Self-inverse and exchangeable random variables

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ABSTRACT

A random variable \( Z \) will be called self-inverse if it has the same distribution as its reciprocal \( Z^{-1} \). It is shown that if \( Z \) is defined as a ratio, \( X/Y \), of two rv's \( X \) and \( Y \) (with \( P[X = 0] = P[Y = 0] = 0 \)), then \( Z \) is self-inverse if and only if \( X \) and \( Y \) are (or can be chosen to be) exchangeable. In general, however, there may not exist iid \( X \) and \( Y \) in the ratio representation of \( Z \).

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1. Introduction

The definition of a self-inverse random variable (rv) is motivated by the observation that several known classical distributions are defined as the ratio of two independent and identically distributed (iid) rv's \( X \) and \( Y \), continuous as a rule, so that \( P[X = 0] = P[Y = 0] = 0 \). Clearly, in this case \( Z \) is self-inverse, that is,

\[
Z = \frac{X}{Y} \overset{d}{=} \frac{Y}{X} = Z^{-1},
\]

where \( X_1 \overset{d}{=} X_2 \) denotes that \( X_1 \) and \( X_2 \) have the same distribution.

A classical example of a self-inverse rv \( Z \) is the Cauchy with density

\[
f_Z(z) = \frac{1}{\pi} \frac{1}{1 + z^2}, \quad z \in \mathbb{R},
\]

since \( Z \) is defined as the ratio of two iid \( N(0, \sigma^2) \) rv's. The usual symmetry of \( Z, Z \overset{d}{=} -Z \), is also obvious in (2).

It may be added that not only do such ratios of iid \( N(0, \sigma^2) \) rv's have the Cauchy density; Laha (1958) showed that if \( X \) and \( Y \) are iid rv's with common density \( f(x) = \sqrt{2}(1 + x^4)^{-1}/\pi \), their ratio also follows (2). In fact, interestingly enough, Jones (2008a) showed that the ratios \( X/Y \) for all centered elliptically symmetrically distributed random vectors \( (X, Y) \) follow a
general (relocated, \( \mu \neq 0 \), and rescaled, \( \sigma \neq 1 \)) Cauchy, \( C(\mu, \sigma) \). Such is the well-known case of a bivariate normal \((X, Y)\) with \( X \sim N(0, \sigma^2) \) and \( Y \sim N(0, \sigma^2) \); the ratio \( Z = X/Y \) has the Cauchy density

\[
f_Z(z) = \frac{1}{\pi} \frac{\sqrt{1 - \rho^2}}{1 + z^2 - 2\rho z} = \frac{1}{\pi} \frac{\sqrt{1 - \rho^2}}{(1 - \rho^2) + (z - \rho)^2}, \quad z \in \mathbb{R},
\]

with \( \mu = \rho \), the correlation coefficient, and scale parameter \( \sigma = \sqrt{1 - \rho^2} \).

Arnold and Brockett (1992) showed that any random scale mixture of elliptically symmetric random vectors has a general Cauchy-type ratio from any bivariate subvectors. Along the same lines, we add the very interesting article of Jones (1999), who used simple trigonometric formulas and polar coordinates to obtain Cauchy-distributed functions of spherically symmetrically distributed random vectors \((X, Y)\).

In the present note we are not concerned with Cauchy-distributed ratios \( X/Y \), known to be self-inverse, but with the question of when a random variable \( Z \) has the same distribution as its reciprocal \( Z^{-1} \), and of whether it is representable as a ratio \( X/Y \). Seshadri (1965) considered the problem for a continuous \( \text{rv} \) \( Z > 0 \), and characterized the density \( f_Z(z) \) of \( Z \) in terms of the density \( f_W(w) \) of \( W = \log Z \); \( f_W \) should be symmetric about the origin. This coincides with what Jones (2008b) refers to as “log-symmetry” about \( \theta > 0 \):

\[
Z/\theta \overset{d}{=} \theta/Z;
\]

cf. the so-called “\( k \)-symmetry”, introduced by Mudholkar and Wang (2007). Thus, our “self-inverse” symmetry for \( Z > 0 \) coincides with log-symmetry about \( \theta = 1 \). Moreover, Seshadri (1965) showed that if \( X \) and \( Y \) are iid, then \( Z = X/Y \) is self-inverse; he also pointed out that the ratio decomposition of \( Z \) into iid \( X \) and \( Y \) is not always possible. As already stated, we show below (Propositions 1 and 2) that the ratio representation of any self-inverse \( Z \) is always possible in terms of two exchangeable \( \text{rv} \)'s \( X \) and \( Y \). Also, two simple examples, showing that \( X \) and \( Y \) cannot always be chosen to be iid \( \text{rv} \)'s, are given at the end of Section 3.

2. Examples of identically distributed \( \text{rv} \)'s whose ratio is not self-inverse

Ratios \( X/Y \) leading to (2) or the \( F \)-distributed \( F_{n,n} \overset{d}{=} F_{n,n}^{-1} \), where \( X \) and \( Y \) are iid, are clearly self-inverse. In (3), however, \( X \) and \( Y \) have the same distribution, but are not independent. One may, therefore, wrongly conclude that equidistribution of \( X \) and \( Y \) is a sufficient condition for the ratio to be self-inverse. This is not so, as shown by the following two examples, one discrete and one continuous. That \( Z \) in (3) is self-inverse is due to the fact that \( X \) and \( Y \), though not iid, are exchangeable. In Section 3, below, we show that the exchangeability of \( X \) and \( Y \) is all we need to characterize a ratio \( X/Y \) as self-inverse.

(a) Discrete \((X, Y)\). The following table gives \( f(x, y) = \mathbb{P}[X = x, Y = y] \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( f_X(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2/36</td>
<td>9/36</td>
<td>1/36</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1/36</td>
<td>2/36</td>
<td>9/36</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9/36</td>
<td>1/36</td>
<td>2/36</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>( f_Y(y) )</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Clearly, \( X \overset{d}{=} Y \), with \( X \sim U(\{1, 2, 3\}) \), uniform on \( \{1, 2, 3\} \). Yet, (1) does not hold, since, for example,

\[
\mathbb{P}\left[ \frac{X}{Y} = 2 \right] = \frac{1}{36} \neq \mathbb{P}\left[ \frac{Y}{X} = 2 \right] = \frac{9}{36}.
\]

(b) Continuous \((X, Y)\). Let \( U_1, U_2 \) iid \( U(0, 1) \), i.e., uniform on \((0, 1)\), and \( I \) uniform on \( \{0, 1, 2\} \), independent of \((U_1, U_2) \). Define \( J = I + 1 \) if \( I = 0 \) or \( I = 1 \), and \( J = 0 \) if \( I = 2 \), so that \( J \sim U(\{0, 1, 2\}) \), that is, \( J \overset{d}{=} I \). Observing that \((I, J)\) and \((U_1, U_2)\) are independent, and defining

\[
(X, Y) = (I + U_1, J + U_2),
\]

we have \( X \overset{d}{=} Y \sim U(0, 3) \), uniform on \((0, 3)\). The joint density \( f(x, y) \) of \( X \) and \( Y \) is

\[
f(x, y) = \begin{cases} 
1/3, & \text{if } x \in (0, 1) \text{ and } y \in (1, 2), \\
1/3, & \text{if } x \in (1, 2) \text{ and } y \in (2, 3), \\
1/3, & \text{if } x \in (2, 3) \text{ and } y \in (0, 1), \\
0, & \text{otherwise.}
\end{cases}
\]

Yet, the ratios \( X/Y \) and \( Y/X \) do not have the same distribution (1 does not hold), since, e.g.,

\[
\mathbb{P}\left[ \frac{X}{Y} \leq 1 \right] = \frac{2}{3}, \quad \mathbb{P}\left[ \frac{Y}{X} \leq 1 \right] = \frac{1}{3}.
\]

In this example too, though \( X \overset{d}{=} Y \), in fact \( U(0, 3) \), again (1) does not hold and \( Z = X/Y \) is not self-inverse.
In both examples of (4) and (5), $X$ and $Y$ have the same distribution, but they are not exchangeable, and (1) fails. However, if $X$ and $Y$ are iid they are also exchangeable, since $F_X = F_Y$ and by independence,

$$
F_{X,Y}(x, y) = F_X(x)F_Y(y) = F_Y(x)F_X(y) = F_{Y,X}(x, y).
$$

(6)

Such were the cases of (2) and $F_{n,n}$, and (1) holds; this also holds in (3) where $X$ and $Y$ are exchangeable, i.e., $F_{X,Y} = F_{Y,X}$.

3. Representation of a self-inverse random variable as a ratio

We have seen that if $X$ and $Y$ are not exchangeable, (1) may not hold, that is, the ratio $Z = X/Y$ may not be self-inverse. Here it will be shown that $Z$ is self-inverse if and only if it can be defined, or represented, as a ratio of two exchangeable rv's $X$ and $Y$.

First we show

**Proposition 1.** Let $Z$ be defined as a ratio of two exchangeable rv's $X$ and $Y$, i.e.

$$
Z = \frac{X}{Y}, \text{ where } (X, Y) \overset{d}{=} (Y, X) \quad \text{and} \quad P[X = 0] = P[Y = 0] = 0.
$$

(7)

Then $Z$ is self-inverse, that is,

$$
Z = \frac{X}{Y} = \frac{Y}{X} = Z^{-1}.
$$

(8)

**Proof.** In the continuous case where $(X, Y)$ has a density $f_{X,Y}(x, y)$ we may use the elementary formula for the density of $Z = X/Y$:

$$
f_Z(z) = \int_0^\infty yf_{X,Y}(yz, y)dy - \int_0^z yf_{X,Y}(yz, y)dy.
$$

(9)

But $X$ and $Y$ are exchangeable, hence $f_{X,Y} = f_{Y,X}$, and (9) can be written as

$$
f_Z(z) = \int_0^\infty yf_{X,Y}(yz, y)dy - \int_0^0 yf_{Y,X}(yz, y)dy.
$$

(10)

whose right hand side is the density of $Y/X = Z^{-1}$. Hence, (7) $\Rightarrow$ (8).

In the general case, (8) is implied by the fact that if $X$ and $Y$ are exchangeable, then, for any (Borel) function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have

$$
g(X, Y) \overset{d}{=} g(Y, X).
$$

(11)

Hence, taking $g(x, y) = x/y$ (with the convention $g(x, y) = 0$ if $xy = 0$), (7) implies (8). □

We are now going to show that, roughly speaking, (8) implies (7), or more accurately:

**Proposition 2.** If $Z \overset{d}{=} Z^{-1}$, there are exchangeable rv's $X$ and $Y$ (with $P[X = 0] = P[Y = 0] = 0$) such that $Z$ can be written as

$$
Z \overset{d}{=} \frac{X}{Y}.
$$

(12)

**Proof.** Consider the pair

$$(X, Y) = (WZ^i, WZ^{1-i}) = (W[1 - I + iZ], W[I + (1 - i)Z]),
$$

(13)

where $I$ denotes the symmetric Bernoulli, with $P[I = 0] = P[I = 1] = \frac{1}{2}$, $W$ any rv with $P[W = 0] = 0$, e.g., $W \equiv 1$ or $W \sim N(\mu, \sigma^2)$, and $Z, I, W$ are independent. It can be shown (cf. (11), (6)) that

$$(X, Y) \overset{d}{=} (Y, X) \quad \text{and, obviously,} \quad X \overset{d}{=} \frac{Z^i}{Z^{1-i}}.
$$

Hence, for any $z$ we have

$$
P \left[ \frac{X}{Y} \leq z \right] = \frac{1}{2} P \left[ \frac{Z^i}{Z^{1-i}} \leq z \mid I = 1 \right] + \frac{1}{2} P \left[ \frac{Z^i}{Z^{1-i}} \leq z \mid I = 0 \right]
$$

$$
= \frac{1}{2} P[Z \leq z] + \frac{1}{2} P \left[ \frac{1}{Z} \leq z \right] = P[Z \leq z].
$$
since, by hypothesis, $Z \overset{d}{=} Z^{-1}$. Hence,

$$Z \overset{d}{=} \frac{X}{Y}, \quad \text{with } (X, Y) \overset{d}{=} (Y, X) \text{ as defined by (13).}$$

Another question is whether there exist not simply exchangeable rv’s $X$ and $Y$ as in (12), but iid $X$, $Y$ so that every self-inverse $Z$ can be written as in (12). The answer is negative, as shown by the following counterexample:

Let $Z > 0$ with $\log Z \sim U(-1, 1)$ and suppose there are iid rv’s $X$, $Y$ such that $Z$ can be written as in (12). Then it would follow that

$$\log Z = U \overset{d}{=} X_1 - X_2 \quad \text{with } X_1 = \log |X|, \quad X_2 = \log |Y|. \quad (14)$$

Moreover, since $X$ and $Y$ are iid, the $X_1$, $X_2$ will also be iid, in which case, if $\varphi$ is the characteristic function of $X_1$, $X_2$, we have

$$\varphi_{X_1 - X_2}(t) = \varphi(t)\varphi(-t) = \varphi(t)\overline{\varphi(t)} = |\varphi(t)|^2 \geq 0, \quad (15)$$

whereas the characteristic function of $U$ is $\varphi_U(t) = \left(\sin t\right)/t$, taking both positive and negative values.

An analogous (simpler) counterexample is the following: Let $Z > 0$ with $\log Z = U$, $U$ the Bernoulli $\mathbb{P}[U = -1] = \mathbb{P}[U = 1] = \frac{1}{2}$. Then, similarly as in (14) and (15),

$$\varphi_{X_1 - X_2}(t) = |\varphi(t)|^2 \quad \text{whereas } \varphi_U(t) = \cos t.$$

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References


