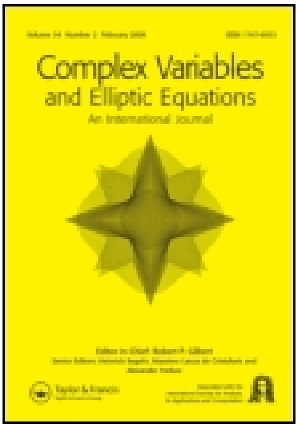

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An extension of the disc algebra, II

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An extension of the disc algebra, II

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We identify the complex plane $\mathbb{C}$ with the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ by the homeomorphism $\mathbb{C} \ni z \mapsto \frac{z}{1+|z|} \in D$. This leads to a compactification $\overline{\mathbb{C}}$ of $\mathbb{C}$, homeomorphic to $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. The Euclidean metric on $\overline{D}$ induces a metric d on $\overline{\mathbb{C}}$. We identify all uniform limits of polynomials on $\overline{D}$ with respect to the metric d. The class of the above limits is an extension of the disc algebra and it is denoted by $\overline{A}(D)$. We study properties of the elements of $\overline{A}(D)$ and topological properties of the class $\overline{A}(D)$ endowed with its natural topology. The class $\overline{A}(D)$ is different and, from the geometric point of view, richer than the class $\widetilde{A}(D)$ introduced, on the basis of the chordal metric χ.

Keywords: disc algebra; Mergelyan's Theorem; polynomial approximation

AMS Subject Classifications: Primary 30J99; secondary 46A99; 30E10

1. Introduction

The uniform limits of the polynomials on the closed unit disc $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ with respect to the usual Euclidean metric on $\mathbb{C}$ are exactly all functions $f : \overline{D} \to \mathbb{C}$, continuous on $\overline{D}$ and holomorphic in $D = \{z \in \mathbb{C} : |z| < 1\}$. The class of the above functions is the disc algebra $A(D)$. In [1,2], the Euclidean metric on $\mathbb{C}$ is replaced by the chordal metric χ on the one-point compactification $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. The set of uniform limits of polynomials, with respect to χ, is an extension of $A(D)$ and it is denoted by $\widetilde{A}(D)$. It contains the constant function equal to ∞ and the following functions: $f : \overline{D} \to \widetilde{\mathbb{C}}$ continuous, such that $f(D) \subset \mathbb{C}$ and $f_{|D}$ is holomorphic. The class $\widetilde{A}(D)$ remains the same if we replace the chordal metric χ by any other metric on $\widetilde{\mathbb{C}}$ generating the same topology – the reason is that any two such metrics are uniformly equivalent, because $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is compact. In this sense, the set of χ-uniform polynomial limits is an *invariant set*, i.e. independent of the specific metric one chooses for generating it.

Instead of the one-point compactification $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we shall consider another, more rich from the geometric viewpoint, compactification, as follows: We identify the complex plane $\mathbb{C}$ with the open unit disc D by the homeomorphism

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$$\mathbb{C} \ni z \mapsto \frac{z}{1+|z|} \in D.$$

In the Euclidean setting, the natural compactification of D is $\overline{D}$. This leads to a compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$ of $\mathbb{C}$, where the set of infinite points $\mathbb{C}^\infty$ is homeomorphic to the unit circle $T = \partial D = \{z \in \mathbb{C} : |z| = 1\}$; more precisely, we write $\mathbb{C}^\infty = \{\infty e^{i\theta} : \theta \in \mathbb{R}\}$. The usual Euclidean metric on $\overline{D}$ induces the metric d on $\overline{\mathbb{C}}$, defined as follows:

$$d(z_1, z_2) = \begin{cases} \left| \dfrac{z_1}{1+|z_1|} - \dfrac{z_2}{1+|z_2|} \right|, & \text{for } z_1, z_2 \in \mathbb{C}, \\ \left| \dfrac{z_1}{1+|z_1|} - e^{i\theta} \right|, & \text{for } z_1 \in \mathbb{C},\ z_2 = \infty e^{i\theta}\ (\theta \in \mathbb{R}), \\ \left| e^{i\theta_1} - e^{i\theta_2} \right|, & \text{for } z_1 = \infty e^{i\theta_1},\ z_2 = \infty e^{i\theta_2}\ (\theta_1, \theta_2 \in \mathbb{R}). \end{cases} \tag{1.1}$$

We shall investigate the uniform limits (on $\overline{D}$) of the polynomials, with respect to the metric d. The class of limit functions is another extension of the disc algebra, different from $\widetilde{A}(D)$, which is denoted by $\overline{A}(D)$. It will be shown that it contains exactly two types of functions. The first type, the *finite type*, consists of all continuous functions $f : \overline{D} \to \overline{\mathbb{C}}$ such that $f(D) \subset \mathbb{C}$ and $f_{|D}$ is holomorphic. The second type, the *infinite type*, consists of all functions $f : \overline{D} \to \mathbb{C}^\infty \subset \overline{\mathbb{C}}$ of the form $f(z) = \infty e^{i\theta(z)}$ where the function $\theta : \overline{D} \to \mathbb{R}$ is continuous on $\overline{D}$ and harmonic in D.

To highlight the difference between $\widetilde{A}(D)$ and $\overline{A}(D)$, we mention that, e.g. the function $f(z) = \frac{1}{1-z}$, with $f(1) = \infty$, belongs to $\widetilde{A}(D)$ but not to $\overline{A}(D)$. An example of an element of $\overline{A}(D)$ of finite type, not belonging to $A(D)$, is given by $f(z) = \log \frac{1}{1-z}$ (here $f(1) = +\infty$ corresponds to the infinite element $\infty e^{i\theta} \in \mathbb{C}^\infty \subset \overline{\mathbb{C}}$ with $\theta = 0$).

Furthermore, we shall investigate some properties of the elements of $\overline{A}(D)$ and some of its topological properties when it is endowed with its natural metric. Finally, we shall consider uniform approximation with respect to the metric d on other compact sets, different from $\overline{D}$. Of course, due to the compactness of $\overline{\mathbb{C}}$, all the above results remain valid if we replace d by any equivalent metric on $\overline{\mathbb{C}}$.

Several open questions are naturally posed and new directions of investigation are indicated. In particular, any result on $A(D)$ and any approximation result with respect to the usual Euclidean metric are worth to be examined in $\overline{A}(D)$ and with respect to d, respectively.

A first version of the present paper can be found in [3].

2. Preliminaries

Let $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane $\mathbb{C}$, endowed with the chordal metric χ, defined by $\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$ for $z_1, z_2 \in \mathbb{C}$, $\chi(z_1, \infty) = \frac{1}{\sqrt{1+|z_1|^2}}$ for $z_1 \in \mathbb{C}$ and, certainly, $\chi(\infty, \infty) = 0$.

We can also define another compactification $\overline{\mathbb{C}}$ of $\mathbb{C}$ with infinitely many points at infinity, as follows: first consider the homomorphism $G : \mathbb{C} \to \{w \in \mathbb{C} : |w| < 1\} = D$ given by $G(z) = \frac{z}{1+|z|}$. Since $\overline{D}$ is a compactification of D, it induces a compactification $\overline{\mathbb{C}}$ of $\mathbb{C}$. The set $\mathbb{C}^\infty$, which consists of all points at infinity, is homeomorphic with the circle $T = \partial D = \{w \in \mathbb{C} : |w| = 1\}$; thus, $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$. Every element of $\mathbb{C}^\infty$ is determined by a unimodular complex number $e^{i\theta}$, $\theta \in \mathbb{R}$, and we shall denote the corresponding element of $\mathbb{C}^\infty \subset \overline{\mathbb{C}}$ by $\infty e^{i\theta}$. In particular, this compactification contains the usual two points of infinity of the real numbers ($\pm\infty$), in the sense that $+\infty$ corresponds to $\theta = 0$ while $-\infty$ is related to $\theta = \pi$. The Euclidean metric on $\overline{D}$ induces the metric d on $\overline{\mathbb{C}}$, which is defined in (1.1). The definition of the metric d can be simplified if we extend $G(z) = \frac{z}{1+|z|}$, defined for $z \in \mathbb{C}$, to the points at infinity $z = \infty e^{i\theta} \in \mathbb{C}^\infty$ ($\theta \in \mathbb{R}$), as $G(z) = G(\infty e^{i\theta}) = e^{i\theta}$. Then, $d(z_1, z_2) = |G(z_1) - G(z_2)|$ for all $z_1, z_2 \in \overline{\mathbb{C}}$.

Both metrics χ and d induce the usual topology on $\mathbb{C}$. For if $W \subset \mathbb{C}$ is compact, then χ, d and the Euclidean metric (all restricted on W) are two-by-two uniformly equivalent. Thus, if S is any set and $f, f_n : S \to W$ ($n = 1, 2, \ldots$) are functions, then the uniform convergence on S, $f_n \to f$, as $n \to \infty$, with respect to any one of these metrics, implies uniform convergence with respect to the other two. Further, we have the following

LEMMA 2.1 *If S is any set and $f, f_n : S \to \mathbb{C}$ ($n = 1, 2, \ldots$) are functions, then the uniform convergence $f_n \to f$, as $n \to \infty$, with respect to the usual Euclidean metric on $\mathbb{C}$, implies the uniform convergence $f_n \to f$, as $n \to \infty$, with respect to the metric d.*

Proof The result is implied by the fact that for all $z_1, z_2 \in \mathbb{C}$,

$$d(z_1, z_2) \leq |z_1 - z_2|. \tag{2.1}$$

This inequality can be proved as follows. Write

$$d(z_1, z_2) = \frac{|z_1 - z_2 + w|}{1 + |z_1| + |z_2| + |z_1 z_2|}, \quad \text{where } w = z_1|z_2| - z_2|z_1|,$$

and observe that $w = \frac{|z_1| + |z_2|}{2}(z_1 - z_2) - \frac{z_1 + z_2}{2}(|z_1| - |z_2|)$. It follows that

$$|z_1 - z_2 + w| \leq \left(1 + \frac{|z_1| + |z_2|}{2}\right)|z_1 - z_2| + \frac{|z_1| + |z_2|}{2}\Big||z_1| - |z_2|\Big| \leq (1 + |z_1| + |z_2|)|z_1 - z_2|,$$

and thus,

$$d(z_1, z_2) \leq \left(\frac{1 + |z_1| + |z_2|}{1 + |z_1| + |z_2| + |z_1 z_2|}\right)|z_1 - z_2| \leq |z_1 - z_2|.$$

□

Clearly, the converse implication of Lemma 2.1 is not true; e.g. consider $S = [0, +\infty)$, $f_n(x) = (1 + 1/n)x$, $f(x) = x$.

Consider now the map $\Phi : (\overline{\mathbb{C}}, d) \to (\widetilde{\mathbb{C}}, \chi)$, defined for all $z \in \overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$ by

$$\Phi(z) = \begin{cases} z, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty e^{i\theta} \in \mathbb{C}^\infty \ (\theta \in \mathbb{R}). \end{cases} \tag{2.2}$$

One can easily see that Φ is continuous and, therefore, uniformly continuous, because $\overline{\mathbb{C}}$ is compact. This immediately implies the following

LEMMA 2.2 *If S is any set and $f, f_n : S \to \overline{\mathbb{C}}$ ($n = 1, 2, \ldots$) are functions, then the uniform convergence (on S) $f_n \to f$, as $n \to \infty$, with respect to the metric d, implies the uniform convergence $\Phi \circ f_n \to \Phi \circ f$, as $n \to \infty$, with respect to the metric χ.*

Remark 2.1 Alternatively, one can prove Lemma 2.2 by making use of the easily proved inequality $\chi(\Phi(z_1), \Phi(z_2)) \leq 2d(z_1, z_2)$, which is valid for all $z_1, z_2 \in \overline{\mathbb{C}}$.

COROLLARY 2.1 *If S is any set and $f, f_n : S \to \mathbb{C}$ $(n = 1, 2, \dots)$ are functions, then the uniform convergence (on S) $f_n \to f$, as $n \to \infty$, with respect to the usual Euclidean metric on $\mathbb{C}$ implies the uniform convergence $f_n \to f$, as $n \to \infty$, with respect to the metric χ.*

Proof It suffices to combine Lemmas 2.1 and 2.2, or to observe the trivial inequality $\chi(z_1, z_2) \leq |z_1 - z_2|$, $z_1, z_2 \in \mathbb{C}$. □

Another useful fact that will be used in the sequel is the following; its simple proof is omitted.

LEMMA 2.3 *Let $R > 0$ be a positive real number and define the map $\Phi_R : \overline{\mathbb{C}} \to \mathbb{C}$ by*

$$\Phi_R(z) = \begin{cases} z, & \text{if } z \in \mathbb{C} \text{ and } |z| < R, \\ R\frac{z}{|z|}, & \text{if } z \in \mathbb{C} \text{ and } |z| \geq R, \\ Re^{i\theta}, & \text{if } z = \infty e^{i\theta} \in \mathbb{C}^{\infty} \ (\theta \in \mathbb{R}). \end{cases}$$

Then, Φ_R is (uniformly) continuous.

3. The definition

In this section, we consider the closed unit disc $S = \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and we identify the set of uniform limits on $\overline{D}$ of polynomial functions with respect to the metric d. Note that the polynomials are continuous with respect to the new metric. Indeed, since for every polynomial P the set $P(\overline{D})$ is a compact subset of $\mathbb{C}$, and on each compact subset of $\mathbb{C}$ the Euclidean metric and the metric d are (uniformly) equivalent, it follows trivially that any polynomial function $P : \overline{D} \to P(\overline{D})$ is uniformly continuous with respect to d.

Suppose f_n, $n = 1, 2, \dots$, is a sequence of complex polynomials. Let $f : \overline{D} \to \overline{\mathbb{C}}$ be a function, where $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^{\infty}$, endowed with the metric d, is the compactification of $\mathbb{C}$ introduced previously. We assume that the sequence f_n, $n = 1, 2, \dots$, converges uniformly on $\overline{D}$ towards *f* with respect to the metric d. Since polynomial functions are continuous and uniform convergence preserves continuity, it follows that the limiting function $f : \overline{D} \to \overline{\mathbb{C}}$ has to be continuous. Furthermore, according to Lemma 2.2, the sequence $\Phi \circ f_n = f_n$, $n = 1, 2, \dots$, converges uniformly (with resect to χ) to the function $\Phi \circ f : \overline{D} \to \widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Here, Φ is the map defined by (2.2) and χ is the chordal metric on $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (see section 2). It follows that the function $\Phi \circ f$ belongs to the class $\widetilde{A}(D)$ introduced in [1,2]. Thus, according to the definition of $\widetilde{A}(D)$, the function $\Phi \circ f$ can be of the following two types:

The first type contains the holomorphic functions $\Phi \circ f : D \to \mathbb{C}$, such that for every boundary point $\zeta \in \partial D = T = \{w \in \mathbb{C} : |w| = 1\}$, the limit

$$\lim_{z \to \zeta,\ z \in D} \Phi(f(z))$$

exists in $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this case, we conclude that the continuous function $f : \overline{D} \to \overline{\mathbb{C}}$ satisfies $f(D) \subset \mathbb{C}$ and $f_{|D}$ is holomorphic. This is the first type, the *finite type* of $\overline{A}(D)$.

The second type of elements of $\widetilde{A}(D)$ is the constant function $\Phi \circ f \equiv \infty$; this means that for every $z \in \overline{D}$, the value $f(z)$ is a point at infinity of $\overline{\mathbb{C}}$. In this case, the continuous function $f : \overline{D} \to \overline{\mathbb{C}}$ satisfies $f(\overline{D}) \subset \mathbb{C}^{\infty}$. Thus, f is of the form $f(z) = \infty e^{i\theta(z)}$ for some function $\theta : \overline{D} \to \mathbb{R}$. Observe that the continuity of the function $f(z) = \infty e^{i\theta(z)}$, with respect to the metric d, is equivalent to the continuity of the function $e^{i\theta(z)}$, with respect to the usual Euclidean metric. Thus, the function

$$\overline{D} \ni z \mapsto e^{i\theta(z)} \in T = \partial D$$

has to be continuous. Since $\overline{D}$ is simply connected, it follows from Theorem 5.1 of [4] (see p.128) that the real function $\overline{D} \ni z \mapsto \theta(z) \in \mathbb{R}$ can be chosen to be continuous. Now, we shall show that any continuous version of the function $\overline{D} \ni z \mapsto \theta(z) \in \mathbb{R}$ is, in fact, harmonic in the open unit disc D.

Indeed, the uniform convergence to the $\mathbb{C}^{\infty}$-valued function f (with respect to the metric d) shows that $|f_n(z)| \to +\infty$, as $n \to \infty$, uniformly on $\overline{D}$. Thus, $|f_n(z)| \geq 1$ for all $z \in \overline{D}$ and for all $n \geq n_0$. Considering in D a branch of $\log f_n$, we conclude that $\operatorname{Arg} f_n(z) = \operatorname{Im}[\log f_n(z)]$ is a harmonic function in D. We also have $\frac{f_n(z)}{|f_n(z)|} \to e^{i\theta(z)}$, uniformly on $\overline{D}$ (with respect to the usual Euclidean metric). Thus, the same holds on $D \subset \overline{D}$. It follows that $e^{i[\theta(z) - \operatorname{Arg} f_n(z)]} \to 1$, as $n \to \infty$, uniformly on D. Thus, there exists $n_1 \geq n_0$, such that for every natural number $n \geq n_1$, there exists an integer $k_n = k_n(z) \in \mathbb{Z}$ such that the function $w_n(z) := \theta(z) - \operatorname{Arg} f_n(z) - 2k_n(z)\pi$ takes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for all $z \in D$. Since $e^{iw_n(z)} = \frac{e^{i\theta(z)}}{e^{i\operatorname{Arg} f_n(z)}}$ is continuous in D and $w_n(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that the function $D \ni z \mapsto w_n(z) \in \mathbb{R}$ is continuous ($n \geq n_1$). Therefore, writing $k_n(z) = \frac{1}{2\pi}(\theta(z) - w_n(z) - \operatorname{Arg} f_n(z))$, we see that the function $D \ni z \mapsto k_n(z) \in \mathbb{Z}$ is also continuous, and hence, constant. Thus, for $z \in D$, we may write $k_n(z) \equiv k_n \in \mathbb{Z}$, independent of $z \in D$ (for $n \geq n_1$). Now, the uniform convergence $e^{iw_n(z)} \to 1$, as $n \to \infty$ (with respect to the Euclidean metric) and the fact that $w_n(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for all $z \in D$ and for all $n \geq n_1$, imply that $w_n(z) \to 0$, uniformly on D (with respect to the Euclidean metric). Equivalently, $2k_n\pi + \operatorname{Arg} f_n(z) \to \theta(z)$, as $n \to \infty$, uniformly on D. Since the functions $2k_n\pi + \operatorname{Arg} f_n(z)$, $n \geq n_1$, are harmonic in D, it follows that the function $\theta(z)$, being a uniform limit of harmonic functions, is also harmonic in D.

Therefore, we have shown that the second type of limiting functions is continuous functions $f : \overline{D} \to \mathbb{C}^{\infty} \subset \overline{\mathbb{C}}$, of the form $f(z) = \infty e^{i\theta(z)}$, where the function $\overline{D} \ni z \mapsto \theta(z) \in \mathbb{R}$ can be chosen to be continuous on $\overline{D}$ and harmonic in D. The above $\mathbb{C}^{\infty}$-valued functions generate the *infinite type* of $\overline{A}(D)$.

Conversely, we shall show that each continuous function $f : \overline{D} \to \overline{\mathbb{C}}$ of the previous two types is, indeed, the uniform limit (with respect to the metric d) of a sequence of polynomials $f_n : \overline{D} \to \mathbb{C}$. The details are as follows:

Suppose first that f is of finite type. That is, $f : \overline{D} \to \overline{\mathbb{C}}$ is continuous (with respect to the metric d), $f(D) \subset \mathbb{C}$ and $f_{|D}$ is holomorphic. Since $\overline{D}$ is compact, it follows that f is uniformly continuous. Therefore, for a given $\epsilon > 0$, we can find a real number $r = r(\epsilon)$, $0 < r < 1$, such that $d(f(z), f(rz)) < \frac{\epsilon}{2}$ for all $z \in \overline{D}$. Since the function $f : D \to \mathbb{C}$ is holomorphic, a partial sum $P(w)$ of the Taylor development of $f(w)$ satisfies $|f(w) - P(w)| < \frac{\epsilon}{2}$ for all $w : |w| \leq r$. Thus, $|f(rz) - P(rz)| < \frac{\epsilon}{2}$ for all $z \in \overline{D}$. Using inequality (2.1), we see that $d(f(rz), P(rz)) < \frac{\epsilon}{2}$ for all $z \in \overline{D}$. Now, the triangle inequality implies that $d(f(z), Q(z)) < \epsilon$, $z \in \overline{D}$, where $Q = Q_\epsilon$ is the polynomial defined by $Q(z) = P(rz)$. Setting $\epsilon = 1/n$, we conclude that the sequence of polynomials $f_n = Q_{1/n}$, $n = 1, 2, \ldots$, approximates f uniformly on $\overline{D}$, with respect to d.

Finally, suppose that f is of infinite type. That is, assume that $f : \overline{D} \to \mathbb{C}^\infty \subset \overline{\mathbb{C}}$ is continuous (with respect to d) and of the form $f(z) = \infty e^{i\theta(z)}$, $z \in \overline{D}$, where the real-valued function $\theta(z)$, $z \in \overline{D}$, is continuous on $\overline{D}$ and harmonic in D. Let $\epsilon > 0$. Since the function $\overline{D} \ni z \mapsto \theta(z) \in \mathbb{R}$ is uniformly continuous, there exists $r = r(\epsilon)$, with $0 < r < 1$, such that $|\theta(z) - \theta(rz)| < \frac{\epsilon}{3}$ for all $z \in \overline{D}$. Therefore, from $d(f(z), f(rz)) = |e^{i\theta(z)} - e^{i\theta(rz)}| \leq |\theta(z) - \theta(rz)|$, we conclude that $d(f(z), f(rz)) < \frac{\epsilon}{3}$ for all $z \in \overline{D}$. Since the function $D \ni z \mapsto \theta(z) \in \mathbb{R}$ is harmonic in the simply connected domain D, we can find a holomorphic function $g \in \mathcal{H}(D)$ such that $\mathrm{Im} g(z) = \theta(z)$, $z \in D$. Setting $\delta = \min\{e^{\mathrm{Re} g(w)} : |w| \leq r\}$, we see that $\delta > 0$. Thus, for any $n \in \{1, 2, \ldots\}$ and any $w \in \mathbb{C}$ with $|w| \leq r$, we have

$$d\left(ne^{g(w)}, f(w)\right) = \frac{1}{1 + ne^{\mathrm{Re} g(w)}} \leq \frac{1}{1 + \delta n}.$$

It follows that we can fix a large enough $n = n(\epsilon)$ so that $d(ne^{g(rz)}, f(rz)) < \frac{\epsilon}{3}$ for all $z \in \overline{D}$. On the other hand, since the function $ne^{g(w)}$ is holomorphic in D, it can be approximated, uniformly over $\overline{D}_r = \{w \in \mathbb{C} : |w| \leq r\}$, by a partial sum of its Taylor expansion, with respect to the Euclidean metric. Therefore, there exists a polynomial $P(w)$ such that $|ne^{g(w)} - P(w)| < \frac{\epsilon}{3}$ for all $w : |w| \leq r$. Using the inequality (2.1) (cf. Lemma 2.1), we conclude that $d(ne^{g(rz)}, P(rz)) < \frac{\epsilon}{3}$ for all $z \in \overline{D}$. By considering the polynomial $Q = Q_\epsilon$, defined by $Q(z) = P(rz)$, we conclude that for all $z \in \overline{D}$,

$$d\big(f(z), Q_\epsilon(z)\big) \leq d\big(f(z), f(rz)\big) + d\big(f(rz), ne^{g(rz)}\big) + d\big(ne^{g(rz)}, P(rz)\big) < \epsilon.$$

It follows that the sequence of polynomials $f_n = Q_{1/n}$, $n = 1, 2, \ldots$, approximates f uniformly on $\overline{D}$, with respect to d.

Therefore, we naturally arrived at the following definition.

Definition 3.1 Let $\mathbb{C}^\infty = \{\infty e^{i\theta} : \theta \in \mathbb{R}\}$ and set $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$, endowed with the metric d defined in (1.1). Let also $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be, as usually, the open and the closed unit disc, respectively. We denote by $\overline{A}(D)$ the class of continuous functions $f : \overline{D} \to \overline{\mathbb{C}}$ of the following two types.

(a) **The finite type:** It contains the continuous functions $f : \overline{D} \to \overline{\mathbb{C}}$ such that $f(D) \subset \mathbb{C}$ and $f_{|D}$ is holomorphic in D.

(b) **The infinite type:** It contains the continuous functions $f : \overline{D} \to \mathbb{C}^\infty \subset \overline{\mathbb{C}}$ of the form $f(z) = \infty e^{i\theta(z)}$, $z \in \overline{D}$, where the real-valued function $\theta : \overline{D} \to \mathbb{R}$ is harmonic in D and continuous (with respect to the usual Euclidean metric) on $\overline{D}$.

Thus, we have shown the following

THEOREM 3.1 *Let $\overline{D}$ be the closed unit disc. The set of uniform limits (on $\overline{D}$) of the complex-valued polynomials – with respect to the metric d, defined on $\overline{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$ by* (1.1) *– coincides with the class $\overline{A}(D)$.*

Obviously, $\overline{A}(D)$ contains the disc algebra $A(D) = \{f : \overline{D} \to \mathbb{C}$, continuous on $\overline{D}$, holomorphic in $D\}$. It is an extension of $A(D)$, essentially different from the extension $\widetilde{A}(D)$ obtained in [1,2], using the chordal metric χ. Indeed, it is easily seen that the function $f(z) = \frac{1}{1-z}$, $|z| < 1$, is neither a restriction in D of any element of $A(D)$, nor a restriction in D of any element of $\overline{A}(D)$. However, $f(z)$ can be extended to $\overline{D}$ in an obvious manner, so that the resulting $\widetilde{\mathbb{C}}$-valued function is χ-continuous and forms an element of $\widetilde{A}(D)$. Loosely speaking, $f(z) = \frac{1}{1-z}$ belongs to $\widetilde{A}(D)$ but not to $\overline{A}(D)$.

Note that any conformal mapping of D onto an open strip (or half-strip) belongs to $\overline{A}(D)$ and not to $A(D)$. More generally, it is also true that for any non-zero complex numbers c_k, $k = 1, 2, \dots, n$, the function

$$f(z) = \sum_{k=1}^{n} c_k \log \frac{1}{e^{i\theta_k} - z}, \quad (\text{where } 0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi)$$

is an element of $\overline{A}(D)$, and not of $A(D)$. In fact, the above examples belong to the finite type of $\overline{A}(D)$. It is easy to show that if f belongs to $\overline{A}(D)$ (and is of finite type) then $\Phi \circ f$ belongs to $\widetilde{A}(D)$ – see (2.2), Lemma 2.2 and Remark 2.1. On the other hand the converse does not hold, as we can see by the example $f(z) = \frac{1}{1-z}$; obviously, this function cannot be written as $f = \Phi \circ g$ for some $g \in \overline{A}(D)$.

Some simple examples of elements of $\overline{A}(D)$ of infinite type can be constructed as follows: consider a polynomial P not vanishing at any point of $\overline{D}$. Then, the sequence $nP(z)$, $n = 1, 2, \ldots$, converges uniformly on $\overline{D}$, with resect to the metric d, to the function $f(z) = \infty e^{i \operatorname{Arg} P(z)}$ which, of course, belongs to the infinite type of $\overline{A}(D)$. In particular, taking $P(z) = (2+z)^k$ with large enough $k \in \mathbb{N}$ (in fact, $k = 6$ suffices), we see that the image of the limiting function f covers the whole $\mathbb{C}^\infty$.

4. Some properties of the elements of $\overline{A}(D)$

Let $f \in \overline{A}(D)$. Then $\Phi \circ f \in \widetilde{A}(D)$ and, applying Proposition 3.1 of [1,2], we obtain the following.

Proposition 4.1 *Let $T = \partial D = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the unit circle and assume that $f \in \overline{A}(D)$.*

(a) *If for some $c \in \mathbb{C}$ the set $\{\zeta \in T : f(\zeta) = c\}$ has positive Lebesgue measure then f is constant.*
(b) *If the set $\{\zeta \in T : f(\zeta) \in \mathbb{C}^\infty\}$ has positive Lebesgue measure then f is of infinite type.*

Remark 4.1 If $f(z) = \infty e^{i\theta(z)}$ belongs to the infinite type of $\overline{A}(D)$ then $\theta(z)$ can be constant on a subarc of T with strictly positive length without being constant on $\overline{D}$. In fact, any continuous function $T \ni \zeta \mapsto \theta(\zeta) \in \mathbb{R}$ has a unique extension $\overline{D} \ni z \mapsto \theta(z) \in \mathbb{R}$ which is continuous on $\overline{D}$ and harmonic in D; it defines a unique $f \in \overline{A}(D)$ of infinite type.

Proposition 4.2 *Let K be a compact subset of T having Lebesgue measure zero. Then any continuous (with respect to the metric d) function $\phi : K \to \mathbb{C}^\infty$ is the restriction of some $f \in \overline{A}(D)$, of finite type, such that $f^{-1}(\mathbb{C}^\infty) = K$.*

Proof There exists a function $g \in A(D)$ such that $g(z) = 1$ on K and $|g(z)| < 1$ on $\overline{D} \setminus K$ (see [5], p.81; see also [6], p.42–43, where it is written that the first who constructed such a function was Fatou). Also, since $\phi(K) \subset \mathbb{C}^{\infty}$, we can write $\phi(\zeta) = \infty e^{i\theta(\zeta)}$, $\zeta \in K$, for some real-valued function $\theta : K \to \mathbb{R}$. Since the function ϕ is continuous (on K) with respect to d, the function $e^{i\theta(\zeta)}$ is continuous (on K) with respect to the usual Euclidean metric. It follows from [5,7,8] that there exists a function $h \in A(D)$ such that $h(\zeta) = e^{i\theta(\zeta)}$ for $\zeta \in K$. Now, it is easy to verify that the function given by

$$f(z) = \begin{cases} h(z) \log \frac{1}{1-g(z)}, & \text{if } z \in \overline{D} \setminus K, \\ \phi(z), & \text{if } z \in K, \end{cases}$$

has the desired properties. □

If $E \subset T$ is compact with positive Lebesgue measure, then E is not a compact of interpolation for $\overline{A}(D)$. That is, there exists a continuous function $\eta : E \to \overline{\mathbb{C}}$ which does not have an extension $f \in \overline{A}(D)$. Indeed, let $\zeta_0 \in E$ and let V be an arc with middle point ζ_0 and length less than half of the Lebesgue measure of E. We set $\eta \equiv 0$ on $E \setminus V$, $\eta(\zeta_0) = 1$, and we extend η linearly on V. Assume now that for some $f \in \overline{A}(D)$, it is true that $f_{|E} \equiv \eta$. Then, $f_{|E \setminus V} \equiv 0$. Since $E \setminus V$ has positive Lebesgue measure, Proposition 4.1(a) implies that $f \equiv 0$, which contradicts $f(\zeta_0) = \eta(\zeta_0) = 1$.

Question 1 If $E \subset T$ is a compact set with Lebesgue measure zero, is it true that E is a compact of interpolation of $\overline{A}(D)$? That is, is it true that every continuous function $\eta : E \to \overline{\mathbb{C}}$ has an extension in $\overline{A}(D)$?

We refer to [7,8] for the corresponding result for $A(D)$. If Question 1 had a positive answer, then we would have the following characterization: A compact set $E \subset T$ is a compact set of interpolation for $\overline{A}(D)$ if and only if E has Lebesgue measure zero. Certainly, one could ask questions similar to Question 1, replacing "Lebesgue measure zero" by other assumptions as, for instance, "logarithmic capacity zero".

One can easily see that for any compact set $E \subset T$, every continuous function $\eta : E \to \mathbb{C}^{\infty}$ has an extension in $\overline{A}(D)$ of infinite type; this follows from the fact that every continuous function $\theta : E \to \mathbb{R}$ has a continuous extension on $\overline{D}$ which is harmonic in D. Of course, this extension is unique only in the case $E = T$.

Another question is as follows:

Question 2 Characterize the compact sets $E \subset \overline{D}$ having the property that every continuous function $\eta : E \to \overline{\mathbb{C}}$, with $\eta(E \cap D) \subset \mathbb{C}$, has an extension in $\overline{A}(D)$.
[We refer to [9] for the corresponding result for $A(D)$.]

One can also pose questions on the nature of the zero set of a function $f \in \overline{A}(D)$ of finite type. Also, what can be said about the nature of the set $\{z \in \overline{D} : f(z) = \infty e^{i0}\} = \{z \in \overline{D} : f(z) = +\infty\} = f^{-1}(+\infty)$, when f is of infinite type? This is related to the zero sets (in $\overline{D}$) of functions $\theta : \overline{D} \to \mathbb{R}$ which are continuous on $\overline{D}$ and harmonic in D.

The maximum principle does not hold in $\overline{A}(D)$. Indeed, consider the polynomials $f(z) = z$ and $g(z) = 2z$, which certainly belong to $A(D) \subset \overline{A}(D)$. Then, $d(f(z), g(z)) =$

$\frac{|z|}{(1+|z|)(1+2|z|)}$. For $|z| = 1$ we find $d(f(z), g(z)) = \frac{1}{6} < \frac{1}{3+2\sqrt{2}} = d\left(f\left(\frac{1}{\sqrt{2}}\right), g\left(\frac{1}{\sqrt{2}}\right)\right)$. However, we have the following:

THEOREM 4.1 *Let $f, g \in \overline{A}(D)$ and suppose that $f(\zeta) = g(\zeta)$ for all $\zeta \in T$. Then $f \equiv g$.*

Proof Consider the set $A = \{\zeta \in T : f(\zeta) \in \mathbb{C}^{\infty}\} = \{\zeta \in T : g(\zeta) \in \mathbb{C}^{\infty}\}$. If A has positive Lebesgue measure then both f and g are of infinite type. Write $f(z) = \infty e^{i\theta(z)}$ and $g(z) = \infty e^{i\phi(z)}$ where $\theta, \phi : \overline{D} \to \mathbb{R}$ are continuous functions on $\overline{D}$, harmonic in D. Since $f(\zeta) = g(\zeta)$ for all $\zeta \in T$, we conclude that $\theta(\zeta) = \phi(\zeta) + 2k\pi$ for all $\zeta \in T$, where k is an integer independent of $\zeta \in T$. This implies that $\theta(z) = \phi(z) + 2k\pi$ for all $z \in \overline{D}$ and, thus, $f \equiv g$.

Suppose now that A has Lebesgue measure zero. Then, f and g are both of finite type. Thus, $f(D) \subset \mathbb{C}$, $g(D) \subset \mathbb{C}$ and both f, g are holomorphic in D. Therefore, the function $f - g$ is holomorphic in D with zero limits on $T \setminus A$. Since $T \setminus A$ contains a compact set of positive Lebesgue measure, Privalov's Theorem ([6], p.84) implies $f \equiv g$. This completes the proof. □

Remark 4.2 Assume that $f, g \in \overline{A}(D)$ coincide on a compact set $E \subset T$ with positive Lebesgue measure. If f is of finite type then g is also of finite type and Privalov's Theorem ([6], p.84) implies $f \equiv g$. If, however, f and g are of infinite type, it may happen $f \neq g$. For example, set $\theta(\zeta) = 0$ on $\{\zeta \in T : \zeta = e^{it},\ 0 \leq t \leq \pi\}$ and consider two different continuous (real-valued) extensions θ_1, θ_2 on T. Extending θ_1 and θ_2 on $\overline{D}$, using the Poisson kernel, we find that the functions $f(z) = \infty e^{i\theta_1(z)}$ and $g(z) = \infty e^{i\theta_2(z)}$ belong to $\overline{A}(D)$, coincide on $\{e^{it}, 0 \leq t \leq \pi\}$ but $f \neq g$.

5. Some topological properties of $\overline{A}(D)$

We recall that $A(D) = \{f : \overline{D} \to \mathbb{C}$, continuous on $\overline{D}$ and holomorphic in $D\}$ is a Banach algebra if it is endowed with the usual supremum norm. Furthermore, $\widetilde{A}(D) = \{f : \overline{D} \to \widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, continuous on $\overline{D}$, $f(D) \subset \mathbb{C}$, $f_{|D}$ holomorphic$\} \cup \{f : f \equiv \infty\}$ is a complete metric space if it is endowed with the metric $\widetilde{\chi}$ given by (see [1,2])

$$\widetilde{\chi}(f, g) = \sup_{|z| \leq 1} \chi(f(z), g(z)), \quad f, g \in \widetilde{A}(D).$$

Finally, $\overline{A}(D)$ is naturally endowed with the metric

$$\overline{d}(f, g) = \sup_{|z| \leq 1} d(f(z), g(z)), \quad f, g \in \overline{A}(D).$$

PROPOSITION 5.1 *The metric space $(\overline{A}(D), \overline{d})$ is complete. The disc algebra $A(D)$ is an open and dense subset of $\overline{A}(D)$. The relative topology of $A(D)$ from $\overline{A}(D)$ coincides with the usual topology of $A(D)$.*

Proof Consider the set $\mathcal{B} = \left\{f : \overline{D} \to \overline{\mathbb{C}}\right\}$ endowed with the metric

$$\beta(f, g) = \sup_{|z| \leq 1} d(f(z), g(z)), \quad f, g \in \mathcal{B}.$$

Since $(\overline{\mathbb{C}}, d)$ is complete, it follows that $(\mathcal{B}, \beta)$ is complete. According to Theorem 3.1, $\overline{A}(D)$ is the closure in $\mathcal{B}$ of the set of polynomials. Thus, $\overline{A}(D)$ is a closed subset of the complete metric space $\mathcal{B}$. It follows that $\overline{A}(D)$ is also complete. Let $f \in A(D)$; then $f(\overline{D})$ is a compact subset of $\mathbb{C}$ and, obviously, the compact sets $f(\overline{D})$ and $\mathbb{C}^{\infty}$ are disjoint. Thus, $\operatorname{dist}\Big(f(\overline{D}), \mathbb{C}^{\infty}\Big) = \delta > 0$. It is easily seen that if a function $g \in \overline{A}(D)$ satisfies $\overline{d}(f, g) < \delta$, then $g(\overline{D}) \subset \mathbb{C}$ and hence, $g \in A(D)$. Therefore, $A(D)$ is an open subset of $\overline{A}(D)$. It is also dense because it contains the set of polynomials which is dense, according to Theorem 3.1.

Let $f, f_n \in A(D)$ $(n = 1, 2, \ldots)$ and assume that $f_n \to f$, as $n \to \infty$, in $A(D)$. From Lemma 2.1 we easily see that $f_n \to f$, as $n \to \infty$, in $\overline{A}(D)$. Conversely, assume that $f_n \to f$, as $n \to \infty$, in $\overline{A}(D)$, for some functions $f, f_n \in A(D)$ $(n = 1, 2, \ldots)$. Then, the set $f(\overline{D})$ is a compact subset of $\mathbb{C}$ which is disjoint from the compact set $\mathbb{C}^{\infty}$, so that $\operatorname{dist}(f(\overline{D}), \mathbb{C}^{\infty}) = \delta > 0$. Thus, for $n \geq n_0$, we have $f_n(\overline{D}) \subset \{w \in \overline{\mathbb{C}} : \operatorname{dist}(w, f(\overline{D})) \leq \delta/2\} = E$, say, which is a compact subset of $\mathbb{C}$. On E the usual Euclidean metric and the metric d are uniformly equivalent. It follows that $f_n \to f$, as $n \to \infty$, in $A(D)$. This completes the proof. □

Consider now the function $\mathbb{F} : \overline{A}(D) \to \widetilde{A}(D)$ defined by $\mathbb{F}(f)(z) = \Phi(f(z))$, i.e.

$$\overline{A}(D) \ni f \mapsto \mathbb{F}(f) = \Phi \circ f \in \widetilde{A}(D),$$

(see Lemma 2.2 and (2.2) for the definition of the map $\Phi : \overline{\mathbb{C}} \to \widetilde{\mathbb{C}}$). According to Lemma 2.2 the function $\mathbb{F}$ is continuous.

A set in a complete metric space is said residual if it contains a dense G_δ set. By Baire's theorem dense G_δ sets are exactly the denumerable intersections of open dense sets. Under this terminology we have the following result.

Corollary 5.1 *The set $\mathbb{F}(\overline{A}(D))$ is a dense subset of $\widetilde{A}(D)$; in fact it is residual.*

Proof Since $\mathbb{F}(\overline{A}(D))$ contains the set of polynomials, it is dense in $\widetilde{A}(D)$.[1,2] Also, $\mathbb{F}(\overline{A}(D))$ contains $A(D)$ which is open and dense in $\widetilde{A}(D)$.[1,2] Thus, $\mathbb{F}(\overline{A}(D))$ is residual in $\widetilde{A}(D)$. □

Also, it is easily seen that the elements of finite type of $\overline{A}(D)$ form an open dense subset, say $\overline{A}_{\circ}(D)$, of $\overline{A}(D)$. It follows that the elements of infinite type of $\overline{A}(D)$ form the closed subset $\overline{A}_{\infty}(D) = \overline{A}(D) \setminus \overline{A}_{\circ}(D)$, which is of the first category. Since $\overline{A}_{\infty}(D)$ is closed, it is a G_δ set.

In the following proposition we use some notation from [10].

Proposition 5.2 *Let η be any Hausdorff measure function. The set of all $f \in \overline{A}_{\circ}(D)$ such that $\Lambda_\eta(E_f) = 0$ is dense and G_δ in $\overline{A}(D)$, where $E_f = \{\zeta \in T : f(\zeta) \notin f(D)\}$.*

The proof is similar to the proof of Proposition 4.3 of [1,2], the only difference being that one has to consider $f^{-1}(\mathbb{C}^{\infty})$ in place of $f^{-1}(\infty)$.

Next, we define $\mathcal{Y} = \{f \in \overline{A}(D) : f(D) \subset f(T)\} \subset \overline{A}(D)$ and $\mathcal{W} = \{f \in \overline{A}(D) : f(T) = \overline{\mathbb{C}}\} \subset \overline{A}(D)$. Arguments similar to those given in Proposition 4.5 of [1,2] show that $\mathcal{Y}$ is a non-empty closed subset of $\overline{A}(D)$ of the first category. With a proof similar to the proof of Proposition 4.6 in [1,2], we can show that $\mathcal{W}$ is also a closed subset of $\overline{A}(D)$ of the first category, but we do not know if $\mathcal{W}$ is non-empty. However, if we assume that every compact set $K \subset T$ with zero Lebesgue measure is a compact of interpolation for $\overline{A}(D)$, then we can show that $\mathcal{W} \neq \emptyset$. Indeed, let $K \subset T$ be a Cantor-type set with Lebesgue measure zero. It is well known that there exists a continuous surjection $\phi : K \to [0, 1]$. Let $\Gamma : [0, 1] \to \overline{D}$ be a Peano curve with $\Gamma([0, 1]) = \overline{D}$. Finally, let $L : \overline{D} \to \overline{\mathbb{C}}$ be a homeomorphism. Then, $L \circ \Gamma \circ \phi$ is continuous on K with $(L \circ \Gamma \circ \phi)(K) = \overline{\mathbb{C}}$. Therefore, the assumption that K is a compact of interpolation for $\overline{A}(D)$ implies that there exists an $f \in \overline{A}(D)$ such that $f_{|K} = L \circ \Gamma \circ \phi$. This f belongs to $\mathcal{W}$.

6. Concluding remarks and questions

In the previous sections, we considered uniform approximation by polynomials on the compact set $\overline{D}$. However, we can also consider uniform approximation on other compact sets with respect to the metric D. Also, the approximating functions do not necessarily have to be polynomials.

Proposition 6.1 *Let $L \subset \mathbb{C}$ be a compact set and let $z_0 \in L^0$. We assume that for every boundary point $\zeta \in \partial L$ the segment $[z_0, \zeta]$ satisfies $[z_0, \zeta] \setminus \{\zeta\} \subset L^0$. Then, the uniform limits, with respect to the metric d, of polynomials on L are exactly the functions $f : L \to \overline{\mathbb{C}}$ of the following two types:*

(a) *The first type (the finite type) contains the continuous functions $f : L \to \overline{\mathbb{C}}$ with $f(L^0) \subset \mathbb{C}$ such that $f_{|L^0}$ is holomorphic.*
(b) *The second type (the infinite type) contains the continuous functions $f : L \to \mathbb{C}^{\infty}$ of the form $f(z) = \infty e^{i\theta(z)}$, where the function $\theta : L \to \mathbb{R}$ can be chosen to be continuous on L and harmonic in L^0.*

For the proof, we may assume $z_0 = 0$ and imitate the proof for the case $L = \overline{D}$. The difference is that when we approximate f on the compact set rL, $0 < r < 1$, we have to use Runge's Theorem rather than considering the Taylor expansion of f. Since the approximation is uniform on rL with respect to the Euclidean distance on $\mathbb{C}$, Lemma 2.1 shows that it is also uniform with respect to the metric d.

Theorem 6.1 *Let $f : T \to \overline{\mathbb{C}}$ be any continuous function. Then, there exists a sequence of trigonometric polynomials converging to f uniformly on T with respect to the metric d.*

Proof Let $\epsilon > 0$. We wish to find a complex-valued trigonometric polynomial $Q = Q_\epsilon$ such that $d(f(\zeta), Q(\zeta)) < \epsilon$ for all $\zeta \in T$. According to Lemma 2.3, for any $R > 0$ the composition $\Phi_R \circ f : T \to \mathbb{C}$ is continuous on T. Fix a large enough $R > 0$ so that $d(f(\zeta), \Phi_R(f(\zeta))) < \frac{\epsilon}{2}$ for all $\zeta \in T$. Now, since $\Phi_R \circ f$ takes only finite complex values, it can be uniformly approximated on T by a trigonometric polynomial Q with respect to the usual Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$. According to Lemma 2.1, the approximation remains uniform for the metric d. Thus, we have found a trigonometric polynomial Q such that $d(Q(\zeta), \Phi_R(f(\zeta))) < \frac{\epsilon}{2}$ for all $\zeta \in T$. The triangle inequality now yields the desired result. □

We mention here that we do not know what is the set of the uniform limits of polynomials, with respect to the metric d, on a circle.

Let now I be a compact segment in $\mathbb{C}$ or, more generally, a homeomorphic image of the segment [0, 1] in $\mathbb{C}$. Then, the uniform limits of the polynomials on I, with respect to the metric d, are exactly all continuous functions $f : I \to \overline{\mathbb{C}}$. The proof is similar to that of Theorem 6.1, with the difference that we make use of the classical Mergelyan's Theorem to approximate $\Phi_R \circ f$ by a (complex) polynomial. This is possible because $I^0 = \emptyset$ and $\mathbb{C} \setminus I$ is connected.

Fix now I to be the compact interval $[-1, 1]$. So far, we have seen approximations of $\overline{\mathbb{C}}$-valued functions by complex-valued polynomials. However, when a continuous function $f : [-1, 1] \to \overline{\mathbb{C}}$ is $\overline{\mathbb{R}}$-valued, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \subset \overline{\mathbb{C}}$, it is reasonable to approximate it, if possible, by real-valued polynomials, with respect to the metric d. For the same reasoning, any χ-continuous $\widetilde{\mathbb{R}}$-valued function $f : [-1, 1] \to \widetilde{\mathbb{R}} \subset \widetilde{\mathbb{C}}$, where $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, should be approximated by real-valued polynomials with respect to the metric χ. According to [1,2], any continuous function $f : [-1, 1] \to \widetilde{\mathbb{C}}$, and hence, any continuous function $f : [-1, 1] \to \widetilde{\mathbb{R}}$, can be uniformly approximated by *complex-valued* polynomials with respect to the metric χ; however, the approximating polynomials need not be real and, sometimes, they cannot be real. Consider, for example, the χ-continuous function $f : [-1, 1] \to \widetilde{\mathbb{R}}$, given by

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \in [-1, 0) \cup (0, 1], \\ \infty, & \text{if } x = 0. \end{cases}$$

Although there exist polynomial approximations for this f, it is easily seen that the approximating polynomials cannot be real-valued – the above function is χ-continuous and not d-continuous. In fact, one can show that a function $f : [-1, 1] \to \widetilde{\mathbb{R}}$ can be uniformly approximated by *real-valued* polynomials, with respect to the metric χ, if and only if it is of the form $f = \Phi \circ g$ for some d-continuous function $g : [-1, 1] \to \overline{\mathbb{R}}$; here, the map $\Phi : \overline{\mathbb{R}} \to \widetilde{\mathbb{R}}$ is the restriction on $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$ of the map Φ defined in (2.2). Also, it is easy to see that a function $f : [-1, 1] \to \overline{\mathbb{R}}$ can be uniformly approximated by *real-valued* polynomials, with respect to the metric d, if and only if it is d-continuous. In other words, uniform real polynomial approximations (with respect to the metric d) can be found for a function $f : [-1, 1] \to [-\infty, +\infty]$ if and only if for each $x_0 \in [-1, 1]$, $\lim_{x \to x_0} f(x) = f(x_0) \in [-\infty, +\infty]$. If this is true, then the same real polynomials approximate $\Phi \circ f$ in the χ-metric. For example, the function $f : [-1, 0) \cup (0, 1] \to \mathbb{R}$ with $f(x) = \frac{1}{x^2}$ (for $x \in [-1, 1]$, $x \neq 0$) can be extended, in an obvious manner, to a d-continuous function on $[-1, 1]$ (setting $f(0) = +\infty$) and to a χ-continuous function on $[-1, 1]$ (setting $f(0) = \infty$). It follows that this f can be uniformly approximated by real polynomials with respect to the metric d (and, hence, also with respect to χ).

Finally, it is natural to ask about the uniform limits of polynomials on L, with respect to the metric d, when L is a compact subset of $\mathbb{C}$ with connected complement. Specifically, we have the following:

Question 3 Let $L \subset \mathbb{C}$ be a compact set with connected complement. Let $f : L \to \overline{\mathbb{C}}$ be a continuous function, such that for every component V of L^0, the following holds: either $f(V) \subset \mathbb{C}$ and $f_{|V}$ is holomorphic, or $f(V) \subset \mathbb{C}^{\infty}$ and f is of the form $f(z) = \infty e^{i\theta(z)}$ for all $z \in V$, where the real-valued function θ is harmonic in V. Does there exist a sequence of (complex-valued) polynomials converging to f uniformly on L with respect to the metric d?

The existence of such a sequence of polynomials would lead to an extension of the classical Mergelyan Theorem in the case of the metric d; we refer to [11] for the classical Mergelyan's Theorem.

We notice that the converse is true and the proof is the same as the one given here for the particular case $L = \overline{D}$. Indeed, in the proof of Theorem 3.1, we have only used the fact that D is a simply connected domain; this is the case for every component V of L^0.

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