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UPPER BOUND FOR THE COVARIANCE OF EXTREME ORDER STATISTICS FROM A SAMPLE OF SIZE THREE*

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SUMMARY. Papathanasiou (1990, *Statist. Probab. Lett.* **9** 145–147) proved that the covariance of the ordered pair from a random sample of size two does not exceed the one third of the population variance. In the present note, by using Legendre polynomials, it is proved that a similar result holds for minimum and maximum from a sample of size three, and the equality characterizes the hyperbolic sine density.

1. **Introduction**

Terrell (1983) proved that the coefficient of correlation for the ordered pair from a random sample of size two is at most one half and that equality characterizes the rectangular (uniform over some interval) distributions. This result was extended by Szekely and Mori (1985), who proved that for $1 \leq i < j \leq n$,

$$\text{Corr}\,[X_{i:n}, X_{j:n}] \leq \sqrt{\frac{i(n+1-j)}{j(n+1-i)}}, \qquad \ldots (1.1)$$

and that equality characterizes the rectangular distributions ($X_{i:n}$, $X_{j:n}$ denote the i-th and j-th smallest order statistics from a sample of size n, respectively). Motivated by these inequalities, Papathanasiou (1990) showed a similar result for the covariance, namely,

$$\text{Cov}\,[X_{1:2}, X_{2:2}] \leq \frac{1}{3}\sigma^2, \qquad \ldots (1.2)$$

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where σ^2 is the population variance and the equality again holds only for rectangular distributions. Ma (1992) extended Papathanasiou's result and Balakrishnan and Balasubramanian (1993) observed that the preceding inequality stated by Papathanasiou (and the generalization stated by Ma) is, in fact, equivalent to the classical Hartley-David-Gumbel bound (1954) for the expectation of $X_{n:n}$ from a sample of size n with population mean μ and population variance σ^2, namely

$$0 \le \mathbb{E}[X_{n:n} - \mu] \le \sigma \frac{n-1}{\sqrt{2n-1}},$$

applied for $n = 2$. This is so because of the simple equalities $X_{1:2} + X_{2:2} = X_1 + X_2$ and $X_{1:2}X_{2:2} = X_1X_2$, in view of which we may write $\mathbb{E}[X_{1:2}X_{2:2}] = \mu^2$ and $\mathbb{E}[X_{1:2}] = 2\mu - \mathbb{E}[X_{2:2}]$, obtaining

$$\mathrm{Cov}\,[X_{1:2}, X_{2:2}] = \mu^2 - (2\mu - \mathbb{E}[X_{2:2}])\,\mathbb{E}[X_{2:2}] = \mathbb{E}^2[X_{2:2} - \mu].$$

Unfortunately, we cannot have similar relations when $n > 2$, so a general result of this kind for all i, j and n, although exists, seems difficult to be found.

In this note, by using some particular properties of the Legendre polynomials in $[0,1]$, we prove that a relation, similar to (1.2) holds for the covariance of $X_{1:3}$ and $X_{3:3}$ (see Theorem 3.1 and Corollary 3.1, below) and we give the corresponding characterization. Although the techniques used here (Legendre polynomials) have become fairly standard in this area, the main result provides an unexpected characterization of *hyperbolic sine density*, which is far from obvious.

2. Some Properties of Legendre Polynomials

Let P_n, $n = 0, 1, \dots$ be the usual orthogonal Legendre polynomials in the interval $[-1, 1]$ and consider the corresponding orthonormalized Legendre polynomials $\phi_n(u) = \sqrt{2n+1}P_n(2u-1)$ in $[0,1]$, such that $\phi_n(1) > 0$ for all n (see Terrell, 1983 and Sugiura, 1962 and 1964). For the proof of the main result we will make use of the properties (2.1)–(2.11), listed below, which are satisfied by ϕ_n and P_n. These properties can be found (or they are simple by-products) in Seaborn (1991, p. 163), Luke (1969, p. 284) and Sansone (1959, p. 195).

For $n = 0, 1, \dots$ and $k = 0, \dots, n$, let

$$c_k(n) = \frac{(n!)^2}{(n-k)!(n+k+1)!}.$$

Then, we have

$$\phi_n(u) = \sqrt{2n+1} \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{(n-k)!(k!)^2} u^k, \qquad \dots (2.1)$$

$$\int_0^1 u^n \phi_k(u)du = \sqrt{2k+1}c_k(n), \qquad \text{for } n \geq k, \qquad \ldots(2.2)$$

$$\int_0^1 (1-u)^n \phi_k(u)du = \sqrt{2k+1}(-1)^k c_k(n), \qquad \text{for } n \geq k, \qquad \ldots(2.3)$$

$$\int_0^1 u^n \phi_k(u)du = \int_0^1 (1-u)^n \phi_k(u)du = 0, \qquad \text{for } n < k. \qquad \ldots(2.4)$$

For $k, s = 1, 2, \ldots$ set

$$A(k,s) = \int\int_{0<u<v<1} (v-u)\phi_k(u)\phi_s(v)dvdu.$$

Then,

$$A(k,s) = A(s,k), \qquad \ldots(2.5)$$

$$A(k,k) = \frac{-1}{2(2k-1)(2k+3)}, \qquad \ldots(2.6)$$

$$A(k,k+2) = \frac{1}{4(2k+3)\sqrt{(2k+1)(2k+5)}}, \qquad \ldots(2.7)$$

$$A(k,s) = 0 \quad \text{for all } k < s, \quad s \neq k+2. \qquad \ldots(2.8)$$

Finally, the following simple properties for the P_n will be used in the sequel.

$$P_{2k+1}(1) = 1, \qquad P_{2k+1}(0) = 0, \qquad \ldots(2.9)$$

$$P'_{2k+1}(1) = (k+1)(2k+1), \qquad \ldots(2.10)$$

$$P''_{2k+1}(u) = \sum_{n=1}^{k}(4n-1)(k-n+1)(2k+2n+1)P_{2n-1}(u). \qquad \ldots(2.11)$$

3. **Main Result**

Let U be a Uniform$(0,1)$ r.v. and consider the order statistics $U_{1:n} < U_{2:n} < \cdots < U_{n:n}$ corresponding to a random sample of size n drawn from U. It is well-known that any function $g \in L^2(0,1)$ admits the representation

$$g(u) = \sum_{k=0}^{\infty} a_k \phi_k(u), \qquad \ldots(3.1)$$

where $a_k = \int_0^1 g(u)\phi_k(u)du$ and the polynomial series (3.1) converges in the corresponding $L^2(0,1)$ metric space.

From the definition it follows immediately that the condition $g \in L^2(0,1)$ is equivalent to $\mathrm{Var}[g(U)] < \infty$ and the statements $\mathbb{E}[g(U)] = 0$ and $\mathrm{Var}[g(U)] = 1$ are equivalent to

$$a_0 = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} a_k^2 = 1. \qquad \ldots(3.2)$$

In the last standardized case, the following result holds.

LEMMA 3.1. *If* $\mathbb{E}[g(U)] = 0$ *and* $\mathrm{Var}[g(U)] = 1$*, then*

$$\mathbb{E}\,[g(U_{1:n})]\,\mathbb{E}\,[g(U_{n:n})] = n^2 \left(\sum_{k=1}^{[(n-1)/2]} \sqrt{4k+1}\, a_{2k} c_{2k}(n-1) \right)^2 + + - n^2 \left(\sum_{k=1}^{[n/2]} \sqrt{4k-1}\, a_{2k-1} c_{2k-1}(n-1) \right)^2 .$$

PROOF. From (3.1) and the fact that

$$\mathbb{E}\,[g(U_{1:n})] = n \int_0^1 (1-u)^{n-1} g(u) du, \quad \mathbb{E}\,[g(U_{n:n})] = n \int_0^1 u^{n-1} g(u) du,$$

we conclude the desired result, taking into account (2.2), (2.3) and (2.4).

REMARK 3.1. Let X be any standardized (with mean zero and variance one) r.v. with d.f. F. Let us take $g(u) = F^{-1}(u) := \inf\{x : F(x) \geq u\}$, $0 < u < 1$, in the previous Lemma. In view of the inverse probability transformation (that the r.v. $F^{-1}(U)$ is distributed like X), it follows the well-known fact that

$$\left(F^{-1}(U_{1:n}), \ldots, F^{-1}(U_{n:n})\right) \stackrel{\mathrm{d}}{=} (X_{1:n}, \ldots, X_{n:n}),$$

where $X_{1:n}, \ldots, X_{n:n}$ is an ordered sample from F (i.e., the two random vectors have the same multivariate d.f.). It then follows from the Lemma that the product of the expectations of the minimum and the maximum from a sample of any standardized d.f. F depends upon F^{-1} only on the first $n-1$ Legendre coefficients $a_1, \ldots, a_{n-1}$ of F^{-1} (c.f. Terrell, 1983). For example, if $n = 3$, we have the simple expression

$$\mathbb{E}\,[X_{1:3}]\,\mathbb{E}\,[X_{3:3}] = \frac{1}{20} a_2^2 - \frac{3}{4} a_1^2. \qquad \ldots(3.3)$$

Similarly, for the expectation of the product we have the following

LEMMA 3.2. *Under the assumptions of Lemma 3.1,*

$$\mathbb{E}\,[g(U_{1:3}) g(U_{3:3})] = -\frac{1}{2} a_1^2 - \frac{1}{10} a_2^2 - \frac{3}{2} \sum_{k=1}^{\infty} \frac{1}{2k+3} \left(\frac{a_k}{\sqrt{2k+1}} - \frac{a_{k+2}}{\sqrt{2k+5}} \right)^2 .$$

Proof. Let $f(u,v) = 6(v-u)$, $0 < u < v < 1$, be the joint density of $(U_{1:3}, U_{3:3})$. By using (3.1) we have

$$A = 6 \int\int_{0<u<v<1} (v-u) g(u) g(v) dv du = 6 \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} a_k a_s A(k,s), \qquad \ldots (3.4)$$

where $A = \mathbb{E}\,[g(U_{1:3}) g(U_{3:3})]$. By properties (2.5), (2.6), (2.7) and (2.8), most of the terms in the double sum (3.4) vanish and the remaining terms yield

$$A = 6 \left(\sum_{k=1}^{\infty} a_k^2 A(k,k) + 2 \sum_{k=1}^{\infty} a_k a_{k+2} A(k, k+2) \right),$$

which reduces to the desired result if we observe that

$$\sum_{k=1}^{\infty} \frac{2a_k^2}{(2k-1)(2k+3)} = \frac{a_1^2}{3} + \frac{a_2^2}{15} + \sum_{k=1}^{\infty} \frac{1}{2k+3} \left(\frac{a_k^2}{2k+1} + \frac{a_{k+2}^2}{2k+5} \right).$$

Remark 3.2. Observe that the above Lemma (if we take $g = F^{-1}$) implies that for any standardized d.f. F,

$$\mathbb{E}\,[X_{1:3} X_{3:3}] < 0.$$

Combining Lemmas 3.1 and 3.2, we readily obtain the following Legendre representation for the covariance of $g(U_{1:3})$ and $g(U_{3:3})$, provided that $\mathbb{E}\,[g(U)] = 0$ and $\mathrm{Var}\,[g(U)] = 1$:

$$\mathrm{Cov}\,[g(U_{1:3}), g(U_{3:3})] = \frac{1}{4} a_1^2 - \frac{3}{20} a_2^2 - \frac{3}{2} \sum_{k=1}^{\infty} \frac{1}{2k+3} \left(\frac{a_k}{\sqrt{2k+1}} - \frac{a_{k+2}}{\sqrt{2k+5}} \right)^2, \qquad \ldots (3.5)$$

where a_k, $k \geq 0$, are the Legendre coefficients of g (satisfying (3.2)). It should be noted that (3.5) is also true even in the case where g is completely arbitrary (provided that $g \in L^2(0,1)$), since for any constants B, C, $\mathrm{Cov}\,[Cg\,(U_{1:3}) + B,\ Cg\,(U_{3:3}) + B] = C^2\,\mathrm{Cov}\,[g\,(U_{1:3})\,, g\,(U_{3:3})]$, and the Legendre coefficients of the function $Cg + B$ are simply $B + Ca_0$, Ca_1, $Ca_2, \ldots$, where a_0, a_1, $a_2, \ldots$ are the coefficients of g.

Using this representation, we are ready now to prove the main result of this note, stated as follows.

Theorem 3.1. *Let U be a Uniform$(0,1)$ r.v. and suppose that $U_{1:3} < U_{3:3}$ are the minimum and the maximum from a random sample of size three drawn from U. Then, for any arbitrary function $g \in L^2(0,1)$ we have*

$$\mathrm{Cov}\,[g(U_{1:3}), g(U_{3:3})] \leq \frac{6}{a^2}\,\mathrm{Var}\,[g(U)], \qquad \ldots (3.6)$$

where $a \approx 5.96941$ *is the unique positive root of the equation* $\tanh(a/2) = a/6$. *Equality in (3.6) holds if and only if there exist some constants* B, C, *such that* $g(u) = C\sinh(a(u-1/2)) + B$ *for almost all* $u \in (0,1)$.

In order to avoid any confusions, it should be noted that here and everywhere in this article, $a(\approx 5.96941)$ is a completely specified constant **and not** a parameter of a function or a distribution (see Corollary 3.1, below). More specifically, it is easy to see that the even function $h(y) = \tanh(y)/y$ decreases continuously from 1 to 0 as y varies from 0 to $+\infty$, so that the equation $\tanh(x/2) = x/6$ has exactly three solutions as x varies: $x = 0$ and $x = \pm a$, where we denote with a the unique positive one. An alternative definition could be given from the fact that a is the unique positive number satisfying $e^a = (6+a)/(6-a)$. Note also that a satisfies a number of characteristic properties that will be used in the sequel, e.g., $\sinh(a/2) = a/\sqrt{36-a^2}$, $\cosh(a/2) = 6/\sqrt{36-a^2}$ and $\sinh(a) = 12a/\left(36-a^2\right)$, as it can be easily seen from the elementary identities $\cosh^2(x) - \sinh^2(x) = 1$ and $\sinh(2x) = 2\sinh(x)\cosh(x)$. Finally, observe that if the equality is attained in (3.6) then the constants B, C determine the values of $\mathbb{E}[g(U)]$ and $\mathrm{Var}[g(U)]$ (and conversely) via the relations $\mathbb{E}[g(U)] = B$ and $\mathrm{Var}[g(U)] = C^2(\sinh(a)-a)/(2a) = C^2\left(a^2-24\right)/\left(2\left(36-a^2\right)\right)$.

A simple implication of the above Theorem is the following Corollary, which presents a characterization of the hyperbolic sine distribution.

COROLLARY 3.1. *Let* $X_{1:3} \le X_{3:3}$ *be the minimum and the maximum corresponding to a random sample of size three drawn from an arbitrary d.f. with mean* μ *and finite variance* $\sigma^2 > 0$. *Then,*

$$\mathrm{Cov}\left[X_{1:3}, X_{3:3}\right] \le \frac{6}{a^2}\,\sigma^2 \approx 0.16838\,\sigma^2, \qquad \ldots (3.7)$$

and the equality in (3.7) characterizes the hyperbolic sine distribution *with density*

$$f(x) = \frac{1/a}{\sqrt{(x-\mu)^2 + \lambda^2\sigma^2}}, \qquad \mu - a\sigma\sqrt{\frac{2}{a^2-24}} < x < \mu + a\sigma\sqrt{\frac{2}{a^2-24}},$$

where $\lambda = \sqrt{2(36-a^2)/(a^2-24)} \approx 0.25089$ *and* a *is as in Theorem 3.1.*

PROOF. Since $(X_{1:3}, X_{3:3}) \stackrel{\mathrm{d}}{=} \left(F^{-1}(U_{1:3}), F^{-1}(U_{3:3})\right)$, Theorem 3.1 (with $g = F^{-1}$) shows that

$$\mathrm{Cov}\left[X_{1:3}, X_{3:3}\right] = \mathrm{Cov}\left[F^{-1}(U_{1:3}), F^{-1}(U_{3:3})\right] \le \frac{6}{a^2}\,\mathrm{Var}\left[F^{-1}(U)\right] = \frac{6}{a^2}\sigma^2,$$

which proves (3.7). Assume now that the equality holds in (3.7). From Theorem 3.1 follows that this is equivalent to the fact that for some constants B, C,

$$F^{-1}(u) = C\sinh(a(u-1/2)) + B$$

for almost all $u \in (0,1)$. Since F is non-degenerate (because $\sigma^2 > 0$ by the assumptions), C must be non-zero. Therefore, $C > 0$ (because $F^{-1}(u)$ should be non-decreasing in u). On the other hand, since F^{-1} is always left-continuous, it follows that $F^{-1}(u) = C \sinh(a(u-1/2)) + B$ for some $C > 0$ and **for all** $u \in (0,1)$. Observing that $F^{-1}(0_+) = B - C\sinh(a/2)$ and $F^{-1}(1_-) = B + C\sinh(a/2)$, we conclude that F is concentrated on the finite range $|x - B| < C\sinh(a/2)$. Therefore, inverting F^{-1} we find that

$$F(x) = \frac{1}{2} + \frac{1}{a}\log\left(\frac{1}{C}\left(x - B + \sqrt{C^2 + (x-B)^2}\right)\right), \qquad |x - B| < C\sinh(a/2),$$

and, since this F is absolutely continuous, the desired result follows by a simple calculation of the derivative of F, observing that

$$\mu = \int_0^1 F^{-1}(u)du = B$$

and

$$\sigma^2 = \int_0^1 \left(F^{-1}(u) - \mu\right)^2 du = C^2\,\frac{\sinh(a) - a}{2a} = C^2\,\frac{a^2 - 24}{2(36 - a^2)}.$$

Proof of theorem 3.1. If $\mathrm{Var}[g(U)] = 0$ the result is obvious (and (3.6) becomes equality in the trivial sense $0 = 0$, since $\mathrm{Var}[g(U)] = 0$ if and only if there exists a constant B such that $g(u) = B$ for almost all $u \in (0,1)$, and this corresponds to the case $C = 0$). Suppose then that $0 < \mathrm{Var}[g(U)] < \infty$. Without any loss of generality we may further assume that $\mathbb{E}[g(U)] = 0$ and $\mathrm{Var}[g(U)] = 1$ since, using standard arguments, if the result holds true in this particular case then the general one will follow immediately if we apply it to the function

$$\hat{g}(u) = \frac{g(u) - \mathbb{E}[g(U)]}{\sqrt{\mathrm{Var}[g(U)]}}, \qquad 0 < u < 1,$$

with $\mathbb{E}[\hat{g}(U)] = 0$ and $\mathrm{Var}[\hat{g}(U)] = 1$. Hence, from now on, assume that $\mathbb{E}[g(U)] = 0$ and $\mathrm{Var}[g(U)] = 1$. Then, (3.5) presents the covariance to be maximized under (3.2).

The rest of the proof is divided in four parts as follows: **(i)** We show that (3.5) attains its maximum value M for at least one $\mathbf{a} = (a_1, a_2, \ldots)$ satisfying (3.2). **(ii)** We show that all the even coefficients of any such maximizing $\mathbf{a}$ must be zero. **(iii)** We find the difference equation (3.10), below, satisfied by any such

$$\mathbf{a} = \left(x_1\sqrt{3}, 0, x_2\sqrt{7}, 0, x_3\sqrt{11}, 0, \ldots\right),$$

which also implies the uniqueness of the solution (up to the sign). **(iv)** Finally, we show that the Legendre coefficients of the $L^2(0,1)$ function of the form $h(u) = \lambda\sinh(a(u - 1/2))$, $0 < u < 1$, with $a = \sqrt{6/M}$ and λ a suitable constant, satisfy the same difference equation (3.10) with the same initial conditions as the solution does, and thus, it coincides with the unique extremal of the problem. This will identify both the extremal function and the value of M (via the value of a), completing the proof.

(i) For the sake of simplicity, set $z_k = a_k/\sqrt{2k+1}$ for all $k \geq 1$. Then, we have the equivalent maximization problem:

$$\text{Maximize} \quad G_1(\mathbf{z}) = \frac{3}{4}z_1^2 - \frac{3}{4}z_2^2 - \frac{3}{2}\sum_{k=1}^{\infty}\frac{(z_k - z_{k+2})^2}{2k+3}$$

$$\text{Over} \quad S_1 = \left\{\mathbf{z} = (z_1, z_2, \ldots) : \sum_{k=1}^{\infty}(2k+1)z_k^2 = 1\right\}.$$

Let $M = \sup_{\mathbf{z}\in S_1} G_1(\mathbf{z})$. For $\mathbf{z} = \big(1/\sqrt{3}, 0, 0, \ldots\big) \in S_1$ we have $G_1(\mathbf{z}) = 3/20$, so that $M \geq 3/20$. On the other hand, for any $\mathbf{z} \in S_1$, $G_1(\mathbf{z}) \leq (3/4)\, z_1^2 \leq 1/4$. Thus, M is a well-defined number in $[3/20, 1/4]$. In order to prove that there exists at least one $\mathbf{z} \in S_1$ such that $G_1(\mathbf{z}) = M$, consider an arbitrary $\mathbf{z} \in S_1$ and observe that

$$\sum_{k=n}^{\infty} z_k^2 = \sum_{k=n}^{\infty}\frac{2k+1}{2k+1}z_k^2 \leq \frac{1}{2n+1}\sum_{k=n}^{\infty}(2k+1)z_k^2 \leq \frac{1}{2n+1},$$

which shows that $\lim_{n\to\infty}\sum_{k\geq n} z_k^2 = 0$ uniformly for $\mathbf{z} \in S_1$. It follows that S_1 is a relatively compact (i.e., its closure is compact; see for example Diestel, 1984, p. 6) subset of ℓ^2, the space of sequences $\mathbf{z}$ satisfying $\|\mathbf{z}\|_2^2 = \sum_{k\geq 1} z_k^2 < \infty$. Hence, S_1 is a compact subset of ℓ^2 (because it is a closed). Therefore, the continuous function $G_1 : S_1 \to \mathbb{R}$ takes on its maximum (and minimum) value on S_1 (see, for example, Reed and Simon (1980), p. 99).

(ii) Let $\mathbf{z} = (z_1, z_2, \ldots) \in S_1$ be any maximizing point of G_1 (i.e., $G_1(\mathbf{z}) = M \geq G_1(\mathbf{y})$ for all $\mathbf{y} \in S_1$; such a $\mathbf{z}$ exists from **(i)**). Since $\sum_{k\geq 1}(2k+1)z_k^2 = 1$, it follows that $\sum_{k\geq 1}(4k+1)z_{2k}^2 = \delta$ and $\sum_{k\geq 1}(4k-1)z_{2k-1}^2 = 1-\delta$ for some $\delta \in [0,1)$ (observe that $\delta \neq 1$; otherwise (since $\delta = 1$ implies $z_1 = 0$) we are lead to the contradiction $G_1(\mathbf{z}) = M \leq 0$). Then, we may construct a new point $\mathbf{z}^* = (z_1^*, z_2^*, \ldots) \in S_1$ with $z_{2k}^* = 0$ and $z_{2k-1}^* = z_{2k-1}/\sqrt{1-\delta}$ for all $k \geq 1$. Since $G_1(\mathbf{z}) \geq G_1(\mathbf{z}^*)$ (because $\mathbf{z}$ maximizes G_1 over S_1 and $\mathbf{z}^* \in S_1$), it follows that

$$\begin{aligned} M &= G_1(\mathbf{z}) \geq G_1(\mathbf{z}^*) = \frac{1}{1-\delta}\left(\frac{3}{4}z_1^2 - \frac{3}{2}\sum_{k=1}^{\infty}\frac{(z_{2k-1} - z_{2k+1})^2}{4k+1}\right) \\ &\geq \frac{1}{1-\delta}\left(\frac{3}{4}z_1^2 - \frac{3}{2}\sum_{k=1}^{\infty}\frac{(z_{2k-1} - z_{2k+1})^2}{4k+1}\right) \\ &\quad -\frac{1}{1-\delta}\left(\frac{3}{4}z_2^2 + \frac{3}{2}\sum_{k=1}^{\infty}\frac{(z_{2k} - z_{2k+2})^2}{4k+3}\right) = \frac{1}{1-\delta}G_1(\mathbf{z}) = \frac{M}{1-\delta} \geq M, \end{aligned}$$

and therefore, $\delta = 0$ (this shows that the Legendre series of any maximizing function g includes only odd polynomials, i.e., g is odd with respect to $1/2$).

(iii) Canceling the even terms in (3.5) and using the new substitution $x_k = a_{2k-1}/\sqrt{4k-1}$, we have the equivalent maximization problem:

$$\text{Maximize} \quad G(\mathbf{x}) = \frac{3}{4}x_1^2 - \frac{3}{2}\sum_{k=1}^{\infty}\frac{(x_k - x_{k+1})^2}{4k+1} \qquad \ldots(3.8)$$

$$\text{Under the restriction} \quad H(\mathbf{x}) = \sum_{k=1}^{\infty}(4k-1)x_k^2 = 1, \qquad \ldots(3.9)$$

where $\mathbf{x} = (x_1, x_2, \ldots)$ may again be regarded as a point of ℓ^2. Since it can be shown, as in **(i)**, that the set $S = \{\mathbf{x} : H(\mathbf{x}) \le 1\}$ is a compact subset of ℓ^2, we conclude that the continuous function $G : \ell^2 \to \mathbb{R}$ attains its maximum and minimum value on S. Therefore, the supremum of G over S ($= M^*$, say) is attained at some point $\mathbf{x} \in S$, and $M^* \in [3/20, 1/4]$. Observe that for any such maximizing point $\mathbf{x} = (x_1, x_2, \ldots)$ we must have $H(\mathbf{x}) = 1$; indeed, $H(\mathbf{x}) = 0$ is impossible (because, in this case, $\mathbf{x} = (0, 0, \ldots)$ and $M^* = G(\mathbf{x}) = 0$ is impossible) and if $0 < H(\mathbf{x}) < 1$, the point $\mathbf{z} = \mathbf{x}/\sqrt{H(\mathbf{x})} = \left(x_1/\sqrt{H(\mathbf{x})}, x_2/\sqrt{H(\mathbf{x})}, \ldots\right)$ obviously satisfies $H(\mathbf{z}) = 1$ (hence $\mathbf{z} \in S$) and $G(\mathbf{z}) = G(\mathbf{x})/H(\mathbf{x}) > G(\mathbf{x}) = M^*$, which is impossible since M^* is, by definition, the supremum of G over S. Therefore, any maximizing point $\mathbf{x}$ must belong to the boundary $\partial S = \{\mathbf{x} : H(\mathbf{x}) = 1\}$ of S. It follows then from **(i)** and **(ii)** that $M = M^*$. Hence, for any maximizing point $\mathbf{x}$ we have $G(\mathbf{x}) = M = MH(\mathbf{x})$. In addition, one can easily show that $G(\mathbf{y}) \le MH(\mathbf{y})$ for all $\mathbf{y}$ with $H(\mathbf{y}) < \infty$. Indeed, if $G(\mathbf{y}) > MH(\mathbf{y})$ for some $\mathbf{y}$ with $H(\mathbf{y}) < \infty$, it would follow that $H(\mathbf{y}) > 0$ (otherwise, $G(\mathbf{y}) = 0$) and the point $\mathbf{z} = \mathbf{y}/\sqrt{H(\mathbf{y})}$ satisfies $\mathbf{z} \in S$ and $G(\mathbf{z}) = G(\mathbf{y})/H(\mathbf{y}) > M = M^*$, contradicting the definition of M^*.

Fix now a maximizing point $\mathbf{x} = (x_1, x_2, \ldots)$ (with $H(\mathbf{x}) = 1$) and for $k \ge 1$ fixed consider the quadratic in λ:

$$Q_k(\lambda) = MH\left(x_1, \ldots x_{k-1}, \lambda, x_{k+1}, \ldots\right) - G\left(x_1, \ldots x_{k-1}, \lambda, x_{k+1}, \ldots\right).$$

Easy calculations show that for $k > 1$,

$$Q_k(\lambda) = (4k-1)\left(M + \frac{3}{(4k-3)(4k+1)}\right)\lambda^2 - 3\left(\frac{x_{k+1}}{4k+1} + \frac{x_{k-1}}{4k-3}\right)\lambda \\ + \text{terms independent of } \lambda,$$

while for $k = 1$,

$$Q_1(\lambda) = 3\left(M - \frac{3}{20}\right)\lambda^2 - \frac{3x_2}{5}\lambda + \text{terms independent of } \lambda.$$

It follows from the previous analysis that for all $k \geq 1$, $Q_k(\lambda) = a(k)\lambda^2 + b(k)\lambda + c(k) \geq 0$ for all $\lambda \in \mathbb{R}$, while $Q_k(x_k) = 0$. Since $a(k) > 0$, this implies that $x_k = -b(k)/(2a(k))$, yielding the following difference equation (satisfied by any maximizing point $\mathbf{x} = (x_1, x_2, \ldots)$) for all $k \geq 1$:

$$x_{k+2} = \frac{2(4k+3)}{4k+1}\left(1 + \frac{M}{3}(4k+1)(4k+5)\right)x_{k+1} - \frac{4k+5}{4k+1}x_k, \qquad \ldots (3.10)$$

where $x_2 = 10\,(M - 3/20)\,x_1$ and x_1 is such that $H(\mathbf{x}) = 1$. This shows that the solution $\mathbf{x}$ is unique (up to the sign) and, consequently, the maximizing function $g \in L^2(0,1)$ of the initial problem is also unique (apart from the fact that it is defined almost everywhere in $(0,1)$ and, also, the fact that if g is a solution (satisfying $\mathbb{E}[g(U)] = 0$ and $\mathrm{Var}[g(U)] = 1$), then $-g$ is also a solution (satisfying the same conditions)). Furthermore, g is odd with respect to $1/2$.

(iv) Consider the function $h(u) = \lambda \sinh(a(u - 1/2))$, $0 < u < 1$, with $a = \sqrt{6/M}$ and λ an arbitrary constant to be specified later (note that a, although well-defined from **(i)**, is yet unknown). Since $h \in L^2(0,1)$ is odd with respect to $1/2$, we can expand it as an odd-degree series of Legendre polynomials in the form $h = \sum_{k\geq 1} b_{2k-1}\phi_{2k-1}$, where

$$b_{2k-1} = \lambda\sqrt{4k-1}\int_0^1 \sinh\left(\frac{au}{2}\right) P_{2k-1}(u)du, \qquad k \geq 1.$$

Making, as in **(iii)**, the substitution $y_k = b_{2k-1}/\sqrt{4k-1}$, integrating twice by parts and using (2.9), (2.10) and (2.11), we conclude that for all $k \geq 1$,

$$y_1 = \frac{2\lambda}{a}\left(\cosh(a/2) - \frac{2}{a}\sinh(a/2)\right), \qquad \ldots (3.11)$$

$$y_{k+1} = \frac{2\lambda}{a}\left(\cosh(a/2) - \frac{2}{a}(k+1)(2k+1)\sinh(a/2)\right) + \frac{4}{a^2}\sum_{n=1}^{k}(4n-1)(k-n+1)(2k+2n+1)y_n. \qquad \ldots (3.12)$$

Hence, for $k \geq 1$,

$$y_{k+1} - y_k = \frac{4(4k+1)}{a^2}\left(\sum_{n=1}^{k}(4n-1)y_n - \lambda\sinh(a/2)\right),$$

and therefore,

$$\frac{y_{k+2} - y_{k+1}}{4k+5} - \frac{y_{k+1} - y_k}{4k+1} = \frac{4}{a^2}(4k+3)y_{k+1} = \frac{2M}{3}(4k+3)y_{k+1},$$

i.e., the y's satisfy the difference equation (3.10). Furthermore, y_1 is given by (3.11), while from (3.11) and (3.12),

$$y_2 = \frac{2\lambda}{a}\left(\left(1+\frac{60}{a^2}\right)\cosh(a/2) - \frac{2}{a}\left(6+\frac{60}{a^2}\right)\sinh(a/2)\right). \qquad \ldots (3.13)$$

Choose now λ such that $y_1 = x_1$, where $\mathbf{x} = (x_1, x_2, \ldots)$ is any maximizing point of G (see **(iii)**) with $H(\mathbf{x}) = 1$ (this can be done because $\tanh(t) < t$ for all $t > 0$; note also that by **(iii)**, there are exactly two such points $\mathbf{x}$, each one opposite to the other). Then, if we prove that $y_2 = x_2$, we will have $x_k = y_k$ for all $k \geq 1$; that is, if g is any maximizing function corresponding to $\mathbf{x}$ (i.e., $g = \sum_{k\geq 1}\sqrt{4k-1}x_k\phi_{2k-1}$) then $g(u) = h(u)$ for almost all $u \in (0,1)$, since the Legendre series of an $L^2(0,1)$ function is unique. In order to prove this last detail, suppose in contrary that $x_2 - y_2 = \epsilon > 0$. Since the function $g - h \in L^2(0,1)$ has the Legendre representation

$$g - h = \sum_{k=1}^{\infty}\sqrt{4k-1}(x_k - y_k)\phi_{2k-1},$$

we have from Parseval's identity that

$$\sum_{k=1}^{\infty}(4k-1)\,(x_k - y_k)^2 = \int_0^1 (g(u) - h(u))^2\,du < \infty.$$

On the other hand, since $x_k - y_k$ satisfies (3.10) with $x_1 - y_1 = 0$ and $x_2 - y_2 = \epsilon > 0$, it is easily verified (by using (3.10) and induction on k) that the sequence $x_k - y_k$ is non-decreasing. This implies that $x_k - y_k \geq \epsilon$ for all $k \geq 2$; a contradiction to Parseval's identity. Similar arguments apply to the case $y_2 - x_2 = \epsilon > 0$.

Finally, from the relations $y_1 = x_1$ and $y_2 = x_2$ we have (see (3.10)) $y_2 = x_2 = 10\,(M - 3/20)\,x_1 = 30\left(2/a^2 - 1/20\right)y_1$ (note that $M = 6/a^2$ by the definition of a), and from (3.11) and (3.13) we get the equation $\tanh(a/2) = a/6$, from which we conclude the value of $\sqrt{6/M} = a \approx 5.96941$ (hence $M = 6/a^2 \approx 0.16838$), and the proof is complete.

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