Self-Inverse and Exchangeable Random Variables*

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Abstract

A random variable Z will be called self-inverse if it has the same distribution as its reciprocal Z^{-1}. It is shown that if Z is defined as a ratio, X/Y, of two rv's X and Y (with $\mathbb{P}[X=0]=\mathbb{P}[Y=0]=0$), then Z is self-inverse if and only if X and Y are (or can be chosen to be) exchangeable. In general, however, there may not exist iid X and Y in the ratio representation of Z.

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1 Introduction

The definition of a self-inverse random variable (rv) is motivated by the observation that several known classical distributions are defined as the ratio of two independent and identically distributed (iid) rv's X and Y, continuous as a rule, so that $\mathbb{P}[X=0]=\mathbb{P}[Y=0]=0$. Clearly, in this case Z is self-inverse, that is,

$$Z=\frac{X}{Y}\stackrel{\mathrm{d}}{=}\frac{Y}{X}=Z^{-1}, \tag{1}$$

where $X_1\stackrel{\mathrm{d}}{=}X_2$ denotes that X_1 and X_2 have the same distribution.

A classical example of a self-inverse rv Z is the Cauchy with density

$$f_Z(z)=\frac{1}{\pi}\,\frac{1}{1+z^2},\quad z\in\mathbb{R}, \tag{2}$$

since Z is defined as the ratio of two iid $N(0,\sigma^2)$ rv's. The usual symmetry of Z, $Z\stackrel{\mathrm{d}}{=}-Z$, is also obvious in (2).

It may be added that not only such ratios of iid $N(0,\sigma^2)$ rv's have the Cauchy density; Laha (1958) showed that if X and Y are iid rv's with common density

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$f(x) = \sqrt{2}(1+x^4)^{-1}/\pi$, their ratio also follows (2). In fact, interestingly enough, Jones (2008a) showed that the ratios X/Y for all centered elliptically symmetrically distributed random vectors (X, Y) follow a general (relocated, $\mu \neq 0$, and rescaled, $\sigma \neq 1$) Cauchy, $C(\mu, \sigma)$. Such is the well-known case of a bivariate normal (X, Y) with $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$; the ratio $Z = X/Y$ has the Cauchy density

$$f_Z(z) = \frac{1}{\pi} \frac{\sqrt{1-\rho^2}}{1+z^2-2\rho z} = \frac{1}{\pi} \frac{\sqrt{1-\rho^2}}{(1-\rho^2)+(z-\rho)^2}, \quad z \in \mathbb{R}, \tag{3}$$

with $\mu = \rho$, the correlation coefficient, and scale parameter $\sigma = \sqrt{1-\rho^2}$.

Arnold and Brockett (1992) showed that any random scale mixture of elliptically symmetric random vectors has a general Cauchy-type ratio (from any bivariate subvectors). Along the same lines, we add the very interesting article of Jones (1999), who used simple trigonometric formulas and polar coordinates to obtain Cauchy-distributed functions of spherically symmetrically distributed random vectors (X, Y).

In the present note we are not concerned with Cauchy-distributed ratios X/Y, known to be self-inverse, but with the question of when a random variable Z has the same distribution as its reciprocal Z^{-1}, and of whether it is representable as a ratio X/Y. Seshadri (1965) considered the problem for a continuous rv $Z > 0$, and characterized the density $f_Z(z)$ of Z in terms of the density $f_W(w)$ of $W = \log Z$: f_W should be symmetric about the origin. This coincides with what Jones (2008b) refers to as "log-symmetry" about $\theta > 0$:

$$Z/\theta \stackrel{\mathrm{d}}{=} \theta/Z;$$

cf. the so called "R-symmetry", introduced by Mudholkar and Wang (2007). Thus, our "self-inverse" symmetry for $Z > 0$ coincides with log-symmetry about $\theta = 1$. Moreover, Seshadri (1965) showed that if X and Y are iid, then $Z = X/Y$ is self-inverse; he also pointed out that the ratio decomposition of Z into iid X and Y is not always possible. As already stated, we show below (Propositions 1 and 2) that the ratio representation of any self-inverse Z is always possible in terms of two exchangeable rv's X and Y. Also, two simple examples, showing that X and Y cannot always be chosen to be iid rv's, are given at the end of Section 3.

2 Examples of identically distributed rv's whose ratio is not self-inverse

Ratios X/Y leading to (2) or the F-distributed $F_{n,n} \stackrel{\mathrm{d}}{=} F_{n,n}^{-1}$, where X and Y are iid, are clearly self-inverse. In (3), however, X and Y have the same distribution, but are not independent. One may, therefore, wrongly conclude that equidistribution of X and Y is a sufficient condition for the ratio to be self-inverse. This is not so, as shown by the following two examples, one discrete and one continuous. That Z in (3) is self-inverse is due to the fact that X and Y, though not iid, are exchangeable. In Section 3, below, we show that the exchangeability of X and Y is all we need to characterize a ratio X/Y as self-inverse.

(a) Discrete (X,Y). The following table gives $f(x, y) = \mathbb{P}[X = x, Y = y]$:

$$
\begin{array}{|ll|ccc|c|}
\hline
 & y & 1 & 2 & 3 & \\
x & & & & & f_X(x) \\
\hline
1 & & 2/36 & 9/36 & 1/36 & 1/3 \\
2 & & 1/36 & 2/36 & 9/36 & 1/3 \\
3 & & 9/36 & 1/36 & 2/36 & 1/3 \\
\hline
f_Y(y) & & 1/3 & 1/3 & 1/3 & 1 \\
\hline
\end{array}
\tag{4}
$$

Clearly, $X \stackrel{\mathrm{d}}{=} Y$, with $X \sim U(\{1, 2, 3\})$, uniform on $\{1, 2, 3\}$. Yet, (1) does not hold, since, for example,

$$\mathbb{P}\left[\frac{X}{Y} = 2\right] = \frac{1}{36} \neq \mathbb{P}\left[\frac{Y}{X} = 2\right] = \frac{9}{36}.$$

(b) Continuous (X,Y). Let U_1, U_2 iid $U(0, 1)$, i.e., uniform on $(0, 1)$, and I uniform on $\{0, 1, 2\}$, independent of (U_1, U_2). Define $J = I + 1$ if $I = 0$ or $I = 1$, and $J = 0$ if $I = 2$, so that $J \sim U(\{0, 1, 2\})$, that is, $J \stackrel{\mathrm{d}}{=} I$. Observing that (I, J) and (U_1, U_2) are independent, and defining

$$(X, Y) = (I + U_1, J + U_2),$$

we have $X \stackrel{d}{=} Y \sim U(0, 3)$, uniform on $(0, 3)$. The joint density $f(x, y)$ of X and Y is

$$f(x, y) = \begin{cases} 1/3, & \text{if } x \in (0, 1) \text{ and } y \in (1, 2), \\ 1/3, & \text{if } x \in (1, 2) \text{ and } y \in (2, 3), \\ 1/3, & \text{if } x \in (2, 3) \text{ and } y \in (0, 1), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Yet, the ratios X/Y and Y/X do not have the same distribution ((1) does not hold), since, e.g.,

$$\mathbb{P}\left[\frac{X}{Y} \le 1\right] = \frac{2}{3}, \quad \mathbb{P}\left[\frac{Y}{X} \le 1\right] = \frac{1}{3}.$$

In this example too, though $X \stackrel{\mathrm{d}}{=} Y$, in fact $U(0, 3)$, again (1) does not hold and $Z = X/Y$ is not self-inverse.

In both examples of (4) and (5), X and Y have the same distribution, but they are not exchangeable, and (1) fails. However, if X and Y are iid they are also exchangeable, since $F_X = F_Y$ and by independence,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) = F_Y(x)F_X(y) = F_{Y,X}(x, y). \tag{6}$$

Such were the cases of (2) and $F_{n,n}$, and (1) holds; this also holds in (3) where X and Y are exchangeable, i.e., $F_{X,Y} = F_{Y,X}$.

3 Representation of a self-inverse random variable as a ratio

We have seen that if X and Y are not exchangeable, (1) may not hold, that is, the ratio $Z = X/Y$ may not be self-inverse. Here it will be shown that Z is self-inverse if and only if it can be defined, or represented, as a ratio of two exchangeable rv's X and Y.

First we show

Proposition 1. Let Z be defined as a ratio of two exchangeable rv's X and Y, i.e.

$$Z = \frac{X}{Y}, \quad \text{where } \ (X,Y) \stackrel{\mathrm{d}}{=} (Y,X) \ \text{ and } \ \mathbb{P}[X=0] = \mathbb{P}[Y=0] = 0. \tag{7}$$

Then Z is self-inverse, that is,

$$Z = \frac{X}{Y} \stackrel{\mathrm{d}}{=} \frac{Y}{X} = Z^{-1}. \tag{8}$$

Proof: In the continuous case where (X,Y) has a density $f_{X,Y}(x,y)$ we may use the elementary formula for the density of $Z = X/Y$:

$$f_Z(z) = \int_0^\infty y f_{X,Y}(yz,y)dy - \int_{-\infty}^0 y f_{X,Y}(yz,y)dy. \tag{9}$$

But X and Y are exchangeable, hence $f_{X,Y} = f_{Y,X}$, and (9) can be written as

$$f_Z(z) = \int_0^\infty y f_{Y,X}(yz,y)dy - \int_{-\infty}^0 y f_{Y,X}(yz,y)dy, \tag{10}$$

whose right hand side is the density of $Y/X = Z^{-1}$. Hence, (7) $\Rightarrow$ (8).

In the general case, (8) is implied by the fact that if X and Y are exchangeable, then, for any (Borel) function $g : \mathbb{R}^2 \to \mathbb{R}$, we have

$$g(X,Y) \stackrel{\mathrm{d}}{=} g(Y,X). \tag{11}$$

Hence, taking $g(x,y) = x/y$ (with the convention $g(x,y) = 0$ if $xy = 0$), (7) implies (8). $\Box$

We are now going to show that, roughly speaking, (8) implies (7), or more accurately:

Proposition 2. If $Z \stackrel{\mathrm{d}}{=} Z^{-1}$, there are exchangeable rv's X and Y (with $\mathbb{P}[X=0] = \mathbb{P}[Y=0] = 0$) such that Z can be written as

$$Z \stackrel{\mathrm{d}}{=} \frac{X}{Y}. \tag{12}$$

Proof: Consider the pair

$$(X,Y) = (WZ^I, WZ^{1-I}) = (W[(1-I)+IZ], W[I+(1-I)Z]), \tag{13}$$

where I denotes the symmetric Bernoulli, with $\mathbb{P}[I=0]=\mathbb{P}[I=1]=\frac{1}{2}$, W any rv with $\mathbb{P}[W=0]=0$, e.g., $W\equiv 1$ or $W\sim N(\mu,\sigma^2)$, and Z,I,W are independent. It can be shown (cf. (11), (6)) that

$$(X,Y) \stackrel{\mathrm{d}}{=} (Y,X) \quad \text{and, obviously,} \quad \frac{X}{Y} = \frac{Z^I}{Z^{1-I}}.$$

Hence, for any z we have

$$\begin{aligned}\mathbb{P}\left[\frac{X}{Y}\le z\right] &= \frac{1}{2}\mathbb{P}\left[\frac{Z^I}{Z^{1-I}}\le z \middle| I=1\right] + \frac{1}{2}\mathbb{P}\left[\frac{Z^I}{Z^{1-I}}\le z \middle| I=0\right]\\ &= \frac{1}{2}\mathbb{P}[Z\le z] + \frac{1}{2}\mathbb{P}\left[\frac{1}{Z}\le z\right] = \mathbb{P}[Z\le z],\end{aligned}$$

since, by hypothesis, $Z \stackrel{\mathrm{d}}{=} Z^{-1}$. Hence,

$$Z \stackrel{\mathrm{d}}{=} \frac{X}{Y}, \quad \text{with } (X,Y) \stackrel{\mathrm{d}}{=} (Y,X) \text{ as defined by (13).}$$

□

Another question is whether there exist not simply exchangeable rv's X and Y as in (12), but iid X,Y so that every self-inverse Z can be written as in (12). The answer is negative, as shown by the following counterexample:

Let $Z>0$ with $\log Z \sim U(-1,1)$ and suppose there are iid rv's X,Y such that Z can be written as in (12). Then it would follow that

$$\log Z = U \stackrel{\mathrm{d}}{=} X_1 - X_2 \quad \text{with} \quad X_1 = \log|X|,\ X_2 = \log|Y|. \tag{14}$$

Moreover, since X and Y are iid, the X_1,X_2 will also be iid, in which case, if φ is the characteristic function of X_1,X_2, we have

$$\varphi_{X_1-X_2}(t) = \varphi(t)\varphi(-t) = \varphi(t)\overline{\varphi(t)} = |\varphi(t)|^2 \ge 0, \tag{15}$$

whereas the characteristic function of U is $\varphi_U(t) = (\sin t)/t$, taking both positive and negative values.

An analogous (simpler) counterexample is the following: Let $Z>0$ with $\log Z = U$, U the Bernoulli $\mathbb{P}[U=-1]=\mathbb{P}[U=1]=\frac{1}{2}$. Then, similarly as in (14) and (15),

$$\varphi_{X_1-X_2}(t) = |\varphi(t)|^2 \quad \text{whereas } \varphi_U(t) = \cos t.$$

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