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Intermediate order statistics with applications to nonparametric estimation

N. Papadatos

Department of Mathematics, University of Athens, 15784 Panepistemiopolis, Athens, Greece

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Abstract

A generalization of order statistics, is presented. Using this generalization, nonparametric confidence intervals are constructed for the quantiles of an absolutely continuous distribution. Finally, an application, concerning confidence intervals for the unique median, is given.

Keywords: Intermediate order statistics; Nonparametric confidence intervals; Quantiles; Median

1. Introduction

Suppose $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ are the order statistics corresponding to independent, identically distributed (iid) random variables (rv's) with an absolutely continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$.

Let δ be the unique median of F (i.e. $F(\delta) = \frac{1}{2}$). It is well known that the interval $[X_{(i)}, X_{(n+1-i)}]$ (for any $i = 1, \dots, [n/2]$) is a nonparametric (distribution free) confidence interval for δ. The confidence coefficient for this interval is

$$\gamma_n(i) = \frac{1}{2^n} \sum_{k=i}^{n-i} \binom{n}{k},$$

and it is obvious that the binomial probability $\gamma_n(i)$ takes values from a discrete subset of $[0, 1]$ only (for example, if $n = 5$, $\gamma_n(i) \in \{0.625, 0.9375\}$).

In order to form confidence intervals for δ with a fixed but arbitrary confidence coefficient $\gamma \in [0, 1 - 1/2^{n-1}]$, we have to interpolate between adjacent order statistics of the initial sample. Hettmansperger and Sheather (1986) studied this problem, but their intervals are no longer distribution free. We also refer to Noether (1973).

In this paper we define the intermediate order statistics, a class of rv's $X_{(a)}$, $a \in [1, n]$, having the properties of order statistics, and we construct confidence intervals for δ of the form $[X_{(a)}, X_{(n+1-a)}]$. It is shown that these intervals are nonparametric, so that $X_{(a)}$ is a generalization of $X_{(i)}$ when $i \notin \mathbb{N}$. Finally, an application to the nonparametric estimation of the unique median is given, using simulation with several df's F.

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2. Definitions of intermediate order statistics

Before giving the main definition, we restate some important relations satisfied by the order statistics from an absolutely continuous df F (these properties can be found, for example, in Balakrishnan and Cohen (1991) and Arnold et al. (1992)).

Property 2.1. Let $1 \leqslant s < j < k \leqslant n$ be integers and $x < z$. Then, the conditional pdf of $X_{(j)}$ given $X_{(s)} = x$, $X_{(k)} = z$, is

$$f_{j\mid s,k}(y \mid x,z) = \frac{f(y)}{B(j-s,k-j)} \frac{(F(y)-F(x))^{j-s-1}(F(z)-F(y))^{k-j-1}}{(F(z)-F(x))^{k-s-1}} \quad \text{for } y \in (x,z). \tag{2.1}$$

Property 2.2. Let $1 \leqslant s_1 < s_2 < \cdots < s_r < s < j < k < k_1 < \cdots < k_t \leqslant n$ be integers and $x_1 < x_2 < \cdots < x_r < x < z < z_1 < \cdots < z_t$. Then the conditional pdf of $X_{(j)}$ given

$$X_{(s_1)} = x_1, \ldots, X_{(s_r)} = x_r, \quad X_{(s)} = x, \quad X_{(k)} = z, \quad X_{(k_1)} = z_1, \ldots, X_{(k_t)} = z_t$$

is given by (2.1).

Motivated by these properties, we give the following

Definition 2.1 (*Intermediate order statistics*). Consider r real numbers $a_1, a_2, \ldots, a_r \in (1,n)\setminus\mathbb{N}$ such that $a_i \neq a_j$ for $i \neq j$, $i,j = 1,2,\ldots,r$. Suppose $x_1 < x_2 < \cdots < x_n$ are the observed values of the order sample $X_{(1)}, \ldots, X_{(n)}$ drawn from F. We define the intermediate order statistics, $X_{(a_1)}, \ldots, X_{(a_r)}$, say, of order $a_1, a_2, \ldots, a_r$, respectively, in a sequential way.

(i) Set $i = [a_1]$ (the integer part of a_1). Then we define $X_{(a_1)}$ to be a rv having conditional pdf

$$\begin{aligned} &f_{X_{(a_1)}\mid X_{(1)},\ldots,X_{(n)}}(y_1 \mid x_1, \ldots, x_n) \\ &\quad = \frac{f(y_1)}{B(a_1 - i, i+1-a_1)} (F(y_1) - F(x_i))^{a_1 - i - 1} (F(x_{i+1}) - F(y_1))^{i - a_1}, \quad y_1 \in (x_i, x_{i+1}). \end{aligned} \tag{2.2}$$

Relation (2.2) specifies simply the pdf of $X_{(a_1)}$, without, of course, giving any actual (sample) information about its possible value y_1. We may regard y_1 as an observation on $X_{(a_1)}$, so that we interpret the value y_1 as the "observed" value of $X_{(a_1)}$.

(ii) Suppose we have defined the rv's $X_{(a_1)}, \ldots, X_{(a_s)}$ and we have "observed" their values $y_1, \ldots, y_s$ (in the preceding sense) for some $s \leqslant r-1$. Set

$$\beta = \max\{t \in \{1, \ldots, n, a_1, \ldots, a_s\}: t < a_{s+1}\}, \qquad \gamma = \min\{t \in \{1, \ldots, n, a_1, \ldots, a_s\}: t > a_{s+1}\}.$$

Obviously, $\beta < a_{s+1} < \gamma$ and $\beta, \gamma \in \{1, 2, \ldots, n\} \cup \{a_1, \ldots, a_s\}$. Let x, z be the "observed" values of the rv's $X_{(\beta)}, X_{(\gamma)}$, respectively. It is clear that $x < z$ and $x, z \in \{x_1, \ldots, x_n, y_1, \ldots, y_s\}$. We define the conditional pdf of $X_{(a_{s+1})}$ with respect to $X_{(1)}, \ldots, X_{(n)}$, $X_{(a_1)}, \ldots, X_{(a_s)}$ by the relation

$$\begin{aligned} &f_{X_{(a_{s+1})}\mid X_{(1)},\ldots,X_{(n)},X_{(a_1)},\ldots,X_{(a_s)}}(y_{s+1} \mid x_1, \ldots, x_n, y_1, \ldots, y_s) \\ &\quad = \frac{f(y_{s+1})}{B(a_{s+1} - \beta, \gamma - a_{s+1})} \frac{[F(y_{s+1}) - F(x)]^{a_{s+1} - \beta - 1} [F(z) - F(y_{s+1})]^{\gamma - a_{s+1} - 1}}{[F(z) - F(x)]^{\gamma - \beta - 1}}, \quad y_{s+1} \in (x,z). \end{aligned} \tag{2.3}$$

We assume now that an observation y_{s+1} of a rv, having pdf as in (2.3), is taken, and we continue the recurrence step (ii).

From Definition 2.1 it is clear that for every pair of real numbers $a, b \in [1, n]$ with $a < b$, we have $P[X_{(a)} < X_{(b)}] = 1$. Hence, the rv's

$$X_{(1)}, \ldots, X_{(n)}, X_{(a_1)}, \ldots, X_{(a_r)}$$

are in ascending order of magnitude. Furthermore, the Properties 2.1 and 2.2 continue to hold. It is also clear that the joint pdf of $X_{(a_1)}, \ldots, X_{(a_r)}$ is well defined from Definition 2.1, since it is the same for every rearrangement of $\{a_1, \ldots, a_r\}$. Therefore, similar to the usual order statistics, the following properties hold for the intermediate order statistics (the proofs are no different, so they are omitted (see, for example, David, 1981)).

Let $1 \leqslant a_1 < \cdots < a_r \leqslant n$ be any real numbers. Then,

$$f_{X_{(a_1)}, \ldots, X_{(a_r)}}(y_1, \ldots, y_r) = \frac{1}{B(a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, n + 1 - a_r)} \cdot \left[\prod_{j=1}^{r} f(y_j) \right] \prod_{j=0}^{r} [F(y_{j+1}) - F(y_j)]^{a_{j+1} - a_j - 1},$$
$$-\infty = y_0 < y_1 < \cdots < y_r < y_{r+1} = +\infty, \tag{2.4}$$

where $a_0 = 0$, $a_{r+1} = n + 1$, $B(u_1, \ldots, u_k) = \Gamma(u_1) \cdots \Gamma(u_k) / \Gamma(u_1 + \cdots + u_k)$.

The special case $r = 1$ leads to ($a \in [1, n]$)

$$f_{X_{(a)}}(y) = \frac{f(y)}{B(a, n + 1 - a)} F(y)^{a-1} [1 - F(y)]^{n-a}, \quad y \in \mathbb{R}. \tag{2.5}$$

Although, in practical applications order statistics from continuous distributions are more useful, for theoretical work, order statistics from any distribution, play an important role.

It is well known that any df $F(x)$, has an inverse, namely,

$$F^{-1}(u) = \inf\{x \colon F(x) \geqslant u\}, \quad 0 < u < 1. \tag{2.6}$$

Let $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ be order statistics corresponding to iid rv's drawn from F. It is well known that $F^{-1}(U)$ has the df $F(x)$, where U is uniform on (0, 1). Hence, $F^{-1}(U_{(1)}) \leqslant F^{-1}(U_{(2)}) \leqslant \cdots \leqslant F^{1}(U_{(n)})$ is an order sample from F, whenever $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ is an order sample from U(0, 1). It follows that for any integers $1 \leqslant s_1 < \cdots < s_k \leqslant n$,

$$(X_{(s_1)}, \ldots, X_{(s_k)})' \stackrel{\mathrm{d}}{=} (F^{-1}(U_{(s_1)}), \ldots, F^{-1}(U_{(s_k)}))' \tag{2.7}$$

($X \stackrel{\mathrm{d}}{=} Y$ means: X has the same distribution as Y). From (2.4) we see that (2.7) continues to hold for noninteger indices, provided F is absolutely continuous. We extend this result to any df by the definition given below.

Definition 2.2. Let F be any df with inverse F^{-1} as in (2.6). Let $1 \leqslant a_1 < \cdots < a_r \leqslant n$ be any real numbers and consider the random vector $(U_{(a_1)}, \ldots, U_{(a_r)})'$ having pdf

$$f(u_1, \ldots, u_r) = \frac{u_1^{a_1 - 1} (u_2 - u_1)^{a_2 - a_1 - 1} \cdots (u_r - u_{r-1})^{a_r - a_{r-1} - 1} (1 - u_r)^{n - a_r}}{B(a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, n + 1 - a_r)}, \quad 0 < u_1 < \cdots < u_r < 1 \tag{2.8}$$

(that is, according to Definition 2.1 and in view of (2.4), $U_{(a_1)}, \ldots, U_{(a_r)}$ are the order statistics (intermediate or not) from U(0, 1), of order $a_1, \ldots, a_r$, respectively). We define $X_{(a_1)}, \ldots, X_{(a_r)}$ by the relation

$$(X_{(a_1)}, \ldots, X_{(a_r)})' \stackrel{\mathrm{d}}{=} (F^{-1}(U_{(a_1)}), \ldots, F^{-1}(U_{(a_r)}))'$$

to be the order statistics corresponding to n iid variates drawn from F.

It is clear that Definitions 2.1 and 2.2 are consistent, whenever F is an absolutely continuous df. However, the first one is more useful for statistical purposes, since it can be used to construct order statistics based on observed values of a sample, while the second one is more general.

3. Nonparametric properties of intermediate order statistics

Return now to the absolutely continuous case, and suppose F has an unique p-quantile ξ_p (that is $x = \xi_p$ is the unique solution of $F(x) = p$).

Theorem 3.1. *For* $a \in [1, n]$,

$$P[X_{(a)} \leqslant \xi_p] = I_p(a, n+1-a), \tag{3.1}$$

where $I_p(a,b)$ *denotes, as usual, the incomplete beta function*

$$I_p(a,b) = \frac{1}{B(a,b)} \int_0^p u^{a-1}(1-u)^{b-1}\,\mathrm{d}u, \quad 0 \leqslant p \leqslant 1.$$

Proof. We obtain (3.1) integrating (2.5) over $(-\infty, \xi_p]$ and putting $F(y) = u$. □

It follows that $P[X_{(a)} \leqslant \xi_p]$ does not depend on F. Furthermore it is easy to see that $I_p(a, n+1-a)$ is a strictly decreasing function of $a \in [1, n]$, since it has a derivative

$$\frac{\mathrm{d}}{\mathrm{d}a} I_p(a, n+1-a) = -\frac{\int_0^p \int_p^1 (xy)^{a-1}[(1-x)(1-y)]^{n-a} \log\left(\dfrac{y(1-x)}{x(1-y)}\right)\mathrm{d}y\,\mathrm{d}x}{B^2(a, n+1-a)} < 0.$$

Thus, we conclude the following corollary.

Corollary 3.1. *For n and p fixed, $P[X_{(a)} \leqslant \xi_p]$ decreases in a continuous fashion from $1-(1-p)^n$ to p^n as a varies from 1 to n.*

Theorem 3.2. *Let* $a \in [1, (n+1)/2]$. *The interval*

$$[X_{(a)}, X_{(n+1-a)}]$$

is a symmetric nonparametric confidence interval for the unique median $\delta = \xi_{1/2}$ of F, having confidence coefficient $\gamma_n(a) = 2I_{1/2}(a, n+1-a) - 1$. Furthermore, $\gamma_n(a)$ is a continuous, strictly decreasing function of a, taking every value in $[0, 1 - 1/2^{n-1}]$ as a varies from 1 to $(n+1)/2$.

Proof. By applying (3.1) with $p = \frac{1}{2}$, we have

$$\gamma_n(a) = I_{1/2}(a, n+1-a) - I_{1/2}(n+1-a, a) = 2I_{1/2}(a, n+1-a) - 1,$$

thus $\gamma_n(a)$ is strictly decreasing and continuous because so does $I_{1/2}(a, n+1-a)$. Finally, we observe that $\gamma_n((n+1)/2) = 0$ and $\gamma_n(1) = 1 - 1/2^{n-1}$ which completes the proof. □

Corollary 3.2. *Let $a_n(\gamma)$ be the inverse function of $\gamma_n(a)$, as in Theorem 3.2. The interval*

$$[X_{(a_n(\gamma))}, X_{(n+1-a_n(\gamma))}]$$

is a γ-nonparametric confidence interval for the unique median δ of F, for any $\gamma \in [0, 1 - 1/2^{n-1}]$.

Table 1
$a_n(\gamma)$-values satisfying $P[X_{(a_n(\gamma))} \leqslant \delta \leqslant X_{(n+1-a_n(\gamma))}] = \gamma$

γ_n	3	4	5	10	20
0.50	1.40	1.81	2.24	4.43	8.99
0.60	1.26	1.65	2.05	4.16	8.62
0.70	1.09	1.46	1.84	3.86	8.18
0.80		1.23	1.58	3.48	7.64
0.90			1.21	2.93	6.85
0.95				2.47	6.17

We recall here that values for $a_n(\gamma)$ can be evaluated from Pearson and Hartley (1970, Table 16), only when $a = k$ or $a = k + 1/2$, $k \in \mathbb{N}$. So, for the application of Section 4, we give a short table (Table 1) of $a_n(\gamma)$-values. For instance, if $n = 5$ and $\gamma = 0.9$, we find from Table 1, $a_5(0.9) = 1.21$, hence, according to Corollary 3.2, the 0.9-confidence interval for δ is

$$[X_{(1.21)}, X_{(4.79)}],$$

where $X_{(1.21)}$, $X_{(4.79)}$ are the intermediate order statistics, as defined by Definition 2.1.

4. An application for the median

Although, the intervals described by Corollary 3.2 are distribution free, we cannot construct them, because F is unknown. Hence, in order to apply the previous results, we have to approximate the df of $X_{(a)}$.

To do this, we denote by $B_{a,b}$ ($a > 0$, $b > 0$) a rv having df $I_x(a,b)$, $0 < x < 1$. We observe that (2.2) of Definition 2.1 can be written in the equivalent form ($i = [a_1]$)

$$X_{(a_1)} \mid X_{(1)} = x_1, \ldots, X_{(n)} = x_n \overset{\mathrm{d}}{=} F^{-1}[F(x_i) + (F(x_{i+1}) - F(x_i)) \cdot B_{a_1 - i, i+1-a_1}] \tag{4.1}$$

and similarly (2.3) has an analogous expression.

It follows that an estimator $\hat{F}$ of F can be used in (4.1) to give (approximately) the conditional pdf of $X_{(a_1)}$ and also of $X_{(a_2)}, \ldots, X_{(a_r)}$. But this estimator must have an inverse over the range $[X_{(1)}, X_{(n)}]$. For this reason,

$$F_n(x) = \#\{X\text{'s} \leqslant x\}/n$$

is not a proper estimator.

A reasonable choice of $\hat{F}$ is given below.

Definition 4.1. Let $x_1 < x_2 < \cdots < x_n$ be the ordered observations of a sample drawn from F. Then, we define

$$G_n(x) = \begin{cases} 0, & \text{if } x \leqslant c_0, \\ 1, & \text{if } x \geqslant c_n, \\ [x - (i+1)c_i + i \cdot c_{i+1}] \cdot [n(c_{i+1} - c_i)]^{-1} & \text{if } x \in [c_i, c_{i+1}],\ i = 0, 1, \ldots, n-1, \end{cases} \tag{4.2}$$

where $c_0 = x_1 - \frac{1}{2}(x_2 - x_1) < x_1$, $c_n = x_n + \frac{1}{2}(x_n - x_{n-1}) > x_n$ and

$$c_i = \tfrac{1}{2}(x_i + x_{i+1}), \quad i = 1, 2, \ldots, n-1.$$

It is obvious that $G_n(x)$ is an absolutely continuous df, having an inverse over (c_0, c_n), namely,

$$G_n^{-1}(u) = n \cdot u \cdot (c_{i+1} - c_i) + (i+1)c_i - i \cdot c_{i+1} \tag{4.3}$$

for $u \in [i/n, (i+1)/n] \setminus \{0, 1\}$, $i = 0, 1, \ldots, n-1$.

For $G_n(x)$, $\sup_x |G_n(x) - F(x)| \to 0$, with probability one, as $n \to \infty$, which is a consequence of the analogous property for $F_n(x)$ and the relation $\sup_x |F_n(x) - G_n(x)| \leqslant 1/n$.

Now (4.1) leads to

$$\hat{X}_{(a_1)} \stackrel{d}{=} G_n^{-1}[G_n(x_i) + (G_n(x_{i+1}) - G_n(x_i)) \cdot B_{a_1 - i, i+1-a_1}], \tag{4.4}$$

where $i = [a_1]$ and G_n, G_n^{-1} are defined by (4.2) and (4.3), respectively.

The RHS of (4.4) is distribution free, hence, it produces an estimator of $X_{(a_1)}$. It is easy to generate a variate $B_{a_1 - i, i+1-a_1}$ from a random number generator (for instance, see Atkinson and Pearce, 1976). Substituting the observed value of this beta variate in (4.4), we obtain an "observed" value of $\hat{X}_{(a_1)}$, say $\hat{y}_1$, which is always a number in the interval (x_i, x_{i+1}).

Using the same arguments, and assuming that the previous estimate is sharp, that is $\hat{X}_{(a_1)} \stackrel{d}{=} X_{(a_1)}$, we can continue this procedure for the rv's $X_{(a_2)}, \ldots, X_{(a_r)}$, according to step (ii) of Definition 2.1 (note that the assumption $\hat{X}_{(a_1)} \stackrel{d}{=} X_{(a_1)}$ is not satisfied, but this statement does not influence the estimation of $X_{(a_2)}, \ldots, X_{(a_r)}$, provided $[a_i] \neq [a_j]$ for $i \neq j$).

Next, we apply this method to 10 000 samples, for $n = 3, 4, 5, 10, 20$, $\gamma = 0.5(0.1)0.9(0.05)0.95$ and several df's F. We construct the interval

$$[\hat{X}_{(a_n(\gamma))}, \hat{X}_{(n+1-a_n(\gamma))}]$$

by the method stated above, based on each sample, where $a_n(\gamma)$ is described in Corollary 3.2. Tables 2 (a)–(f) give the total number of such intervals including δ. This number must be close to $10000 \cdot \gamma$.

Of course, these intervals are not exact (since F is unknown) and they only achieve the confidence level approximately. Note that one can form exact confidence intervals by the usual randomization method, namely,

$$[(1-Z)X_{(i)} + ZX_{(i+1)}, (1-Z)X_{(n+1-i)} + ZX_{(n-i)}],$$

where Z is a (independent of the X's) Bernoulli 0–1 rv with $p = P\{Z = 1\} = p(a)$. It follows that p, i, γ, n and a are related through the equalities $a = a_n(\gamma)$, $i = [a]$, $p = [I_{1/2}(i, n+1-i) - I_{1/2}(a, n+1-a)]/[I_{1/2}(i, n+1-i) - I_{1/2}(i+1, n-i)]$.

Hence, the mean length of the "randomized" intervals is

$$\mu_1 = (1-p)g(i) + pg(i+1), \quad i = [a], \ p = p(a),$$

while, the mean length of the "intermediate" intervals is simply

$$\mu_2 = g(a),$$

where $g(t) = E[X_{(n+1-t)} - X_{(t)}]$, $1 \leqslant t \leqslant (n+1)/2$.

Therefore, in Table 2, we also give for comparison, the ratio $\mu_1/\hat{\mu}_2$ (the number in parentheses), where $\hat{\mu}_2$ is the average length of the 10 000 "intermediate" intervals.

We observe that this method can be used to practical situations, since the error did not exceed 0.03 in any case. Furthermore, the intervals constructed by this method, are, in general, shorter than the "randomized" intervals. Note that Hettmansperger and Sheather's intervals are also shorter than the "randomized" intervals, but it seems that their behaviour is more parametric (in the sense that the mean error has a stronger dependence on the underlying distribution).

In order to improve the results, we have to use in (4.4) a stronger empirical df. Such a df is found by Gilat and Hill (1992).

Table 2
Total number of intervals (from 10 000) including δ (the number in parentheses is the ratio $\mu_1/\hat{\mu}_2$)

	n				
γ	3	4	5	10	20
(a) Cauchy $f(x) = [\pi(1+x^2)]^{-1}$ $x \in \mathbb{R}$, $\delta = 0$, $X \overset{d}{=} \tan(\pi(U-0.5))$					
0.50	5242 (+∞)	5220 (+∞)	4724 (1.138)	4895 (1.065)	5074 (1.014)
0.60	6135 (+∞)	6243 (+∞)	5985 (1.007)	5880 (1.009)	6029 (1.027)
0.70	7014 (+∞)	7153 (+∞)	7104 (+∞)	7021 (1.060)	7027 (1.019)
0.80		7965 (+∞)	8076 (+∞)	7957 (1.086)	7977 (1.035)
0.90			8983 (+∞)	8982 (1.070)	8934 (1.024)
0.95				9475 (1.182)	9492 (1.018)
(b) Normal $f(x) = [2\pi e^{x^2}]^{-1/2}$ $x \in \mathbb{R}$, $\delta = 0$, $X \overset{d}{=} \sqrt{-2\log U_1}\sin(2\pi U_2)$					
0.50	4907 (1.140)	4933 (1.123)	4858 (1.070)	4888 (1.046)	4915 (1.001)
0.60	5858 (1.111)	5991 (1.170)	5898 (1.011)	5944 (1.025)	5999 (1.023)
0.70	7035 (1.032)	6879 (1.132)	6973 (1.101)	6946 (1.036)	7037 (1.013)
0.80		7904 (1.086)	7937 (1.148)	8004 (1.054)	7959 (1.021)
0.90			8913 (1.075)	8970 (1.018)	9019 (1.018)
0.95				9418 (1.066)	9471 (1.019)
(c) Exponential $f(x) = e^{-x}$ $x > 0$, $\delta = \log 2$, $X \overset{d}{=} -\log U$					
0.50	4912 (1.158)	4926 (1.132)	4709 (1.063)	5017 (1.055)	5005 (1.003)
0.60	5920 (1.113)	5889 (1.178)	6029 (1.016)	6002 (1.017)	6018 (1.030)
0.70	6962 (1.048)	6900 (1.155)	6980 (1.112)	6991 (1.028)	6901 (1.013)
0.80		7882 (1.080)	7926 (1.156)	7952 (1.056)	8008 (1.029)
0.90			8948 (1.078)	9071 (1.027)	8966 (1.022)
0.95				9473 (1.077)	9491 (1.017)
(d) Uniform $f(x) = 1$ $x \in (0,1)$, $\delta = 0.5$, $X \overset{d}{=} U$					
0.50	4854 (1.141)	4929 (1.090)	4927 (1.049)	4927 (1.048)	5001 (1.001)
0.60	5841 (1.101)	5885 (1.141)	5934 (1.015)	6046 (1.019)	5923 (1.027)
0.70	6990 (1.041)	6918 (1.119)	6966 (1.073)	7025 (1.018)	6972 (1.011)
0.80		7889 (1.075)	7861 (1.113)	7926 (1.041)	7995 (1.021)
0.90			8963 (1.060)	9024 (1.008)	8922 (1.011)
0.95				9422 (1.051)	9488 (1.013)
(e) Logistic $f(x) = e^x(1+e^x)^{-2}$ $x \in \mathbb{R}$, $\delta = 0$, $X \overset{d}{=} \log[U/(1-U)]$					
0.50	4897 (1.130)	5068 (1.121)	4814 (1.073)	4993 (1.051)	4962 (1.001)
0.60	5814 (1.109)	5925 (1.190)	5962 (1.026)	5951 (1.016)	5969 (1.028)
0.70	7029 (1.035)	6864 (1.134)	6905 (1.119)	7009 (1.036)	7037 (1.008)
0.80		7960 (1.086)	7979 (1.156)	7943 (1.052)	8008 (1.030)
0.90			8964 (1.076)	9024 (1.025)	9035 (1.012)
0.95				9479 (1.078)	9477 (1.018)
(f) $f(x) = \begin{cases} 0.5 & x \in (0,1) \\ (2x^2)^{-1} & x > 1 \end{cases}$, $\delta = 1$, $X \overset{d}{=} U_1/U_2$					
0.50	5152 (+∞)	5076 (+∞)	4803 (1.176)	4924 (1.058)	4986 (1.006)
0.60	6043 (+∞)	6141 (+∞)	6035 (1.020)	5913 (1.026)	5991 (1.029)
0.70	7011 (+∞)	7007 (+∞)	7117 (+∞)	6980 (1.063)	6896 (1.018)
0.80		7939 (+∞)	8002 (+∞)	7957 (1.089)	8012 (1.041)
0.90			8981 (+∞)	9013 (1.063)	9021 (1.029)
0.95				9443 (1.163)	9514 (1.036)

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