Some facts concerning ratios of random variables

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Concerning a question of Prof. T. Cacoullos, I pointed out the following:

If X and Y are (simply) identically distributed,

$$X \stackrel{\mathrm{d}}{=} Y,$$

it is not true that

$$\frac{X}{Y} \stackrel{\mathrm{d}}{=} \frac{Y}{X},$$

even for positive rv's X and Y.

Counterexample 1 (Discrete). Let $\mathbb{P}(X = x, Y = y)$ be given by the following table:

$x \backslash y$	1	2	3	$f_X(x)$
1	2/36	9/36	1/36	1/3
2	1/36	2/36	9/36	1/3
3	9/36	1/36	2/36	1/3
$f_Y(y)$	1/3	1/3	1/3	1

Obviously $X \stackrel{\mathrm{d}}{=} Y$, $X \sim U(\{1,2,3\})$, but

$$\mathbb{P}\left(\frac{X}{Y} = 2\right) = \frac{1}{36} \neq \mathbb{P}\left(\frac{Y}{X} = 2\right) = \frac{9}{36}.$$

We construct a similar example for the continuous case:

Counterexample 2 (Continuous). Let U_1, U_2 be i.i.d. and uniformly distributed over the interval $(0,1)$. Let I be independent of (U_1, U_2) such that $I \sim U(\{0,1,2\})$, that is, $\mathbb{P}(I = i) = 1/3$, $i = 0,1,2$. Let also $J = I + 1$ if $I = 0$ or $I = 1$, and $J = 0$ if $I = 2$, so that $J \sim U(\{0,1,2\})$, i.e.,

$$J \stackrel{\mathrm{d}}{=} I.$$

It is clear that (I, J) and (U_1, U_2) are independent random vectors. We now define

$$(X, Y) = (I + U_1, J + U_2).$$

Clearly X and Y have the same distribution, $U(0,3)$. The joint density of (X, Y) is

$$f(x,y) = \begin{cases} 1/3, & \text{if } x \in (0,1) \text{ and } y \in (1,2), \\ 1/3, & \text{if } x \in (1,2) \text{ and } y \in (2,3), \\ 1/3, & \text{if } x \in (2,3) \text{ and } y \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

This density is not exchangeable, $f(x,y) \neq f(y,x)$, and the ratios X/Y and Y/X are not identically distributed, e.g.,

$$\mathbb{P}\left(\frac{X}{Y} \leqslant 1\right) = \frac{2}{3}, \quad \mathbb{P}\left(\frac{Y}{X} \leqslant 1\right) = \frac{1}{3}.$$

In both examples given above, the vector (X,Y) fails to be exchangeable. We say that (X,Y) are exchangeable if

$$(X,Y) \stackrel{\mathrm{d}}{=} (Y,X). \tag{1}$$

Clearly, if (1) holds then for any (Borel) $g : \mathbb{R}^2 \mapsto \mathbb{R}$ we have

$$g(X,Y) \stackrel{\mathrm{d}}{=} g(Y,X),$$

and in particular, for $g(x,y) = x/y$ we get

$$\frac{X}{Y} \stackrel{\mathrm{d}}{=} \frac{Y}{X},$$

provided that $\mathbb{P}(X=0)=0$. Therefore, we have the following.

Fact 1. If the vector (X,Y) is exchangeable and $\mathbb{P}(X=0)=0$ (and thus, $\mathbb{P}(Y=0)=0$, by exchangeability), then

$$\frac{X}{Y} \stackrel{\mathrm{d}}{=} \frac{Y}{X}. \tag{2}$$

In particular, if X and Y are i.i.d. with $\mathbb{P}(X=0)=0$ then (2) holds.

We now pose some questions:

Question 1. If T is an rv with $\mathbb{P}(T=0)=0$ such that

$$T \stackrel{\mathrm{d}}{=} \frac{1}{T}, \tag{3}$$

is it true that there exist an exchangeable vector (X,Y) with $\mathbb{P}(X=0)=0$ such that

$$T \stackrel{\mathrm{d}}{=} \frac{X}{Y} \ ?$$

Question 2 (stronger). If T is an rv with $\mathbb{P}(T=0)=0$ such that (3) holds, is it true that there exist i.i.d. rv's X,Y with $\mathbb{P}(X=0)=0$ such that

$$T \stackrel{\mathrm{d}}{=} \frac{X}{Y} \ ?$$

The answer to the Question 1 is in the affirmative, in contrast to the Question 2. To see this, consider a symmetric Bernoulli r.v. I with $\mathbb{P}(I=0)=\mathbb{P}(I=1)=1/2$, and define the random vector

$$(X,Y)=(W\cdot T^{I},W\cdot T^{1-I})=(W[(1-I)+IT],W[I+(1-I)T]). \tag{4}$$

In (4), W is **any random variable** with $\mathbb{P}(W=0)=0$, for example $W\equiv 1$ or $W\sim N(\mu,\sigma^2)$, and the r.v.'s T,W,I are independent. Since

$$I\overset{\mathrm{d}}{=}1-I$$

it follows that

$$(T,W,I)\overset{\mathrm{d}}{=}(T,W,1-I),$$

and thus, for any Borel $\boldsymbol{g}=(g_1,g_2):\mathbb{R}^3\mapsto\mathbb{R}^2$ we have

$$\boldsymbol{g}(T,W,I)\overset{\mathrm{d}}{=}\boldsymbol{g}(T,W,1-I)$$

i.e.,

$$(g_1(T,W,I),g_2(T,W,I))\overset{\mathrm{d}}{=}(g_1(T,W,1-I),g_2(T,W,1-I)). \tag{5}$$

Applying (5) to $\boldsymbol{g}(t,w,i)=(wt^{i},wt^{1-i})$ we get, in view of (4), that

$$(X,Y)\overset{\mathrm{d}}{=}(Y,X),$$

so that (X,Y) is an exchangeable random vector. Moreover, by definition,

$$\frac{X}{Y}=\frac{T^{I}}{T^{1-I}}.$$

Finally, for any $t\in\mathbb{R}$ we have

$$\begin{aligned}\mathbb{P}\left(\frac{X}{Y}\leqslant t\right) &= \frac{1}{2}\,\mathbb{P}\left(\frac{T^{I}}{T^{1-I}}\leqslant t\;\Big|\;I=0\right)+\frac{1}{2}\,\mathbb{P}\left(\frac{T^{I}}{T^{1-I}}\leqslant t\;\Big|\;I=1\right)\\ &= \frac{1}{2}\,\mathbb{P}\left(\frac{1}{T}\leqslant t\right)+\frac{1}{2}\mathbb{P}(T\leqslant t)\\ &= \mathbb{P}(T\leqslant t),\end{aligned}$$

because, from (3),

$$\mathbb{P}\left(\frac{1}{T}\leqslant t\right)=\mathbb{P}(T\leqslant t).$$

Therefore,

$$\frac{X}{Y}\overset{\mathrm{d}}{=}T,$$

as it was to be shown.

Note. If the r.v. W is taken to be absolutely continuous then the vector (X, Y) is absolutely continuous. This shows that there always exists an absolutely continuous exchangeable solution even if T is discrete.

We now show that there exists an r.v. T which satisfies (3), but T has not a representation of the form

$$T \stackrel{\mathrm{d}}{=} \frac{X}{Y}, \quad \text{with } X, Y \text{ being i.i.d.} \tag{6}$$

Indeed, consider the positive r.v. T for which $\log(T)$ is uniformly distributed over the interval $(-1, 1)$, i.e., $U = \log(T) \sim U(-1, 1)$ — **log-uniform**. Clearly T satisfies (3). If (6) was true then U should have the representation

$$U \stackrel{\mathrm{d}}{=} X_1 - X_2, \quad \text{with } X_1 = \log(|X|),\ X_2 = \log(|Y|). \tag{7}$$

Since X and Y are i.i.d. it follows that X_1 and X_2 in (7) are also i.i.d., but this is impossible. Indeed, if $\phi(t)$ is the common characteristic function of X_1 and X_2 then the characteristic function of $X_1 - X_2$ is, by independence,

$$\phi_{X_1 - X_2}(t) = \phi(t)\phi(-t) = \phi(t)\overline{\phi(t)} = |\phi(t)|^2 \geqslant 0.$$

On the other hand, the characteristic function of U is

$$\phi_U(t) = \frac{\sin(t)}{t},$$

which assumes both positive and negative values.