariance identity. For any function g defined on S, the function $\Delta^k g$ has domain the set S_k; see Proposition 6.2(a). Thus, the values $\Delta^k g(j), j \in S \smallsetminus S_k$ (if exist), that appear in the formula, are immaterial. Note that if S is finite and $k > M(X)$, then both polynomials P_k and $q^{[k]}$ are identically zero on S, and the relation (29) takes the trivial form $0=0$.

It is important to note that the identity (29), combined with Equation (22), enables a convenient calculation of the Fourier coefficient $\alpha_k = \mathbb{E}\phi_k(X)g(X)$ of a function g. Specifically,

$$\alpha_k = \mathbb{E}\phi_k(X)g(X) = [k!c_k(\delta)A_k]^{-1/2}\mathbb{E}q^{[k]}(X)\Delta^k g(X). \tag{30}$$

The rhs of Equation (30) shows that we do not need to know the polynomial ϕ_k in order to calculate α_k.

We now shed some light on the interrelations between the spaces $L^2(\mathbb{R}, X_i)$ and $L^1(\mathbb{R}, X_i)$.

Lemma 8.2: *Let the rvs X and X^* be as in Lemma* 6.1. *Assume that the function g is defined on the support of X. Then,*

(a) $\Delta g \in L^2(\mathbb{R}, X^*) \Rightarrow g \in L^2(\mathbb{R}, X)$;
(b) $\Delta g \in L^1(\mathbb{R}, X^*) \Rightarrow g \in L^1(\mathbb{R}, X)$.

Proof: (a) For $|S| < \infty$, the result is obvious. Thus, assume that $|S| = \infty$ and consider a function g such that $\Delta g \in L^2(\mathbb{R}, X^*)$. It suffices to show that for some $m \in \mathbb{Z}$,

$$\sum_{j=m}^{\infty} g^2(j)p(j) < \infty \quad \text{when } \omega = \infty, \quad \text{and} \quad \sum_{j=-\infty}^{m} g^2(j)p(j) < \infty \quad \text{when } \alpha = -\infty.$$

For the first inequality, it suffices to show that $\Sigma_1(m) \doteq \sum_{j=m}^{\infty}[g(j) - g(m)]^2 p(j) < \infty$. Let $m = \lfloor\mu\rfloor + 1 > \mu$. Then, $\Sigma_1(m) = \sum_{j=m}^{\infty} p(j)[\sum_{i=m}^{j-1} \Delta g(i)]^2 \le \sum_{j=m}^{\infty} p(j)(j-m)\sum_{i=m}^{j-1}[\Delta g(i)]^2 = \sum_{i=m}^{\infty}[\Delta g(i)]^2 \sum_{j=i+1}^{\infty}(j-m)p(j) \le \sum_{i=m}^{\infty}[\Delta g(i)]^2 \sum_{j=i+1}^{\infty}(j-\mu)p(j)$. Since $\sum_{j=i+1}^{\infty}(j-\mu)p(j) = q(i)p(i)$, we get $\Sigma_1(m) \le \sum_{i=m}^{\infty}[\Delta g(i)]^2 q(i)p(i) \le \sum_{i\in\mathbb{Z}}[\Delta g(i)]^2 q(i)p(i) = \mathbb{E}q(X)\mathbb{E}[\Delta g(X^*)]^2 < \infty$. For the second inequality, we use the same arguments with $m = \lfloor\mu\rfloor \le \mu$.

(b) Let $\Delta g \in L^1(\mathbb{R}, X^*)$. Then, $\mathbb{E}q(X)|\Delta g(X)| = \mathbb{E}q(X)\mathbb{E}|\Delta g(X^*)| < \infty$. Applying Theorem 8.1 for $k=1$, and since $P_1(j) = j - \mu$, it follows that $\mathbb{E}|P_1(X)g(X)| = \sum_{j\in\mathbb{Z}} |(j-\mu)g(j)|p(j)$ is finite. Thus, $\sum_{j>\lfloor\mu\rfloor+1} |g(j)|p(j) \le \sum_{j>\lfloor\mu\rfloor+1} |(j-\mu)g(j)|p(j) < \infty$ and $\sum_{j\le\lfloor\mu\rfloor-1} |g(j)|p(j) \le \sum_{j\le\lfloor\mu\rfloor-1} |(j-\mu)g(j)|p(j) < \infty$, completing the proof. ■

Corollary 8.3: *Let the rvs X and X_i, $i = 0, 1, \ldots, n$ be as in Proposition 6.2 and consider a function g defined on the support of X. Then:*

(a) $\Delta^n g \in L^2(\mathbb{R}, X_n) \Rightarrow \Delta^i g \in L^2(\mathbb{R}, X_i)$ *for every* $i = 0, 1, \ldots, n$;
(b) $\Delta^n g \in L^1(\mathbb{R}, X_n) \Rightarrow \Delta^i g \in L^1(\mathbb{R}, X_i)$ *for every* $i = 0, 1, \ldots, n$.

Proof: Follows immediately by an application of Lemma 8.2. ■

It is known (due to M. Riesz) that the real polynomials are dense in $L^2(\mathbb{R}, X)$ whenever the probability measure of X is determined by its moments; see [9,10]. An even simpler sufficient condition is when X has a finite moment generating function at a neighborhood of zero, that is, when there exists $t_0 > 0$ such that

$$M_X(t) = \mathbb{E}\,e^{tX} < \infty, \quad t \in (-t_0, t_0); \tag{31}$$

see [3], cf. [11].

Consider a rv X in the CO family. If the support of X is finite, then (31) holds, and obviously, the real polynomials are dense in the finite-dimensional space $L^2(\mathbb{R}, X)$; in this case, $L^2(\mathbb{R}, X) = \text{span}\{1, x, x^2, \ldots, x^M\}$, and the system of polynomials $\{\phi_k\}_{k=0}^{M}$ is an orthonormal basis of $L^2(\mathbb{R}, X)$. When X has infinite support, then there are two possibilities: If $\delta > 0$, then X does not have finite moments of any order, see Lemma 2.6, and any real polynomial of $L^2(\mathbb{R}, X)$ is of bounded degree; thus, only a finite number of orthonormal polynomials exist, and these polynomials cannot be dense in the infinite-dimensional space $L^2(\mathbb{R}, X)$. If $\delta \le 0$, then Equation (31) holds, see Section 3 or Table 1, so the real polynomials are dense in $L^2(\mathbb{R}, X)$ and the system of polynomials $\{\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of this space. From the above observations, it is natural to define the following subclass of rvs of the CO system:

$$\mathcal{X} \doteq \{X : X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma) \text{ for some } (\mu; \delta, \beta, \gamma), \text{ and } \delta \le 0 \text{ or } |S(X)| < \infty\}.$$

Remark 8.4: Let $X \in \mathcal{X}$. Then:

(a) The set of polynomials $\{\phi_k\}_{k=0}^{M}$ (M is finite or infinite) is an orthonormal basis of $L^2(\mathbb{R}, X)$. Thus, any function $g \in L^2(\mathbb{R}, X)$ can be expanded as

$$g(j) \sim \sum_{k=0}^{M} \alpha_k \phi_k(j), \tag{32}$$

where $\alpha_k = \mathbb{E}\phi_k(X)g(X)$ are the Fourier coefficients of g. The series converges in the norm of $L^2(\mathbb{R}, X)$; that is, $\mathbb{E}[g(X) - \sum_{k=0}^{M} \alpha_k\phi_k(X)]^2 = 0$ (when $M < \infty$) or $\mathbb{E}[g(X) - \sum_{k=0}^{N} \alpha_k\phi_k(X)]^2 \to 0$ as $N \to \infty$ (when $M = \infty$). Parseval's identity shows that

$$\mathsf{Var}\, g(X) = \sum_{k=1}^{M} \alpha_k^2, \quad g \in L^2(\mathbb{R}, X); \tag{33}$$

(b) For every $i = 0, 1, \ldots, M, X_i \in \mathcal{X}$ (see Proposition 6.2), and the corresponding results of (a) hold for each X_i.

One can apply i times the forward difference operator in the series (32) to get, in view of Theorem 7.6, the formal expansion

$$\Delta^i g(j) \sim \sum_{k=i}^{M} \alpha_k \Delta^i \phi_k(j) = \sum_{k=i}^{M} v_{k-i}^{(i)}(\mu; q)\alpha_k \phi_{k-i,i}(j), \tag{34}$$

where $v_{k-i}^{(i)}(\mu; q)$ and $\{\phi_{k-i,i}(j)\}_{k=i}^{M}$ are given by Equations (28) and (25), respectively. Now, if the expansion (34) was indeed correct in the $L^2(\mathbb{R}, X_i)$-sense, then the completeness of the system $\{\phi_{k,i}\}_{k=0}^{M_i}$ in $L^2(\mathbb{R}, X_i)$ would lead to the corresponding Parseval identity,

$$\frac{\mathbb{E}q^{[i]}(X)\left[\Delta^i g(X)\right]^2}{\mathbb{E}q^{[i]}(X)} = \mathbb{E}\left[\Delta^i g(X_i)\right]^2 = \sum_{k=i}^{M}\left[v_{k-i}^{(i)}(\mu; q)\right]^2 \alpha_k^2. \tag{35}$$

Finally, from Equation (28), we have $[v_{k-i}^{(i)}(\mu; q)]^2 = k!\Pi_{\delta}^{[i]}(k-1)/[(k-i)!\mathbb{E}q^{[i]}(X)]$. A combination of the last equation with (35) yields the important identity

$$\mathbb{E}q^{[i]}(X)\left[\Delta^i g(X)\right]^2 = \sum_{k=i}^{M} \frac{k!\Pi_{\delta}^{[i]}(k-1)}{(k-i)!}\alpha_k^2. \tag{36}$$

This should be correct for all g such that $\Delta^i g \in L^2(\mathbb{R}, X_i)$, provided that expansion (32) is valid. We shall show that this is indeed the case. The L^2 convergence of $\sum_{k=0}^{N} \alpha_k\phi_k(X)$ to $g(X)$ implies that

$g(X) = \sum_{k=0}^{M} \alpha_k \phi_k(X)$ with probability 1, that is, $g(j) = \sum_{k=0}^{M} \alpha_k \phi_k(j)$ for all $j \in S(X)$. Therefore, $\Delta^i g(j) = \sum_{k=0}^{M} \alpha_k \Delta^i \phi_k(j) = \sum_{k=i}^{M} v_{k-i}^{(i)}(\mu; q) \alpha_k \phi_{k-i,i}(j)$ for all $j \in S(X_i)$.

However, the same result can be derived by an alternative technique, similar to the one given in [12]. In fact, we shall show more, namely, that an initial segment of the Fourier coefficients for the ith difference of g, suggested by Equation (34), can be derived for any $X \sim \mathrm{CO}(\mu; \delta, \beta, \gamma)$ having a sufficient number of moments. This result holds even if $\delta > 0$ and $|S| = \infty$. We present this technique since Lemma 8.6 and Theorem 8.8 may be of interest on their own right.

Lemma 8.5: *Consider a non-negative sequence $\{a_i\}_{i\in\mathbb{Z}}$ and assume that there is a positive integer n such that $\sum_{i\in\mathbb{Z}} |i|^n a_i$ is finite. For each $k \in \{0, 1, \ldots, n\}$, we define the sequence $\{b_{j;k}\}_{j\in\mathbb{Z}}$ by the relation $b_{j;k} \doteq \sum_{i\geq j} [j-i]_k a_i$. Then:*

(a) *For every $k \in \{1, 2, \ldots, n\}$, $\Delta b_{j;k} = k b_{j+1;k-1}$, where the forward difference is taken with respect to the index j;*

(b) *$\Delta^r b_{j;n} = (n)_r b_{j+r;n-r}$ for each $r \in \{1, 2, \ldots, n\}$. In particular, for $r = n$,*

$$\Delta^n b_{j;n} = n! b_{j+n;0} = n! \sum_{i\geq j+n} a_i.$$

Proof: (a) $\Delta b_{j;k} = \sum_{i\geq j+1} [j+1-i]_k a_i - \sum_{i\geq j} [j-i]_k a_i = \sum_{i\geq j+1} \Delta [j-i]_k a_i - [0]_k$. Since $[0]_k = 0$ $(k > 0)$ and $\Delta [j-i]_k = k[j+1-i]_{k-1}$, the desired result follows.

(b) It follows easily by applying (a) r times inductively. ■

Lemma 8.6: *Let $X \sim \mathrm{CO}(\mu; q) = \mathrm{CO}(\mu; \delta, \beta, \gamma)$ and consider a positive integer $k \leq M$. Then, provided that $\mathbb{E}|X|^{2k-1}$ is finite,*

$$q(j)p(j)\Delta P_k(j) = -\lambda_k(\delta) \sum_{i\leq j} P_k(i)p(i) = \lambda_k(\delta) \sum_{i>j} P_k(i)p(i),$$

where $\lambda_k(\delta) \doteq k[1-(k-1)\delta]$ and P_k is the orthogonal polynomial given by Equation (18). If, in addition, $\mathbb{E}|X|^{2k}$ is finite, then for the standardized polynomial $\phi_k = [\mathbb{E}P_k^2(X)]^{-1/2} P_k$, we have

$$q(j)p(j)\Delta \phi_k(j) = -\lambda_k(\delta) \sum_{i\leq j} \phi_k(i)p(i) = \lambda_k(\delta) \sum_{i>j} \phi_k(i)p(i). \tag{37}$$

Proof: Since $(x)_n = (-1)^n[-x]_n$, applying Equation (21) (replacing j by $j-(k-1)$),

$$q^{[k]}(j-(k-1))p(j-(k-1)) = \frac{(-1)^{k-1}}{(k-1)!} \sum_{i\geq j-k+2} [j-i-k-2]_{k-1} P_k(i)p(i). \tag{38}$$

The lhs of Equation (38) can be written as $(1-2\delta)^{k-1} q_1^{[k-1]}(j-(k-1))p_1(j-(k-1))\mathbb{E}q(X)$. Applying the operator Δ^{k-1} and using Equation (24), we obtain $(-1)^{k-1}(1-2\delta)^{k-1} p_1(j) P_{k-1,1}(j)$ $\mathbb{E}q(X) = (-1)^{k-1}(1-2\delta)^{k-1} q(j)p(j)P_{k-1,1}(j)$. As in Lemma 7.5, we find that $\Delta P_k(j) = B_{k-1} P_{k-1,1}$ (j), where $B_{k-1} = \mathrm{lead}(\Delta P_k)/\mathrm{lead}(P_{k-1,1}) = k\mathrm{lead}(P_k)/\mathrm{lead}(P_{k-1,1}) = kc_k(\delta)/c_{k-1}(\delta_1) = k[1-(k-1)\delta](1-2\delta)^{k-1}$. Therefore, an application of the operator Δ^{k-1} to the lhs of Equation (38) produces the quantity $(-1)^{k-1}\lambda_k^{-1}(\delta)q(j)p(j)\Delta P_k(j)$. Applying the operator Δ^{k-1} to the rhs of Equation (38) and using Lemma 8.5, we arrive at the quantity $(-1)^{k-1}\sum_{i>j} P_k(i)p(i)$, and the result follows from the fact that the last two quantities must be equal to each other. Finally, since $\mathbb{E}P_k(X) = 0$ (because $k \geq 1$), we conclude that $(-1)^{k-1}\sum_{i>j} P_k(i)p(i) = (-1)^k \sum_{i\leq j} P_k(i)p(i)$. ■

Lemma 8.7: *Let the rvs X and X^* be as in Lemma 6.1, and assume that for some integer k with $1 \le k \le M$, $\mathbb{E}|X|^{\max\{2k,3\}} < \infty$. Then, for any function g with $\Delta g \in L^2(\mathbb{R}, X^*)$, we have the identity*

$$\mathbb{E}\phi^*_{k-1}(X^*)\Delta g(X^*) = \upsilon_{k-1}\mathbb{E}\phi_k(X)g(X), \tag{39}$$

where ϕ_k, $\phi_{k,1} \equiv \phi^*_k$ and $\upsilon_{k-1} = \upsilon_{k-1}(\mu; q)$ are as given in Equations (22), (25) *and* (26), *respectively.*

Proof: By an application of Cauchy-Schwarz inequality, we get $\mathbb{E}^2|\phi^*_{k-1}(X^*)\Delta g(X^*)| \le \mathbb{E}[\phi^*_{k-1}(X^*)]^2\mathbb{E}[\Delta g(X^*)]^2 = \mathbb{E}[\Delta g(X^*)]^2 < \infty$. From Corollary 8.3, it follows that $g \in L^2(\mathbb{R}, X)$, and similarly, $\mathbb{E}|\phi_k(X)g(X)| < \infty$. Since $\mathbb{E}\phi_k(X) = 0$, ϕ_k must change its sign in the support of X. Thus, ϕ_k has real roots, say $\rho_1 < \cdots < \rho_m$, that lie in the interval $[\alpha, \omega]$. Fix now an integer $\rho \in \{[\rho_1], \ldots, [\rho_m]\} \subset S$. Then, $\mathbb{E}q(X)\mathbb{E}\phi^*_{k-1}(X^*)\Delta g(X^*) = \sum_{j=\alpha}^{\omega-1} \Delta g(j)q(j)p(j)\phi^*_{k-1}(j) = \upsilon^{-1}_{k-1}\sum_{j=\alpha}^{\omega-1} \Delta g(j)q(j)p(j)\Delta\phi_k(j) = -\lambda_k(\delta)\upsilon^{-1}_{k-1}\sum_{j=\alpha}^{\rho-1}\Delta g(j)\sum_{i=\alpha}^{j} p(i)\phi_k(i) + \lambda_k(\delta)\upsilon^{-1}_{k-1}$ $\times \sum_{j=\rho}^{\omega-1}\Delta g(j)\sum_{i=j+1}^{\omega} p(i)\phi_k(i)$. Observing that $\lambda_k(\delta)\upsilon^{-1}_{k-1} = \upsilon_{k-1}\mathbb{E}q(X)$, the preceding equation can be rewritten as

$$\mathbb{E}\phi^*_{k-1}(X^*)\Delta g(X^*) = \upsilon_{k-1}(\Sigma_2 - \Sigma_1), \quad \text{where}$$
$$\Sigma_1 \doteq \sum_{j=\alpha}^{\rho-1}\Delta g(j)\sum_{i=\alpha}^{j} p(i)\phi_k(i), \quad \Sigma_2 \doteq \sum_{j=\rho}^{\omega-1}\Delta g(j)\sum_{i=j+1}^{\omega} p(i)\phi_k(i). \tag{40}$$

Now, we wish to change the order of summation to both sums Σ_1 and Σ_2. To this end, for Σ_2, it suffices to show that

$$\Sigma^*_2 \doteq \sum_{j=\rho}^{\omega-1}|\Delta g(j)|\sum_{i=j+1}^{\omega} p(i)|\phi_k(i)| < \infty. \tag{41}$$

Similarly, for Σ_1, it suffices to show that $\Sigma^*_1 \doteq \sum_{j=\alpha}^{\rho-1}|\Delta g(j)|\sum_{i=\alpha}^{j} p(i)|\phi_k(i)| < \infty$. Note that, obviously, if $\alpha > -\infty$, then $\Sigma^*_1 < \infty$ and if $\omega < \infty$, then $\Sigma^*_2 < \infty$. We now proceed to verify Equation (41) when $\omega = \infty$. Write $\Sigma^*_2 = \Sigma^*_{21} + \Sigma^*_{22}$, where $\Sigma^*_{21} \doteq \sum_{j=\rho}^{[\rho_m]}|\Delta g(j)|\sum_{i=j+1}^{\infty} p(i)|\phi_k(i)|$, and $\Sigma^*_{22} \doteq \sum_{j=[\rho_m]+1}^{\infty}|\Delta g(j)|\sum_{i=j+1}^{\infty} p(i)|\phi_k(i)|$. Since $\mathbb{E}|X|^k < \infty$, $\sum_{i=j+1}^{\infty} p(i)|\phi_k(i)| < \infty$ for each $j = \rho, \ldots, [\rho_m]$ and thus, $\Sigma^*_{21} < \infty$, being a finite sum of finite terms. On the other hand, since the polynomial ϕ_k does not change its sign in the set $\{[\rho_m]+1, [\rho_m]+2, \ldots\}$, we can define the constant $c \doteq \text{sign}\phi_k(j) \in \{-1, 1\}$, $j \in \{[\rho_m]+1, [\rho_m]+2, \ldots\}$. Then, $c\phi_k(j) = |\phi_k(j)|$ holds for all $j \in \{[\rho_m]+1, [\rho_m]+2, \ldots\}$ and from (37), we get $\Sigma^*_{22} = c\sum_{j=[\rho_m]+1}^{\infty}|\Delta g(j)|\sum_{i=j+1}^{\infty} p(i)$ $\phi_k(i) = c\lambda_k^{-1}(\delta)\sum_{j=[\rho_m]+1}^{\infty}|\Delta g(j)|q(j)p(j)\Delta\phi_k(j) \le \lambda_k^{-1}(\delta)\sum_{j=[\rho_m]+1}^{\infty}|\Delta g(j)|q(j)p(j)|\Delta\phi_k(j)| \le$ $\lambda_k^{-1}(\delta)\sum_{j=\alpha}^{\infty}|\Delta g(j)|q(j)p(j)|\Delta\phi_k(j)| = \upsilon_{k-1}\lambda_k^{-1}(\delta)\mathbb{E}q(X)\sum_{j=\alpha}^{\infty}|\Delta g(j)\phi^*_{k-1}(j)|p^*(j) = \upsilon^{-1}_{k-1}\mathbb{E}|\phi^*_{k-1}$ $(X^*)\Delta g(X^*)| < \infty$. Therefore, Equation (41) follows for both cases ($\omega < \infty$ or $\omega = \infty$). If $\alpha = -\infty$, using similar arguments it can be shown that $\Sigma^*_1 < \infty$. Thus, we can indeed interchange the order of summation to both sums Σ_1 and Σ_2 of Equation (40). It follows that $\Sigma_1 = \sum_{i=\alpha}^{\rho-1} p(i)\phi_k(i)\sum_{j=i}^{\rho-1}\Delta g(j) = g(\rho)\sum_{i=\alpha}^{\rho-1} p(i)\phi_k(i) - \sum_{i=\alpha}^{\rho-1} g(i)p(i)\phi_k(i)$ and $\Sigma_2 = \sum_{i=\rho+1}^{\omega} p(i)\phi_k(i)\sum_{j=\rho}^{i-1}\Delta g(j) = \sum_{i=\rho+1}^{\omega} g(i)p(i)\phi_k(i) - g(\rho)\sum_{i=\rho+1}^{\omega} p(i)\phi_k(i) = \sum_{i=\rho}^{\omega} g(i)p(i)$ $\phi_k(i) - g(\rho)\sum_{i=\rho}^{\omega} p(i)\phi_k(i)$. Taking into account the fact that $\sum_{\alpha}^{\omega} p(i)\phi_k(i) = \mathbb{E}\phi_k(X) = 0$, we get $\Sigma_2 - \Sigma_1 = \sum_{\alpha}^{\omega} g(i)p(i)\phi_k(i) - g(\rho)\sum_{\alpha}^{\omega} p(i)\phi_k(i) = \mathbb{E}\phi_k(X)g(X)$, which completes the proof of the lemma. ■

Theorem 8.8: *Let $X \sim \mathrm{CO}(\mu; q) = \mathrm{CO}(\mu; \delta, \beta, \gamma)$ and fix an integer k with $1 \le k \le M$. Assume that $\mathbb{E}|X|^{2k+1} < \infty$ and consider the rvs X_i, $i = 0, 1, \ldots, k$, as in Proposition 6.2. Then:*

(a) *The Fourier coefficients satisfy the relation*

$$\mathbb{E}\phi_{k-i,i}(X_i)\Delta^i g(X_i) = v_{k-i}^{(i)}\mathbb{E}\phi_k(X)g(X), \quad i = 0, 1, \ldots, k, \tag{42}$$

where ϕ_k, $\phi_{k,i}$ and $v_{k-i}^{(i)} = v_{k-i}^{(i)}(\mu; q)$ are as given in Equations (22), (25) and (28), respectively;

(b) *If, in addition, $X \in \mathcal{X}$ and $\Delta^n g \in L^2(\mathbb{R}, X_n)$ for some fixed integer n with $1 \le n \le M$, then Equation (36) holds for all $i = 0, 1, \ldots, n$.*

Proof: (a) By Corollary 8.3, $\Delta^i g \in L^2(\mathbb{R}, X_i)$ for all $i = 0, 1, \ldots, k$. For $i = 0$, Equation (42) is obvious and for $i = 1$, it follows from Lemma 8.7. Assume that it is true for $i - 1 \in \{0, \ldots, k-1\}$, that is, $\mathbb{E}\phi_{k-i+1,i-1}(X_{i-1})\Delta^{i-1}g(X_{i-1}) = v_{k-i+1}^{(i-1)}\mathbb{E}\phi_k(X)g(X)$. Observe that the assumptions of Lemma 8.7 are satisfied for the rv X_{i-1}, the integer $k-i+1$ and the function $\Delta^{i-1}g$. Using Equation (39), $\mathbb{E}\phi_{k-i,i}(X_i)\Delta^i g(X_i) = \mathbb{E}\phi_{k-i,i}(X_i)\Delta(\Delta^{i-1}g(X_i)) = v_{k-i}(\mu_{i-1}; q_{i-1})\mathbb{E}\phi_{k-i+1,i-1}(X_{i-1})\Delta^{i-1}g(X_{i-1})$. Thus, we get $\mathbb{E}\phi_{k-i,i}(X_i)\Delta^i g(X_i) = v_{k-i}(\mu_{i-1}; q_{i-1})v_{k-i+1}^{(i-1)}\mathbb{E}\phi_k(X)g(X)$. Finally, $v_{k-i}(\mu_{i-1}; q_{i-1}) = \{(k-i+1)[1-(k-i)\delta_{i-1}]/A_1(\mu_{i-1}; q_{i-1})\}^{1/2}$, where $A_1(\mu_{i-1}; q_{i-1}) = A_i/\{[1-2(i-1)\delta]A_{i-1}\}$ and $1-(k-i)\delta_{i-1} = [1-(k+i-2)\delta]/[1-2(i-1)\delta]$. Hence, $v_{k-i}(\mu_{i-1}; q_{i-1}) = \{(k-i+1)[1-(k+i-2)\delta]A_{i-1}/A_i\}^{1/2}$ and a straightforward calculation gives $v_{k-i}(\mu_{i-1}; q_{i-1})v_{k-i+1}^{(i-1)} = v_{k-i}^{(i)}$.

(b) Since $X \in \mathcal{X}$, we have that $X_i \in \mathcal{X}$ and the set of polynomials $\{\phi_{k,i}\}_{k=0}^{M_i}$ (where $M_i = M(X_i) = M - i$) is an orthonormal basis of $L^2(\mathbb{R}, X_i)$; see Remark 8.4(b). Moreover, $\Delta^i g \in L^2(\mathbb{R}, X_i)$. Thus, by Parseval's identity, it follows that $\mathbb{E}[\Delta^i g(X_i)]^2 = \sum_{k=0}^{M_i}\alpha_{k,i}^2 = \sum_{k=i}^{M}\alpha_{k-i,i}^2$, where $\alpha_{k,i} \doteq \mathbb{E}\phi_{k,i}(X_i)\Delta^i g(X_i)$ (with $\alpha_{k,0} = \alpha_k$) is the Fourier coefficient of $\Delta^i g$ with respect to $\phi_{k,i}$. Using Equation (42), $\alpha_{k-i,i}^2 = \mathbb{E}^2\phi_{k-i,i}(X_i)\Delta^i g(X_i) = [v_{k-i}^{(i)}]^2\mathbb{E}^2\phi_k(X)g(X) = [v_{k-i}^{(i)}]^2\alpha_k^2$, which verifies Equation (35) and the proof is complete. ■

9. Applications to variance bounds

We now use the results of Section 8 to present a wide class of variance bounds for a function g of a rv X in the CO family.

Let X be any rv in the CO family and consider two non-negative integers $m, n \le M$ such that $\mathbb{E}X^{2\ell} < \infty$, where $\ell = \max\{m, n\}$. We denote by $\mathcal{H}^{m,n}(X)$ the class of functions $g : S \to \mathbb{R}$ ($S = S(X)$ is the support of X) satisfying the restrictions

$$\mathbb{E}q^{[n]}(X)\left[\Delta^n g(X)\right]^2 < \infty \quad \text{and} \quad \mathbb{E}q^{[m]}(X)|\Delta^m g(X)| < \infty.$$

From Corollary 8.3 and the fact that $\mathbb{E}^2 q^{[i]}(X)|\Delta^i g(X)| \le \mathbb{E}q^{[i]}(X) \cdot \mathbb{E}q^{[i]}(X)[\Delta^i g(X)]^2$ for all $i = 0, 1, \ldots, n$, we conclude the following:

$$\text{If } m \le n \text{ and if } \mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2 < \infty, \text{ then } \mathbb{E}q^{[m]}(X)|\Delta^m g(X)| < \infty.$$

Note that Corollary 8.3 requires $\mathbb{E}|X|^{2\ell+1} < \infty$, but this assumption is needed only for the existence of the pmf p_ℓ; thus, for the validity of the above observation, it is sufficient that $\mathbb{E}X^{2\ell} < \infty$. It follows that $\mathcal{H}^{0,n} = \mathcal{H}^{1,n} = \cdots = \mathcal{H}^{n,n}$ [of course, $\mathcal{H}^{0,0}(X) = L^2(\mathbb{R}, X)$].

Furthermore, when $M = \infty$ and X has finite moments of any order (that is, $\delta \le 0$), we shall denote by $\mathcal{H}^{\infty,n}(X)$ and $\mathcal{H}^{\infty}(X)$ the classes $\cap_{m=0}^{\infty}\mathcal{H}^{m,n}(X) = \cap_{m=n+1}^{\infty}\mathcal{H}^{m,n}(X)$ and $\cap_{n=0}^{\infty}\mathcal{H}^{\infty,n}(X)$, respectively. That is,

$$\mathcal{H}^{\infty,n}(X) = \left\{g : \mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2 < \infty \text{ and } \mathbb{E}q^{[m]}(X)|\Delta^m g(X)| < \infty\ \forall m > n\right\},$$
$$\mathcal{H}^{\infty}(X) = \left\{g : \mathbb{E}q^{[n]}(X)[\Delta^n g(X)]^2 < \infty\ \forall n \in \mathbb{N}\right\}.$$

Note that, by definition, $\mathcal{H}^{m,\infty}(X) \doteq \cap_{n=0}^{\infty}\mathcal{H}^{m,n} \equiv \mathcal{H}^{\infty}(X)$ for arbitrary fixed m.

From Corollary 8.3, we conclude that the (finite or infinite) sequence $\mathcal{H}^{m,n}(X)$ is decreasing in both m and n. In particular, if all moments of X exist, then

$$\begin{array}{llll}
L^2(\mathbb{R}, X) \equiv \mathcal{H}^{0,0}(X) & & & \\
\quad\quad\quad\quad\quad \cup\!\mid & & & \\
\quad\quad\quad\quad \mathcal{H}^{1,0}(X) \supseteq & \mathcal{H}^{1,1}(X) & & \\
\quad\quad\quad\quad\quad \cup\!\mid & \cup\!\mid & & \\
\quad\quad\quad\quad \mathcal{H}^{2,0}(X) \supseteq & \mathcal{H}^{2,1}(X) \supseteq & \mathcal{H}^{2,2}(X) & \\
\quad\quad\quad\quad\quad \cup\!\mid & \cup\!\mid & \cup\!\mid & \\
\quad\quad\quad\quad\quad \vdots & \vdots & \vdots & \\
\quad\quad\quad\quad\quad \cup\!\mid & \cup\!\mid & \cup\!\mid & \\
\quad\quad\quad\quad \mathcal{H}^{M,0}(X) \supseteq & \mathcal{H}^{M,1}(X) \supseteq & \mathcal{H}^{M,2}(X) \supseteq \cdots \supseteq & \mathcal{H}^{M,M}(X).
\end{array}$$

Equations (29) and (36) are almost identical with those given in [13, eq's (2.3) and (2.2)], for the continuous case. Therefore, using similar arguments, the next theorem holds; cf. [13, Theorem 2.1].

Theorem 9.1: *Let $X \in \mathcal{X}$, and fix two non-negative integers m,n with $1 \le m + n \le M$. Assume that the function $g \in \mathcal{H}^{m,n}(X)$. Consider the quantity*

$$S_{m,n}(g) = \sum_{i=1}^{m} \kappa_i \mathbb{E}^2 q^{[i]}(X)\Delta^i g(X) + \sum_{i=1}^{n} (-1)^{i-1} \nu_i \mathbb{E}q^{[i]}(X)\left[\Delta^i g(X)\right]^2, \tag{43}$$

where

$$\kappa_i \doteq \frac{\binom{m}{i}\Pi_\delta^{[n]}(m+i)}{(m+n)_i \Pi_\delta^{[i]}(i-1)\Pi_\delta^{[n]}(m)\mathbb{E}q^{[i]}(X)} \quad \text{and} \quad \nu_i \doteq \frac{\binom{n}{i}}{(m+n)_i \Pi_\delta^{[i]}(m)}$$

are strictly positive constants (depending only on m,n and X), and an empty sum (when $m = 0$ or $n = 0$) should be treated as zero. Then, the following inequality holds:

$$(-1)^n \left[\mathsf{Var}\, g(X) - S_{m,n}(g)\right] \ge 0.$$

Moreover, $S_{m,n}(g)$ becomes equal to $\mathsf{Var}\, g(X)$ if and only if g is identically equal to a polynomial of degree at most $m+n$ on the support of X, that is, if and only if there exists a polynomial H_{m+n} of degree at most $m+n$ such that $\mathbb{P}[g(X) = H_{m+n}(X)] = 1$.

Proof: Let $\alpha_k = \mathbb{E}\phi_k(X)g(X)$ be the Fourier coefficients of g. From Equations (36) and (29), we get, as in [13], that $(-1)^n[\mathsf{Var} g(X) - S_{m,n}(g)] = R_{m,n}(g)$, where

$$R_{m,n}(g) = \sum_{k=m+n+1}^{M} r_{k;m,n}(\delta)\alpha_k^2 \doteq \sum_{k=m+n+1}^{M} \frac{(k-m-1)_n \Pi_\delta^{[n]}(m+k)}{(m+n)_n \Pi_\delta^{[n]}(m)}\alpha_k^2. \tag{44}$$

If $\delta \le 0$, $\Pi_\delta^{[n]}(m+k) > 0$ and $\Pi_\delta^{[n]}(m) > 0$ because $1 - j\delta \ge 1$ for all $j \in \mathbb{N}$, while if $\delta > 0$, the same follows by Remark 2.7. Therefore, the residual $R_{m,n}(g)$ in Equation (44) is non-negative, and it is equal to zero if and only if $\alpha_k = 0$ for all $k > m+n$, i.e. if and only if the function $g : S(X) \to \mathbb{R}$ is a polynomial of degree at most $m+n$. Note that if $m+n=M$ (in the case where M is finite), the sum in Equation (44) is empty and it is treated as zero. ■

Example 9.2: Suppose $X \sim \text{Poisson}(\lambda)$ and consider a function $g : \mathbb{N} \to \mathbb{R}$. Theorem 9.1 produces the inequality $(-1)^n[\mathsf{Var}g(X) - S_{m,n}(g)] \geq 0$, where

$$S_{m,n}(g) = \sum_{i=1}^{m} \frac{\lambda^i}{i!} \frac{\binom{m}{i}}{\binom{m+n}{i}} \mathbb{E}^2 \Delta^i g(X) + \sum_{i=1}^{n} (-1)^{i-1} \frac{\lambda^i}{i!} \frac{\binom{n}{i}}{\binom{m+n}{i}} \mathbb{E}\left[\Delta^i g(X)\right]^2, \quad n, m = 0, 1, \ldots, \quad n+m > 0,$$

provided $\mathbb{E}[\Delta^n g(X)]^2 < \infty$ and $\mathbb{E}|\Delta^m g(X)| < \infty$ (of course, if $m \leq n$, the second restriction is implied by the first one). The equality holds if and only if $g : \mathbb{N} \to \mathbb{R}$ is a polynomial of degree at most $n+m$. For $n=m=1$, we get Equation (6).

Remark 9.3: (a) For fixed n and for any function $g \in \mathcal{H}^{\tilde{m},n}(X)$, where $\tilde{m}$ can be finite or infinite, the variance bounds $\{S_{m,n}(g)\}_{m=0}^{\tilde{m}}$ are of the same kind, i.e. upper bounds when n is odd and lower bounds when n is even;
(b) The bounds $\{S_{m,n}(g)\}_{m=0}^{n}$ require the same condition on g, i.e. $g \in \mathcal{H}^{n,n}(X)$.

Remark 9.4: (a) When $m=0$, the bounds $S_{0,n}(g)$ are the bounds S_n given by Afendras et al. [5, Theorem 4.1, pp.179–180], see Equation (4);
(b) The results of Theorem 9.1 also apply to the special case when $n=0$ (note that the second sum is empty and is treated as zero). In this case, the lower bound $S_{m,0}(g)$ is reduced to the one given by Afendras et al. [3, Theorem 4.1, pp.518–519], see Equation (3).

Remark 9.5: Regarding the conditions of Theorem 9.1 imposed on the function g, we note that $g \in \mathcal{H}^{\max\{m,n\},n-1}(X) \setminus \mathcal{H}^{\max\{m,n\},n}(X)$ implies that the bound $S_{m,n}(g)$ is trivial, i.e. $+\infty$ when n is odd and $-\infty$ when n is even. Of course, such a g exists only when the support is infinite (with $\delta = 0$).

When $M < \infty$ and $m+n=M$, then $R_{m,n}(g) = 0$ and the variance bound $S_{m,n}(g)$ is equal to $\mathsf{Var}\, g(X)$ for any g. In any other case, it is of some interest to find an upper bound for the residual $R_{m,n}(g)$.

Proposition 9.6: *Assume the conditions of Theorem 9.1, with $m+n < M$, and, further, suppose that $g \in \mathcal{H}^{T,T}(X)$ for some $T \in \{n, \ldots, m+n+1\}$. Then, the residual $R_{m,n}(g)$, given by Equation* (44), *is bounded above by*

$$u_\tau \mathbb{E} q^{[\tau]}(X) \left(\Delta^\tau g(X)\right)^2, \quad \tau = n, n+1, \ldots, T, \tag{45}$$

where $u_\tau = u_{m,n,\tau}(X) \doteq \Pi_\delta^{[n]}(2m+n+1)/\{\binom{m+n}{n}(m+n+1)_\tau \Pi_\delta^{[n+\tau]}(m)\}$.

Proof: Using Equation (36), we write the quantity (45) in the form $\sum_{k=\tau}^{M} \pi_{k;\tau}\alpha_k^2$. Next, consider the sequence $\{w_{k;\tau} = \pi_{k;\tau}/r_{k;m,n}(\delta)\}_{k=m+n+1}^{M}$, where the numbers $r_{k;m,n}(\delta)$ are given by Equation (44), and observe that this sequence is increasing in k, with $w_{m+n+1;\tau} = 1$. ■

In general, the upper bounds (when there are at least two) of the residual $R_{m,n}(g)$, given by Equation (45), are not comparable.

Next, for n fixed, we investigate the bounds $S_{m,n}(g)$ as m increases.

Theorem 9.7: *Suppose $X \in \mathcal{X}$ and fix a positive integer n and a function $g \in \mathcal{H}^{\tilde{m},n}(X)$, where $\tilde{m}$ (with $\tilde{m} \ge n$) can be finite or infinite. Then, for each m_1, m_2 such that $0 \le m_1 < m_2 \le \min\{\tilde{m}, M\}$, the following inequality holds:*

$$\left|\mathsf{Var}\, g(X) - S_{m_1,n}(g)\right| \ge \zeta_{m_1,m_2,n}(\mu;q)\left|\mathsf{Var}\, g(X) - S_{m_2,n}(g)\right|, \tag{46}$$

where $\zeta_{m_1,m_2,n}(\mu;q) = \zeta_{m_1,m_2,n}$ is given by

$$\zeta_{m_1,m_2,n} \doteq \begin{cases} \dfrac{(m_2+n)_n(M-m_1-1)_n\Pi_\delta^{[n]}(m_2)\Pi_\delta^{[n]}(m_1+M)}{(m_1+n)_n(M-m_2-1)_n\Pi_\delta^{[n]}(m_1)\Pi_\delta^{[n]}(m_2+M)}, & |M| < \infty, \\ \dfrac{(m_2+n)_n\Pi_\delta^{[n]}(m_2)}{(m_1+n)_n\Pi_\delta^{[n]}(m_1)}, & |M| = \infty. \end{cases} \tag{47}$$

For both cases, $|M| < \infty$ and $|M| = \infty$,

$$\zeta_{m_1,m_2,n} > (m_2+n)_n \big/ (m_1+n)_n. \tag{48}$$

The equality in Equation (46) *holds if and only if the function $g : S \to \mathbb{R}$ is identically equal to a polynomial of degree at most $n + m_1$.*

Proof: Note that if $n + m_2 = M$, then $S_{m_2,n}(g) = \mathsf{Var} g(X)$ for every function g and Equation (46) holds in a trivial way. Otherwise, we consider the finite or infinite positive sequence

$$\left\{\zeta_k = \frac{r_{k;m_1,n}(\delta)}{r_{k;m_2,n}(\delta)} = \frac{(m_2+n)_n(k-m_1-1)_n\Pi_\delta^{[n]}(m_2)\Pi_\delta^{[n]}(m_1+k)}{(m_1+n)_n(k-m_2-1)_n\Pi_\delta^{[n]}(m_1)\Pi_\delta^{[n]}(m_2+k)}\right\}_{k=m_2+n+1}^{M}.$$

Claim: *The sequence $\{\zeta_k\}_{k=m_2+n+1}^{M}$ is strictly decreasing in k.*

Proof: Since $\{r_{k;m_1,n}(\delta)/r_{k+1;m_1,n}(\delta)\}/\{r_{k;m_2,n}(\delta)/r_{k+1;m_2,n}(\delta)\} = \zeta_k/\zeta_{k+1}$, $k = m_2 + n + 1, \ldots, M - 1$, it is sufficient to show that the function $h(m) = r_{k;m,n}(\delta)/r_{k+1;m,n}(\delta) = (k-m-n)[1-(m+k)\delta]/\{(k-m)[1-(m+n+k)\delta]\}$, $0 \le m \le M-n-1$, is strictly decreasing. After some algebra, $h'(m) = -n[1-(2m+n)\delta](1-2k\delta)/\{(k-m)^2[1-(m+n+k)\delta]^2\}$. If $\delta \le 0$, then it is obvious that $h'(m) < 0$; if $\delta > 0$, then it is necessary that $M < \infty$ and, using Remark 2.7, again it follows that $h'(m) < 0$ and the claim is proved. ■

If $M < \infty$, then the Claim shows that $\min_{k\in\{m_2+n+1,\ldots,M\}}\{\zeta_k\} = \zeta_M = \zeta_{m_1,m_2,n}$. If $M = \infty$, then observe that

$$\zeta_k \searrow \frac{(m_2+n)_n\Pi_\delta^{[n]}(m_2)}{(m_1+n)_n\Pi_\delta^{[n]}(m_1)} = \zeta_{m_1,m_2,n}, \quad \text{as } k \to \infty. \tag{49}$$

Moreover, observing that $r_{k;m_1,n}(\delta) > 0$ and $r_{k;m_2,n}(\delta) = 0$ for all $k = n + m_1 + 1, \ldots, n + m_2$, (46) follows.

If $\delta = 0$ and $M = \infty$, then (48) is obvious. For $\delta \le 0$ and $M < \infty$, we observe that $\zeta_M > (m_2+n)_n\Pi_\delta^{[n]}(m_2)\big/[(m_1+n)_n\Pi_\delta^{[n]}(m_1)]$, see (49), and (48) follows. Now, assume $\delta > 0$ ($M < \infty$). Since $\Pi_\delta^{[n+M-k]}(m_1+k) > \Pi_\delta^{[n+M-k]}(m_2+k) > 0$, it is sufficient to show that $(M-m_1-1)_n\Pi_\delta^{[n]}(m_2) \ge (M-m_2-1)_n\Pi_\delta^{[n]}(m_1) > 0$. Observing that $(M-m_1-1)_n\Pi_\delta^{[n]}(m_2)/\{(M-m_2-1)_n\Pi_\delta^{[n]}(m_1)\} = \prod_{j=0}^{n-1}((M-n+j-m_1)_n[1-(m_2+j)\delta]/(M-n+j-m_2)_n[1-(m_1+j)\delta])$, and putting $\eta_j \mapsto M-n+j$ and $\xi_j \mapsto 1-j\delta$, it is sufficient to show that $[(\eta_j - m_1)(\xi_j - m_2)]/[(\eta_j - m_2)(\xi_j - m_1)] > 1$ for all $j = 0, \ldots, n-1$. This is equivalent to $\xi_j - \eta_j\delta >$

0, that is, $\delta < (M-n+2j)^{-1}$ for all $j = 0, \ldots, n-1$. Observe that for each $j = 0, \ldots, n-1$, $(M-n+2j)^{-1} \geq [M-n+2(n-1)]^{-1} = (M+n-2)^{-1} \geq (2M-2)^{-1} = [2(|S|-2)]^{-1} > \delta$; see Remark 2.7. Thus, Equation (47) holds in any case. Finally, writing $|\mathsf{Var}\, g(X) - S_{m_1,n}(g)| - \zeta_{m_1,m_2,n}|\mathsf{Var} g(X) - S_{m_2,n}(g)| = \sum_{k=n+m_1+1}^{M} \theta_k \alpha_k^2$, we observe that $\theta_k > 0$ for all k. Thus, the equality in Equation (46) holds if and only if g is identified with a polynomial of degree at most $n + m_1$. ■

Remark 9.8: Assume the conditions of Theorem 9.7.

(a) In view of Remark 9.3(a), the bounds $\{S_{m,n}(g)\}_{m=0}^{\tilde{m}}$ are of the same kind. From Equation (46), it follows that the bound $S_{m_2,n}(g)$ is better than the bound $S_{m_1,n}(g)$. Thus, writing $n=2r$ (when n is even) and $n=2r+1$ (when n is odd), we have

$$S_{0,2r}(g) \leq S_{1,2r}(g) \leq \cdots \leq \mathsf{Var}\, g(X) \leq \cdots \leq S_{1,2r+1}(g) \leq S_{0,2r+1}(g);$$

(b) For the case $\tilde{m} = M = \infty$, from Equations (33), (43) and (a), it follows that

$$\underset{\text{[when } n \text{ is even]}}{S_{m,n}(g) \nearrow \mathsf{Var}\, g(X)} \quad \text{or} \quad \underset{\text{[when } n \text{ is odd]}}{S_{m,n}(g) \searrow \mathsf{Var}\, g(X),} \quad \text{as} \quad m \to \infty.$$

Now, we compare the existing variance bound $S_{0,n}(g)$, see Remark 9.4(a), with the best proposed bound shown in this section, requiring the same conditions on g, i.e. with the bound $S_{n,n}(g)$, see Remark 9.3(b).

Corollary 9.9: *The variance bounds $S_{n,n}(g)$ and $S_{0,n}(g)$ are of the same kind and require the same assumptions on g. Moreover, the new bound $S_{n,n}(g)$ is better than the existing (see Remark 9.4) bound $S_{0,n}(g)$. Specifically,*

$$\left|\mathsf{Var}\, g(X) - S_{0,n}(g)\right| \geq \zeta_{0,n,n} \left|\mathsf{Var}\, g(X) - S_{n,n}(g)\right|,$$

with $\zeta_{0,n,n} > \binom{2n}{n}$. The equality holds only in the trivial case when $\mathsf{Var}\, g(X) = S_{n,n}(g) = S_{0,n}(g)$, i.e. the function $g : S \to \mathbb{R}$ is identified with a polynomial of degree at most n.

Remark 9.10: Assume that $X_1, \ldots, X_\nu$ is a random sample from the geometric distribution with parameter $\theta \in (0, 1)$, i.e. with pmf $p(j) = \theta(1-\theta)^j$, $j = 0, 1, \ldots$, and let $X = X_1 + \cdots + X_\nu$ be the complete sufficient statistic. The uniformly minimum variance unbiased estimator of $-\log(\theta)$ is $T_\nu = T_\nu(X) = \sum_{j=\nu}^{\nu+X-1} 1/j$. Variance bounds of the kind of Theorem 9.1 have been used for constructing bounds of $\mathsf{Var}(T_\nu)$; see [5, Section 5] and [3, Application 5.1]. In the similar and easy manner, we can use the results of Theorems 9.1 and 9.7 in regard to the approximation of $\mathsf{Var}(T_\nu)$ and its accuracy.

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