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MULTIVARIATE COVARIANCE IDENTITIES WITH AN APPLICATION TO ORDER STATISTICS

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SUMMARY. General mutlivariate covariance identities are obtained, and some applications are given. The Siegel's identity for the Normal is extended to other distributions. Some characterization results are also provided and, furtermore, upper and lower variance bounds for linear estimators based on normal order statistics are obtained.

1. Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_p)'$ be a multivariate normal random vector with arbitrary mean and covariance structure, provided that the variables are distinct with probability 1. In this case, Siegel (1993) established the covariance identity

$$\mathrm{Cov}\,[X_1, \min(X_1, \ldots, X_p)] = \sum_{i=1}^{p} \mathrm{Cov}\,[X_1, X_i]\, \mathbb{P}\,[X_i = \min(X_1, \ldots, X_p)]. \tag{1.1}$$

Liu (1994), using a multivariate version of Stein's identity, obtained a generalization of Siegel's formula to other order statistics as well as other distributions. Other covariance identities and/or variance inequalities similar to that of Stein can be found, for example, in Hudson (1978), Morris (1982), Anderson (1993), Rinott and Samuel-Cahn (1994) and Vitale (1996), among others.

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Furthermore, Wang, Sarkar and Bai (1996) extended the above formulae for the covariance matrix between a random vector and its ordered components to a more general multivariate setup (inclunding a multivariate t distribution).

In this note, some general multivariate covariance identities are obtained. By using them, we extend the results of Wang et al (1996) in a wider class of distributions. Some characterizations of distributions through these identities are also provided, and an application to order statistics is presented.

2. Multidimensional Covariance Identities

Let $\boldsymbol{X} = (X_1, \ldots, X_p)'$ be a random vector belonging to the continuous p-dimensional exponential family, with density of the form

$$f(\boldsymbol{x}) = C(\boldsymbol{\theta}) \exp\{\boldsymbol{\theta}'\boldsymbol{x} - k(\boldsymbol{x})\} I(\boldsymbol{x} \in E), \ \boldsymbol{x} = (x_1, \ldots, x_p)' \in \mathbb{R}^p, \tag{2.1}$$

where $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_p)$ is the natural parameter set of the family, E is a finite union of open connected sets in $\mathbb{R}^p$, and $k(\boldsymbol{x})$ has continuous partial derivatives $k_j(\boldsymbol{x}) = \partial k(\boldsymbol{x})/\partial x_j$, $j = 1, \ldots, p$. The distribution of $\boldsymbol{X}$ is said to belong to the family $\mathcal{F}_0$ if, in addition to (2.1), $f(\boldsymbol{x})$ tends to zero monotonically as $\boldsymbol{x}$ approaches any boundary point of E along the coordinate axes.

Let $\boldsymbol{X} \in \mathcal{F}_0$ and assume that $g : \mathbb{R}^p \to \mathbb{R}$ is an indefinite integral of $g_j = \partial g/\partial x_j$ for all $j = 1, \ldots, p$. If, in addition, the function g satisfies the moment conditions

$$\mathbb{E}|g_j(\boldsymbol{X})| < \infty, \quad \mathbb{E}|(k_j(\boldsymbol{X}) - \theta_j)g(\boldsymbol{X})| < \infty, \quad \text{for } \ j = 1, \ldots, p, \tag{2.2}$$

then Chou (1988) established the identity

$$\mathbb{E}[(k_j(\boldsymbol{X}) - \theta_j)g(\boldsymbol{X})] = \mathbb{E}[g_j(\boldsymbol{X})], \quad \text{for } \ j = 1, \ldots, p. \tag{2.3}$$

Applying (2.3) to the identity function $g(\boldsymbol{X}) \equiv 1$, it follows that $\mathbb{E}[k_j(\boldsymbol{X})] = \theta_j$. Therefore, for arbitrary g with $\nabla g = (g_1, \ldots, g_p)'$ satisfying (2.2), (2.3) can be rewritten in a matrix form as

$$\mathrm{Cov}[\nabla k(\boldsymbol{X}), g(\boldsymbol{X})] = \mathbb{E}[\nabla g(\boldsymbol{X})], \tag{2.4}$$

where $\nabla k = (k_1, \ldots, k_p)'$.

In a different context, Brascamp and Lieb (1976) gave an upper bound for the variance of $g(\boldsymbol{X})$ when the density f of $\boldsymbol{X}$ is log-concave, namely

$$\mathrm{Var}[g(\boldsymbol{X})] \le \mathbb{E}\left[(\nabla g(\boldsymbol{X}))' \left(-\frac{\partial^2 \log f(\boldsymbol{X})}{\partial X_i \partial X_j}\right)^{-1} \nabla g(\boldsymbol{X})\right]. \tag{2.5}$$

This inequality, however, is closely related to the corresponding covariance identity, given in the following

LEMMA 2.1 *If the random vector $\boldsymbol{X}$ has a log-concave density f, then*

$$\text{Cov}\,[\nabla(-\log f(\boldsymbol{X})), g(\boldsymbol{X})] = \mathbb{E}[\nabla g(\boldsymbol{X})] \tag{2.6}$$

for any function $g(\boldsymbol{x})$ such that

$$\mathbb{E}|\nabla g(\boldsymbol{X})| < \infty \quad \textit{and} \quad \mathbb{E}\left|\left(\frac{\partial \log f(\boldsymbol{X})}{\partial X_j}\right) g(\boldsymbol{X})\right| < \infty \quad \textit{for} \quad j = 1, \ldots, p.$$

PROOF. Integrating by parts and using Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^p} -\frac{f_j(\boldsymbol{x})}{f(\boldsymbol{x})} g(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} \\
&\quad = \int_{\mathbb{R}^{p-1}} \left[\int_{\mathbb{R}} -f_j(\boldsymbol{x}) g(\boldsymbol{x}) dx_j\right] dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_p \\
&\quad = \int_{\mathbb{R}^{p-1}} \left[\int_{-\infty}^{+\infty} f(\boldsymbol{x}) g_j(\boldsymbol{x}) dx_j\right] dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_p \\
&\quad = \int_{\mathbb{R}^p} f(\boldsymbol{x}) g_j(\boldsymbol{x}) d\boldsymbol{x},
\end{aligned}$$

which implies the desired result. □

In the special case where $\boldsymbol{X}$ is a multivariate normal with $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and positive definite dispersion matrix $\boldsymbol{\Sigma}$, both (2.4) and (2.6) yield the known identity

$$\text{Cov}[\boldsymbol{X}, g(\boldsymbol{X})] = \boldsymbol{\Sigma}\,\mathbb{E}[\nabla g(\boldsymbol{X})],$$

which is the multivariate version of Stein's identity (cf. Cacoullos and Papathanasiou, 1992). We denote by $\mathcal{F}_1$ the family of random vectors with log-concave densities satisfying identity (2.6).

Let us now present the above identities in a unified form. Let $\boldsymbol{X}$ be a random vector with density $f(\boldsymbol{x})$, where each X_j, $j = 1, \ldots, p$, is supported by an interval (a_j, b_j), where the endpoints a_j anb b_j may be infinite and depend on $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$; in other words, the support of $\boldsymbol{X}$ is assumed to be a convex open subset C_p of $\mathbb{R}^p$. For each component $h^j(\boldsymbol{x})$ of the function $\boldsymbol{h}(\boldsymbol{x}) = (h^1(\boldsymbol{x}), \ldots, h^p(\boldsymbol{x}))' : C_p \to \mathbb{R}^p$ with $\mathbb{E}\|\boldsymbol{h}(\boldsymbol{X})\| = \sum_{j=1}^p \mathbb{E}|h^j(\boldsymbol{X})| < \infty$, define the function $z^j(\boldsymbol{x})$ by the relation

$$z^j(\boldsymbol{x}) f(\boldsymbol{x}) = \int_{a_j}^{x_j} (\mathbb{E}[h^j(\boldsymbol{X})] - h^j(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j)) f(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j) dt_j, \quad \boldsymbol{x} \in C_p,$$

where $\boldsymbol{u}_j = (x_1, \ldots, x_{j-1})'$, $\boldsymbol{v}_j = (x_{j+1}, \ldots, x_p)'$, $j = 1, \ldots, p$, and set $\boldsymbol{z} : C_p \to \mathbb{R}^p$ with $\boldsymbol{z}(\boldsymbol{x}) = (z^1(\boldsymbol{x}), \ldots, z^p(\boldsymbol{x}))'$. Moreover, assume that $z^j(\boldsymbol{x})f(\boldsymbol{x}) \to 0$ as x_j tends to the endpoints a_j, b_j. Then we have the following

THEOREM 2.1 *Under the above conditions, if $g(\boldsymbol{x})$ is an indefinite integral of its partial derivatives $g_j(\boldsymbol{x}) = \partial g(\boldsymbol{x})/\partial x_j$, $j = 1, \ldots, p$, then*

$$\mathrm{Cov}[h^j(\boldsymbol{X}), g(\boldsymbol{X})] = \mathbb{E}[z^j(\boldsymbol{X}) g_j(\boldsymbol{X})], \quad j = 1, \ldots, p, \tag{2.7}$$

provided that

$$\mathbb{E}|(h^j(\boldsymbol{X}) - \mathbb{E}[h^j(\boldsymbol{X})])g(\boldsymbol{X})| < \infty, \ \mathbb{E}|z^j(\boldsymbol{X}) g_j(\boldsymbol{X})| < \infty, \ j = 1, \ldots, p.$$

PROOF. The proof can be easily obtained by observing that

$$-\frac{\partial}{\partial x_j}(z^j(\boldsymbol{x})f(\boldsymbol{x})) = (h^j(\boldsymbol{x}) - \mathbb{E}[h^j(\boldsymbol{X})])f(\boldsymbol{x}),$$

which leads to

$$\begin{aligned}
&\mathrm{Cov}[h^j(\boldsymbol{X}), g(\boldsymbol{X})] \\
&\quad = \mathbb{E}[(h^j(\boldsymbol{X}) - \mathbb{E}[h^j(\boldsymbol{X})])g(\boldsymbol{X})] \\
&\quad = \int_{\mathbb{R}^{p-1}} \left[\int_{a_j}^{b_j} -(z^j(\boldsymbol{x})f(\boldsymbol{x}))_j g(\boldsymbol{x}) dx_j \right] dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_p \\
&\quad = \mathbb{E}[z^j(\boldsymbol{X}) g_j(\boldsymbol{X})].
\end{aligned}$$

□

For a fixed function $\boldsymbol{h}$, identity (2.7) characterizes the corresponding density f through the function $\boldsymbol{z}$. Indeed, the following inverse of Theorem 2.1 holds true.

THEOREM 2.2 *Let $\boldsymbol{h} : C_p \to \mathbb{R}^p$ be an arbitrary function with $\mathbb{E}\|\boldsymbol{h}(\boldsymbol{X})\| < \infty$. If identity (2.7) holds for every $g \in \mathcal{G}$, where the family $\mathcal{G}$ contains the bounded functions $g : C_p \to \mathbb{R}$ which are indefinite integrals of their partial derivatives, and for all $j = 1, \ldots, p$,*

$$\int_{a_j}^{b_j} (\mathbb{E}[h^j(\boldsymbol{X})] - h^j(\boldsymbol{x}))f(\boldsymbol{x}) dx_j = 0,$$

then the density f of $\boldsymbol{X}$ and the function $\boldsymbol{z}$ are related through

$$z^j(\boldsymbol{x})f(\boldsymbol{x}) = \int_{a_j}^{x_j} (\mathbb{E}[h^j(\boldsymbol{X})] - h^j(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j))f(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j) dt_j,$$

provided that for all $j = 1, \ldots, p$,

$$\int_{C_p} \left| \int_{a_j}^{x_j} (\mathbb{E}[h^j(\boldsymbol{X})] - h^j(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j)) f(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j) dt_j - z^j(\boldsymbol{x}) f(\boldsymbol{x}) \right| d\boldsymbol{x} < \infty. \tag{2.8}$$

PROOF. For any bounded function $g \in \mathcal{G}$ we have

$$\begin{aligned}&\mathrm{Cov}[h^j(\boldsymbol{X}), g(\boldsymbol{X})] \\ &= \int_{\mathbb{R}^{p-1}} \left[\int_{a_j}^{b_j} g_j(\boldsymbol{x}) \int_{a_j}^{x_j} (\mathbb{E}[h^j] - h^j(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j)) f(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j) dt_j dx_j \right] d\boldsymbol{u}_j d\boldsymbol{v}_j.\end{aligned}$$

On the other hand,

$$\mathrm{Cov}[h^j(\boldsymbol{X}), g(\boldsymbol{X})] = \mathbb{E}[z^j(\boldsymbol{X}) g_j(\boldsymbol{X})].$$

If we put first $g(\boldsymbol{x}) = \cos(\boldsymbol{t}'\boldsymbol{x})$ and then $g(\boldsymbol{x}) = \sin(\boldsymbol{t}'\boldsymbol{x})$, where $\boldsymbol{t} = (t_1, \ldots, t_p)' \in \mathbb{R}^p$ is arbitrary, the proof is completed by the uniqueness of the Fourier transform. □

It should be noted that, under the general conditions of Theorem 2.1, the function $\boldsymbol{h}(\boldsymbol{x}) = -\nabla \log f(\boldsymbol{x})$ yields the function $\boldsymbol{z}$ with $z^j(\boldsymbol{x}) \equiv 1$ for all $j = 1, \ldots, p$; in other words, the assertion of Theorem 2.1 implies the result of Lemma 2.1. Moreover, if we take $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{\Sigma}^{-1}\boldsymbol{x}$, then the corresponding $\boldsymbol{z}$-function is the multivariate $\boldsymbol{w}$-function, where its components w^j are given by the relations

$$w^j(x) f(\boldsymbol{x}) = \int_{a_j}^{x_j} (\mu^j - q^j(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j)) f(\boldsymbol{u}_j, t_j, \boldsymbol{v}_j) dt_j, \quad j = 1, \ldots, p,$$

where $q^j(\boldsymbol{x})$ is the j-th component of the vector $\boldsymbol{q}(\boldsymbol{x}) = (q^1(\boldsymbol{x}), \ldots, q^p(\boldsymbol{x}))' = \boldsymbol{\Sigma}^{-1}\boldsymbol{x}$ and $\mu^j = \mathbb{E}[q^j(\boldsymbol{X})]$. Therefore, the known identity (see Cacoullos and Papathanasiou, 1992)

$$\mathrm{Cov}[q^j(\boldsymbol{X}), g(\boldsymbol{X})] = \mathbb{E}[w^j(\boldsymbol{X}) g_j(\boldsymbol{X})], \quad j = 1, \ldots, p, \tag{2.9}$$

is a special case of Theorem 2.1.

3. An Application to Siegel's Formula

Let $\boldsymbol{X}$ be a random vector with density $f \in \mathcal{F}_1$. Consider the random vector $\boldsymbol{Y} = \nabla(-\log f(\boldsymbol{X}))$, and let $X_{(j)}$ be the j-th ordered component of $\boldsymbol{X}$. Then, by (2.6),

$$\mathrm{Cov}[Y_i, X_{(j)}] = \mathbb{E}\left[\frac{\partial}{\partial X_i} X_{(j)}\right]. \tag{3.1}$$

Since $X_{(j)} = \sum_{i=1}^{p} X_i I(X_i = X_{(j)})$, it follows that

$$\frac{\partial}{\partial X_i} X_{(j)} = I(X_i = X_{(j)}),$$

provided that the random varianles X_j are distinct with probability 1. Applying the covariance identity (3.1), it follows that

$$\text{Cov}[Y_i, X_{(j)}] = \mathbb{E}[I(X_i = X_{(j)})] = \mathbb{P}[X_i = X_{(j)}], \tag{3.2}$$

which in a matrix form can be rewritten as

$$\text{Cov}[\boldsymbol{Y}, \boldsymbol{X}_{(\ )}] = \boldsymbol{P}, \tag{3.3}$$

where $\boldsymbol{X}_{(\ )} = (X_{(1)}, \ldots, X_{(p)})'$ denotes the ordered vector corresponding to $\boldsymbol{X}$, $\boldsymbol{Y} = \nabla(-\log f(\boldsymbol{X}))$, and $\boldsymbol{P} = (p_{ij})$ is the $p \times p$ matrix with elements $p_{ij} = \mathbb{P}[X_i = X_{(j)}]$. Obviously, if $\boldsymbol{X}$ is normally distributed with mean vector $\boldsymbol{\mu}$ and positive definite dispersion matrix $\boldsymbol{\Sigma}$, then $\boldsymbol{Y} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu})$, and (3.3) is a version of the results of Wang et al (1996). Moreover, (3.3) gives more general results for an arbitrary density $f \in \mathcal{F}_1$. For example, if $\boldsymbol{X}$ follows a Dirichlet density with parameters $a_0 > 1$ and $a_j > 1$, $j = 1, \ldots, p$, namely

$$f(\boldsymbol{x}) = \frac{\Gamma(\sum_{j=0}^{p} a_j)}{\prod_{j=0}^{p} \Gamma(a_j)} \left(\prod_{j=1}^{p} x_j^{a_j - 1} \right) \left(1 - \sum_{j=1}^{p} x_j \right)^{a_0 - 1}, \quad \text{for } \boldsymbol{x} \in C_p,$$

where $C_p = \{\boldsymbol{x} \in \mathbb{R}^p : x_j > 0,\ j = 1, \ldots, p,\ \sum_{j=1}^{p} x_j < 1\}$, then (3.3) holds with

$$Y_j = \frac{a_0 - 1}{1 - \sum_{i=1}^{p} X_i} - \frac{a_j - 1}{X_j}, \quad j = 1, \ldots, p.$$

Moreover, for the n-variate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, some interesting upper and lower bounds can be obtained for the variance of $g(\boldsymbol{X}) = \sum_{k=1}^{n} a_k X_{(k)}$, when the coefficients a_k are arbitrary and $X_{(1)} < \cdots < X_{(n)}$ denote the corresponding order statistics, provided that the variables are distinct with probability 1. To this end, it suffices to observe that

$$g(\boldsymbol{X}) = \sum_{i=1}^{n} X_i \sum_{k=1}^{n} a_k I(X_i = X_{(k)}),$$

yielding

$$g_j(\boldsymbol{X}) = \frac{\partial}{\partial X_j} g(\boldsymbol{X}) = \sum_{k=1}^{n} a_k I(X_j = X_{(k)}).$$

It follows that

$$\mathbb{E}[g_j(\boldsymbol{X})] = \sum_{k=1}^{n} a_k \, \mathbb{P}[X_j = X_{(k)}]$$

and

$$\begin{aligned}\mathbb{E}[g_i(\boldsymbol{X})g_j(\boldsymbol{X})] &= \delta_{ij} \sum_{k=1}^{n} a_k^2 \, \mathbb{P}[X_j = X_{(k)}] \\ &+ (1-\delta_{ij}) \sum_{\substack{k,s=1 \\ k \neq s}}^{n} a_k a_s \, \mathbb{P}[X_i = X_{(k)}, X_j = X_{(s)}],\end{aligned}$$

where δ_{ij} is Kronecker's δ. Therefore, working as in Houdré (1995) and using the variance inequalities

$$\mathbb{E}[(\nabla g(\boldsymbol{X}))']\boldsymbol{\Sigma}\,\mathbb{E}[\nabla g(\boldsymbol{X})] \le \mathrm{Var}[g(\boldsymbol{X})] \le \mathbb{E}[(\nabla g(\boldsymbol{X}))'\boldsymbol{\Sigma}\nabla g(\boldsymbol{X})],$$

where the upper bound is due to Chen (1982) (it is, in fact, a special case of (2.5) applied to the normal), while the lower bound is due to Cacoullos (1982), we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} \sum_{k=1}^{n}\sum_{s=1}^{n} a_k a_s \, \mathbb{P}[X_i = X_{(k)}]\,\mathbb{P}[X_j = X_{(s)}] \le \mathrm{Var}[\sum_{k=1}^{n} a_k X_{(k)}] \\ \le \sum_{j=1}^{n} \sigma_j^2 \sum_{k=1}^{n} a_k^2 \, \mathbb{P}[X_j{=}X_{(k)}] + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sigma_{ij} \sum_{\substack{k,s=1 \\ k \neq s}}^{n} a_k a_s \, \mathbb{P}[X_i{=}X_{(k)}, X_j{=}X_{(s)}], \quad (3.4)\end{aligned}$$

where $\sigma_j^2 = \sigma_{jj}$. Two particular cases follow immediately from (3.4): For any single order statistic $X_{(k)}$ from the multivariate normal with arbitrary mean and covariance structure such that the variables are distinct with probability 1, we have the inequality

$$\mathrm{Var}[\boldsymbol{p}_k'\boldsymbol{X}] = \boldsymbol{p}_k'\boldsymbol{\Sigma}\boldsymbol{p}_k \le \mathrm{Var}[X_{(k)}] \le \sum_{j=1}^{n} \sigma_j^2 p_{jk},$$

where $\boldsymbol{p}_k = (p_{1k}, \ldots, p_{nk})' = \Big(\mathbb{P}[X_1{=}X_{(k)}], \ldots, \mathbb{P}[X_n{=}X_{(k)}]\Big)'$, from which it follows that for all $k = 1, \ldots, n$,

$$\frac{1}{n}\lambda_1(\boldsymbol{\Sigma}) \le \lambda_1(\boldsymbol{\Sigma}) \sum_{j=1}^{n} p_{jk}^2 \le \mathrm{Var}[X_{(k)}] \le \max_{1\le j\le n} \sigma_j^2, \qquad (3.5)$$

with $\lambda_1(\mathbf{\Sigma})$ being the minimal eigenvalue of $\mathbf{\Sigma}$ (note that for the lower bound we used the well-known fact that $(\boldsymbol{x}'\boldsymbol{x})\lambda_1(\mathbf{\Sigma}) \le \boldsymbol{x}'\mathbf{\Sigma}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^p$, and the inequality $\boldsymbol{p}_k'\boldsymbol{p}_k \ge (\sum_{j=1}^n p_{jk})^2/n = 1/n)$. The above upper estimate is also given by Houdré (1995), while, if $\mathbf{\Sigma}$ is diagonal, as when we take a normal random sample of size n, then (3.5) implies the inequality

$$\frac{1}{n}\min_{1\le j\le n}\sigma_j^2 \le \min_{1\le j\le n}\mathrm{Var}[X_{(j)}] \le \max_{1\le j\le n}\mathrm{Var}[X_{(j)}] \le \max_{1\le j\le n}\sigma_j^2.$$

Another interesting application of (3.4) results if we assume that $X_1,\ldots,X_n$ is a random sample (that is, i.i.d.) from $N(\mu,\sigma^2)$ with $\sigma^2>0$. In that case, the variance of any linear estimator $L(\boldsymbol{X}) = \sum_{k=1}^n a_k X_{(k)}$, based on the ordered sample, is bounded in terms of the variance σ^2 and the coefficients a_k, namely,

$$\frac{1}{n}\left(\sum_{k=1}^n a_k\right)^2\sigma^2 \le \mathrm{Var}[L(\boldsymbol{X})] \le \left(\sum_{k=1}^n a_k^2\right)\sigma^2. \tag{3.6}$$

Indeed, (3.6) follows from (3.4) because $\sigma_{jj}=\sigma^2$, $\sigma_{ij}=0$ for $i\ne j$, and, by symmetry, $\mathbb{P}[X_j = X_{(k)}] = 1/n$ and $\mathbb{P}[X_i = X_{(k)}, X_j = X_{(s)}] = 1/(n(n-1))$ for all $i \ne j$ and $k \ne s$.

It should be noted that the same covariance identity (3.3) is also valid for the random vector $\boldsymbol{Y} = \nabla k(\boldsymbol{X})$ (see (2.1)), assuming that the distribution of $\boldsymbol{X}$ belongs to $\mathcal{F}_0$; this can be shown by using (2.4) instead of (2.6).

4. Concluding Remarks

The discrete analogue of (2.9) has been shown by Cacoullos and Papathanasiou (1992). Specifically, for a discrete random vector $\boldsymbol{X}$ they proved the identity

$$\mathrm{Cov}[q^j(\boldsymbol{X}), g(\boldsymbol{X})] = \mathbb{E}[w^j(\boldsymbol{X})\Delta_j g(\boldsymbol{X})], \tag{4.1}$$

where $\Delta_j g(\boldsymbol{x})$ denotes the j-th forward difference of the function g (that is, $\Delta_j g(\boldsymbol{x}) = g(x_1,\ldots,x_j+1,\ldots,x_p) - g(\boldsymbol{x})$), and the functions w^j are defined by

$$w^j(\boldsymbol{x})f(\boldsymbol{x}) = \sum_{k_j=0}^{x_j}(\mathbb{E}[q^j(\boldsymbol{X})] - q^j(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j))f(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j),$$

and $\boldsymbol{q}(\boldsymbol{x}) = (q^1(\boldsymbol{x},\ldots,q^p(\boldsymbol{x}))'$, $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$ are defined as in the continuous case (note that here it is not necessary to assume that $\boldsymbol{X}$ belongs to the family of PSDs).

The above results can be extended as follows: Consider a discrete random vector $\boldsymbol{X}$ with probability function $f(\boldsymbol{x})$, supported by a 'convex' subset C_p of $\{0,1,\ldots\}^p$, with $\mathbf{0}=(0,\ldots,0)' \in C_p$. The term 'convex' here means that if $\boldsymbol{x} \in C_p$ then $(y_1,\ldots,y_p)' \in C_p$ for all $y_j \in \{0,1,\ldots,x_j\}$, $j=1,\ldots,p$. For $\boldsymbol{x} \in C_p$, define the function $\boldsymbol{z}=(z^1,\ldots,z^p)'$ by the relations

$$z^j(\boldsymbol{x})f(\boldsymbol{x}) = \sum_{k_j=0}^{x_j} (\mathbb{E}[h^j(\boldsymbol{X})] - h^j(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j))f(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j), \quad j=1,\ldots,p,$$

where $\boldsymbol{h}(\boldsymbol{x})=(h^1(\boldsymbol{x}),\ldots,h^p(\boldsymbol{x}))'$ is a vector of real valued functions defined on C_p, with $\mathbb{E}\|\boldsymbol{h}(\boldsymbol{X})\| = \mathbb{E}\|h^1(\boldsymbol{X}),\ldots,h^p(\boldsymbol{X})\| < \infty$. Furthermore, assume that for all j,

$$z^j(\boldsymbol{x})f(\boldsymbol{x}) = 0, \text{ for } x_j = b_j, \tag{4.2}$$

where $b_j = \sup\{x_j : \boldsymbol{x} \in C_p\}$ is the upper endpoint of the support of X_j, given $X_1=x_1,\ldots,X_{j-1}=x_{j-1},X_{j+1}=x_{j+1},\ldots,X_p=x_p$; observe that b_j may depends on $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_p$, and may be $+\infty$. In the last case, the above condition (4.2) should be interpreted as $\lim_{x_j\to\infty} z^j(\boldsymbol{x})f(\boldsymbol{x})=0$.

The discrete analogue of Theorem 2.1 is contained in the following theorem (the proof is similar to that of Theorem 2.1 and is omitted).

THEOREM 4.1 *Under the above conditions,*

$$\mathrm{Cov}[h^j(\boldsymbol{X}),g(\boldsymbol{X})] = \mathbb{E}[z^j(\boldsymbol{X})\Delta_j g(\boldsymbol{X})], \quad j=1,\ldots,p, \tag{4.3}$$

provided that g satisfies the conditions

$$\mathbb{E}|(h^j(\boldsymbol{X}) - \mathbb{E}[h^j(\boldsymbol{X})])g(\boldsymbol{X})| < \infty, \quad \mathbb{E}|z^j(\boldsymbol{X})\Delta_j g(\boldsymbol{X})| < \infty, \quad j=1,\ldots,p.$$

Conversely, one can easily verify the following

THEOREM 4.2 *If identity* (4.3) *holds for all bounded functions $g : C_p \to \mathbb{R}$, then the functions z^j, h^j and the probability function f are related through*

$$z^j(\boldsymbol{x})f(\boldsymbol{x}) = \sum_{k_j=0}^{x_j} (\mathbb{E}[h^j] - h^j(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j))f(\boldsymbol{u}_j,k_j,\boldsymbol{v}_j). \tag{4.4}$$

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