Optimal moment inequalities for order statistics from nonnegative random variables

by

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Abstract

We obtain the best possible upper bounds for the moments of a single order statistic from independent, non-negative random variables, in terms of the population mean. The main result covers the independent identically distributed case. Furthermore, the case of the sample minimum for merely independent (not necessarily identically distributed) random variables is treated in detail.

Key-words and phrases: order statistics; optimal moment bounds; nonnegative random variables; sample minimum; reliability systems.

1 Introduction

The investigation of the behavior of expectations of order statistics in a random sample has a long history, since the order statistics have several applications in statistics and reliability. The earliest results in this direction are those by Placket (1947), concerning the sample range, followed by the well-known papers by Hartley and David (1954) and Gumbel (1954), regarding the expected extremes. At those years, a pioneer paper by Moriguti (1953) established a powerful projection method, making possible to evaluate tight expectation bounds for the non-extreme order statistics in terms of the population mean and variance. Since then, a large number of generalizations extensions and improvements have been found, including linear estimators from dependent samples (Arnold and Groeneveld (1979); Rychlik (1992, 1993a, 1993b, 1998); Balakrishnan (1990); Gascuel and Caraux (1992); Papadatos (2001a); Papadatos and Rychlik (2004); Miziula and Navarro (2018)), record values and kth records (Raqab (2004); Raqab and Rychlik (2002)) as well as distribution bounds (Caraux and Gascuel (1992); Papadatos (2001b), Okolewski (2015)), to mention a few. The reader is referred to the monographs by Arnold and Balakrishnan (1989), Rychlik (2001) and Ahsanullah and Raqab (2006) for a comprehensive presentation on characterizations and bounds through order statistics and records.

Beyond the well-developed theory on expectation bounds for order statistics and records, the corresponding theory to other moments does not seem to have receive much attention. Of course, some exceptions exist concerning variances; see, e.g., Papadatos (1995), Jasiński and Rychlik (2012, 2016), Rychlik (2008, 2014). The purpose of the present work is to obtain tight upper bounds for the moments of a single order statistic from a nonnegative population. These bounds are useful at least for reliability systems, since, as is well-known, the kth order statistic, $X_{k:n}$, represents the time-to-failure in a $(n+1-k)$-out-of-n system – clearly, the individual components cannot have negative lifetimes, hence the assumption of nonnegativity is natural for this kind of systems.

In general, let $X_1,\ldots,X_n$ be n iid (independent identically distributed) copies of the random variable (rv) X and consider the corresponding order statistics $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$. It is well-known that if X is integrable then the same is true for any order statistic $X_{i:n}$ (for all n and i). Moreover, an old result by P.K. Sen (1959) showed that the condition

$$\mathbb{E}|X|^{\delta} < \infty \quad \text{for some} \;\; \delta \in (0,1]$$

is sufficient for

$$\mathbb{E}|X_{i:n}| < \infty \quad \text{for all } i \text{ with } \frac{1}{\delta} \leqslant i \leqslant n+1-\frac{1}{\delta}.$$

It is natural to look at similar conditions when X is nonnegative (cf. Papadatos, 1997), since this is the case for several applications including k-out-of-n systems. Hence, the main purpose of the present work is to obtain best possible bounds for the moments of a single order statistic from non-negative populations, in terms of the population mean.

The paper is organized as follows. In Section 2 we provide results on the existence of moments of a single order statistic in the general (not necessarily identically distributed) non-negative independent case. Section 3 presents tight upper bounds for the moments of the sample minimum in the general independent case, which represents the lifetime of a serial system with possibly different components. The main results are given in Section 4, providing tight upper bounds for the moments of order statistics in terms of the population mean in the independent, identically distributed non-negative case.

2 Existence of moments in the independent case

The results of the present section concern the existence of moments in the more general case where the X_i's are merely independent. First we have the following result.

Theorem 1 *If $X_1,\ldots,X_n$ are non-negative independent rv's with $\mathbb{E}X_i = \mu_i \in (0,\infty)$ then*

$$\mathbb{E}(X_{k:n})^{n+1-k} \leqslant \sum_{1\leqslant i_1<\cdots<i_{n+1-k}\leqslant n} \mu_{i_1}\cdots\mu_{i_{n+1-k}}, \quad k=1,\ldots,n. \tag{1}$$

The equality in (1) *is attainable for* $k = 1$*, and is best possible and non-attainable for* $k \geqslant 2$.

Proof: Observe that

$$\begin{aligned} (X_{k:n})^{n+1-k} &\leqslant X_{k:n} \cdots X_{n:n} \\ &\leqslant \sum_{1 \leqslant i_1 < \cdots < i_{n+1-k} \leqslant n} X_{i_1:n} \cdots X_{i_{n+1-k}:n} \\ &= \sum_{1 \leqslant i_1 < \cdots < i_{n+1-k} \leqslant n} X_{i_1} \cdots X_{i_{n+1-k}}. \end{aligned}$$

Hence, taking expectations and using the fact that the X_i's are independent, we deduce (1). We shall now verify that for $k \geqslant 2$ the equality is non-attainable. Indeed, if $k \geqslant 2$, the above sum contains at least two summands. Let

$$Y_1 = (X_{k:n})^{n+1-k}, \quad Y_2 = X_{k:n} \cdots X_{n:n}, \quad Y_3 = \sum_{1 \leqslant i_1 < \cdots < i_{n+1-k} \leqslant n} X_{i_1:n} \cdots X_{i_{n+1-k}:n},$$

so that $Y_1 \leqslant Y_2 \leqslant Y_3$. Assuming equality in (1), that is, $\mathbb{E} Y_1 = \mathbb{E} Y_3$, we see that $\mathbb{E}(Y_3 - Y_2) = 0$. Therefore, taking expectations to the obvious inequalities $0 \leqslant (X_{1:n})^{n+1-k} \leqslant X_{1:n} \cdots X_{n+1-k:n} \leqslant Y_3 - Y_2$ (the last one is valid because $k \geqslant 2$), we obtain

$$0 \leqslant \mathbb{E}(X_{1:n})^{n+1-k} \leqslant \mathbb{E}(Y_3 - Y_2) = 0.$$

Hence, $X_{1:n} = 0$ with probability (w.p.) 1. However, this fact is impossible, since

$$\mathbb{P}(X_{1:n} > 0) = \mathbb{P}(X_1 > 0, \ldots, X_n > 0) = \prod_{j=1}^{n} \mathbb{P}(X_j > 0) > 0,$$

because $\mathbb{E} X_i > 0$.

We now examine the case of equality. For $k = 1$, (1) reads as

$$\mathbb{E}(X_{1:n})^n \leqslant \mu_1 \cdots \mu_n,$$

and it is readily verified that the independent rv's X_i with $\mathbb{P}(X_i = M) = \mu_i/M = 1 - \mathbb{P}(X_i = 0)$ (with $M \geqslant \max\{\mu_i\}$) attain the equality, since $\mathbb{P}(X_{1:n} = M) = \mu_1 \cdots \mu_n / M^n = 1 - \mathbb{P}(X_{1:n} = 0)$. We finally show that inequality (1) is best possible for $k \geqslant 2$. Indeed, fix $M \geqslant \max_i\{\mu_i\}$ and, as before, consider independent two-valued rv's X_i with

$$\mathbb{P}(X_i = 0) = 1 - \frac{\mu_i}{M}, \quad \mathbb{P}(X_i = M) = \frac{\mu_i}{M},$$

so that $\mathbb{E} X_i = \mu_i$ for all i. It is easy to see that

$$\begin{aligned} \mathbb{P}(X_{k:n} = M) &= \mathbb{P}(\text{at least } n+1-k \text{ among } X_1, \ldots, X_n \text{ are equal to } M) \\ &\geqslant \mathbb{P}(\text{exactly } n+1-k \text{ among } X_1, \ldots, X_n \text{ are equal to } M) \\ &= \sum_{1 \leqslant i_1 < \cdots < i_{n+1-k} \leqslant n} \frac{\mu_{i_1} \cdots \mu_{i_{n+1-k}}}{M^{n+1-k}} \prod_{j \in S(i_1, \ldots, i_{n+1-k})} \left(1 - \frac{\mu_j}{M}\right), \end{aligned}$$

where $S(i_1,\dots,i_{n+1-k}) = \{1,\dots,n\}\setminus\{i_1,\dots,i_{n+1-k}\}$. The smallest term in the product is at least $1-\frac{\max_i\{\mu_i\}}{M}$, hence,

$$\mathbb{P}(X_{k:n}=M) \geqslant \left(1-\frac{\max_i\{\mu_i\}}{M}\right)^{k-1}\frac{1}{M^{n+1-k}}\sum_{1\leqslant i_1<\cdots<i_{n+1-k}\leqslant n}\mu_{i_1}\cdots\mu_{i_{n+1-k}}.$$

It follows that

$$\begin{aligned}\mathbb{E}(X_{k:n})^{n+1-k} &= M^{n+1-k}\,\mathbb{P}(X_{k:n}=M)\\ &\geqslant \left(1-\frac{\max_i\{\mu_i\}}{M}\right)^{k-1}\sum_{1\leqslant i_1<\cdots<i_{n+1-k}\leqslant n}\mu_{i_1}\cdots\mu_{i_{n+1-k}}\\ &\to \sum_{1\leqslant i_1<\cdots<i_{n+1-k}\leqslant n}\mu_{i_1}\cdots\mu_{i_{n+1-k}},\quad \text{as } M\to\infty,\end{aligned}$$

and the proof is complete. □

Corollary 1 *If $\mu_1=\cdots=\mu_n=\mu>0$ (in particular, if the X_i's are iid), the best possible upper bound is given by*

$$\mathbb{E}(X_{k:n})^{n+1-k} \leqslant \binom{n}{k-1}\mu^{n+1-k},\quad k=1,\dots,n,$$

and it is attainable only in the case $k=1$.

Corollary 2 *If $X_1,\dots,X_n$ are non-negative independent rv's with $\mathbb{E}X_i<\infty$ then $\mathbb{E}(X_{k:n})^{n+1-k}<\infty$. That is, $X_{1:n}$ has finite n-th moment, $X_{2:n}$ has finite $(n-1)$-th moment, ..., $X_{n:n}$ has finite first moment.*

Finally, a converse to Theorem 1 reads as follows.

Theorem 2 *Given $\mu_1,\dots,\mu_n>0$, there are non-negative independent rv's $X_1,\dots,X_n$ with $\mathbb{E}X_i=\mu_i$ for all i and $\mathbb{E}(X_{k:n})^{n+1-k+\delta}=\infty$ for all $k\in\{1,\dots,n\}$ and for any $\delta>0$. Moreover, if $\mu_1=\cdots=\mu_n$, the rv's $X_1,\dots,X_n$ can be chosen to be iid.*

Proof: For $\mu>0$ consider the function

$$R_\mu(x)=\begin{cases}1, & \text{if } x\leqslant \mu/2,\\ \dfrac{1}{\frac{2x}{\mu}(1+\log\frac{2x}{\mu})^2}, & \text{if } x\geqslant \mu/2.\end{cases}$$

It is easy to check that $R_\mu(x)$ is a reliability function of an rv, Y_μ, say, that is, $Y_\mu\sim F_\mu=1-R_\mu$. Obviously, Y_μ is supported in $(\frac{\mu}{2},\infty)$ and, moreover, $\lambda Y_\mu\sim F_{\lambda\mu}$, $\lambda>0$; hence, $Y_\mu\overset{\mathrm{d}}{=}\mu Y_1$, where $\overset{\mathrm{d}}{=}$ denotes equality in distribution. Furthermore,

$$\mathbb{E}Y_1=\int_0^\infty R_1(x)dx=\frac{1}{2}+\int_{1/2}^\infty\frac{1}{2x(1+\log 2x)^2}dx=\frac{1}{2}+\frac{1}{2}\int_0^\infty\frac{1}{(1+t)^2}dt=1,$$

where we made use of the substitution $\log 2x = t$. For any $\alpha > 0$, a similar calculation yields

$$\mathbb{E}(Y_1)^\alpha = \alpha \int_0^\infty x^{\alpha-1} R_1(x)dx = \frac{1}{2^\alpha} + \frac{\alpha}{2^\alpha}\int_0^\infty \frac{e^{-(1-\alpha)t}}{(1+t)^2}dt;$$

note that this formula holds even if $(Y_1)^\alpha$ is non-integrable (see, e.g., Jones and Balakrishnan 2002). Hence, $\mathbb{E}(Y_1)^\alpha < \infty$ if and only if $\alpha \in (0,1]$.

Without loss of generality assume that $0 < \mu_1 \leqslant \cdots \leqslant \mu_n$ and consider the independent rv's $X_1, \ldots, X_n$ with $X_i \stackrel{\mathrm{d}}{=} \mu_i Y_1$, $i = 1, \ldots, n$. It is clear that the X_i's are iid if and only if the μ_i's are all equal. Moreover, consider the iid rv's $Z_1, \ldots, Z_n$ with $Z_i = \frac{\mu_1}{\mu_i} X_i$, $i = 1, \ldots, n$. Since the function $(x_1, \ldots, x_n) \mapsto x_{k:n}(x_1, \ldots, x_n)$ is non-decreasing in its arguments and $Z_i \leqslant X_i$, we have

$$Z_{k:n} \leqslant X_{k:n}, \quad k = 1, \ldots, n.$$

Hence, it suffices to show that $\mathbb{E}(Z_{k:n})^{n+1-k+\delta} = \infty$ for $\delta > 0$. To this end, observe that the Z_i's are iid from F_{μ_1} and

$$\mathbb{P}(Z_{k:n} > x) = \sum_{j=n+1-k}^{n} \binom{n}{j} R_{\mu_1}(x)^j F_{\mu_1}(x)^{n-j} \geqslant \binom{n}{k-1} R_{\mu_1}(x)^{n+1-k} F_{\mu_1}(x)^{k-1};$$

the lower bound is just the first term of the sum. Therefore, since $F_{\mu_1}(x) > \frac{1}{2}$ for $x \geqslant \mu_1$, we have

$$\begin{aligned}
\mathbb{E}(Z_{k:n})^{n+1-k+\delta} &= (n+1-k+\delta)\int_0^\infty x^{n-k+\delta}\,\mathbb{P}(Z_{k:n} > x)dx \\
&\geqslant \frac{n+1-k+\delta}{2^{k-1}}\binom{n}{k-1}\int_{\mu_1}^\infty x^{n-k+\delta} R_{\mu_1}(x)^{n+1-k}dx \\
&= \frac{(n+1-k+\delta)(\mu_1)^{n+1-k+\delta}}{2^{n+\delta}}\binom{n}{k-1}\int_{\log 2}^\infty \frac{e^{\delta t}}{(1+t)^{2(n+1-k)}}dt \\
&= \infty,
\end{aligned}$$

completing the proof. □

3 Moment bounds for the minimum in the independent case

Through this section we assume that $X_1, \ldots, X_n$ are independent, non-negative rv's with finite means $\mathbb{E}X_i = \mu_i > 0$ $(i = 1, \ldots, n)$, and we set $X_{1:n} = \min\{X_1, \ldots, X_n\}$. Our purpose is to derive best possible upper bounds for the moments of $X_{1:n}$, and to provide the populations that attain the bounds. Reformulating Theorem 1 for $k = 1$ we get:

Theorem 3 *The random variable $X_{1:n}$ has finite n-th moment and, moreover, the inequality*

$$\mathbb{E}(X_{1:n})^n \leqslant \mu_1 \cdots \mu_n \tag{2}$$

is valid, with equality if and only

$$\mathbb{P}(X_i = M) = \frac{\mu_i}{M} = 1 - \mathbb{P}(X_i = 0), \quad i = 1, \ldots, n,$$

for some $M \geqslant \max_i\{\mu_i\}$.

Proof: The inequality (2) is the same as (1) for $k = 1$, obtained by taking expectations to the obvious (deterministic) inequality

$$(X_{1:n})^n \leqslant X_1 \cdots X_n.$$

Moreover, observe that $X_1 \cdots X_n = X_{1:n} \cdots X_{n:n}$, where $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ are the corresponding order statistics of $X_1, \ldots, X_n$. Thus, for the equality to hold, it is necessary and sufficient that

$$\mathbb{E}\left[X_{1:n}\Big(X_{2:n} \cdots X_{n:n} - (X_{1:n})^{n-1}\Big)\right] = 0.$$

This implies the relation

$$\mathbb{P}\Big(\{X_{1:n} = 0\} \cup \{X_1 = \cdots = X_n > 0\}\Big) = 1. \tag{3}$$

Let $p_i = \mathbb{P}(X_i > 0) > 0$ (since $\mu_i > 0$). It follows that $\mathbb{P}(X_{1:n} > 0) = \prod_{i=1}^n p_i > 0$ and, from (3),

$$\mathbb{P}(X_1 = \cdots = X_n > 0 \mid X_{1:n} > 0) = 1. \tag{4}$$

Define now the independent rv's Y_i with $Y_i \overset{\mathrm{d}}{=} (X_i \mid X_i > 0)$; that is, $F_{Y_i}(y) = (F_{X_i}(y) - 1 + p_i)/p_i$, $y \geqslant 0$. Then (4) reads as $\mathbb{P}(Y_1 = \cdots = Y_n) = 1$ and, by the independence of Y_i, it follows that we can find a constant $M > 0$ such that $\mathbb{P}(Y_i = M) = 1$ for all i; hence, $\mathbb{P}(X_i = 0) + \mathbb{P}(X_i = M) = 1$. From $\mu_i = \mathbb{E}X_i = Mp_i$ we get $p_i = \mu_i/M$ and, thus, $M \geqslant \max_i\{\mu_i\}$. As a final check, it is easily verified that the rv's X_i with $\mathbb{P}(X_i = M) = \mu_i/M = 1 - \mathbb{P}(X_i = 0)$ attain the equality in (2). □

Remark 1 Applied to the case of sample minimum ($k = 1$), Theorem 2 yields the following result: For given strictly positive numbers $\mu_1, \ldots, \mu_n$ ($n \geqslant 2$) we can find independent non-negative rv's $X_1, \ldots, X_n$ such that

$$\mathbb{E}X_i = \mu_i \ (i = 1, \ldots, n) \quad \text{and} \quad \mathbb{E}(X_{1:n})^{n+\delta} = \infty \ \text{ for all } \delta \in (0, \infty).$$

Furthermore, if $\mu_1 = \cdots = \mu_n$, the rv's $X_1, \ldots, X_n$ can be chosen to be iid.

Therefore, in the particular iid case we have obtained the following corollary.

Corollary 3 *Let X be a nonnegative rv with $\mathbb{E}X = \mu \in (0, \infty)$, and assume that $X_1, \ldots, X_n$ ($n \geqslant 2$) are iid rv's distributed like X. Then, the random variable $X_{1:n} = \min\{X_1, \ldots, X_n\}$ has finite n-th moment. Moreover, the inequality*

$$\mathbb{E}(X_{1:n})^n \leqslant \mu^n \tag{5}$$

holds true, and the equality is attained if and only if $\mathbb{P}(X = \mu/p) = p = 1 - \mathbb{P}(X = 0)$ for some $p \in (0, 1]$.

Remark 2 Corollary 3 can be viewed in another form, as follows: If X is a nonnegative rv with $\mathbb{E}X^{1/n} < \infty$ for some n, then the minimum $X_{1:N}$ is integrable for all $N \geqslant n$, and, moreover,

$$\mathbb{E}X_{1:N} \leqslant \mathbb{E}X_{1:n} \leqslant \left(\mathbb{E}X^{1/n}\right)^n, \quad N \geqslant n.$$

Note that, for any $N \geqslant n$, the upper bound $\left(\mathbb{E}X^{1/n}\right)^n$ is best possible for $\mathbb{E}X_{1:N}$; this happens because we did not exclude a degenerate rv X.

Remark 3 The result of Corollary 3 cannot be extended to any higher moment; see Remark 1. A somewhat more direct computation is as follows: consider the rv X with df

$$F(x) = 1 - \frac{\mathrm{e}}{x(\log x)^2}, \quad x \geqslant \mathrm{e}.$$

Using the well-known formula

$$\mathbb{E}X = \int_0^\infty (1 - F(t))dt, \tag{6}$$

which is valid for any nonnegative rv, it is easily seen that $\mathbb{E}X = 2\mathrm{e} < \infty$. Also, since for any $\delta \in (0, \infty)$ and $n \in \{2, 3, \ldots\}$, the df of $(X_{1:n})^{n+\delta}$ is $1 - \left(1 - F(t^{1/(n+\delta)})\right)^n$, $t \geqslant 0$, (6) yields

$$\mathbb{E}(X_{1:n})^{n+\delta} = (n + \delta) \int_0^\infty x^{n+\delta-1}(1 - F(x))^n dx \geqslant \int_{\mathrm{e}}^\infty \frac{(n + \delta)\mathrm{e}^n}{x^{1-\delta}(\log x)^{2n}} dx = \infty.$$

It is clear that for arbitrary $\mu > 0$, the rv $Y = \mu X/(2\mathrm{e}) \geqslant 0$ has mean μ, and the rv $(Y_{1:n})^{n+\delta}$ is non-integrable for any $\delta \in (0, \infty)$ and for any $n \in \{2, 3, \ldots\}$.

Remark 4 Corollary 3 yields the best upper bound for any fractional moment of $X_{1:n}$ as follows: Since $x \mapsto x^p$ ($0 < p < 1$) is concave in $[0, \infty)$, Jensen (or Lyapounov) inequality, combined with (5), yields

$$\mathbb{E}(X_{1:n})^\alpha = \mathbb{E}\left[((X_{1:n})^n)^{\alpha/n}\right] \leqslant \Big(\mathbb{E}(X_{1:n})^n\Big)^{\alpha/n} \leqslant \mu^\alpha, \quad 0 < \alpha \leqslant n.$$

The upper bound μ^α is clearly best possible, since it is attained (uniquely, unless $\alpha = n$) by a degenerate X at μ.

It became clear from Remarks 1 and 3 that we cannot hope for finiteness of moments of order higher than n (for $X_{1:n}$) without additional assumptions. It is, thus, desirable, to derive upper bounds for lower moments. Indeed, turning to the general independent case we have the following result.

Theorem 4 *Let $X_1, \ldots, X_n$ be independent, non-negative, rv's with finite expectations $\mathbb{E}X_i = \mu_i > 0$ and, without loss of generality, assume that $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$. Then, for every $\alpha \in (0, n]$ we have*

$$\mathbb{E}(X_{1:n})^\alpha \leqslant \mu_1 \cdots \mu_{k-1}(\mu_k)^{\alpha-k+1}, \quad \alpha \in (k-1, k], \ k = 1, \ldots, n. \tag{7}$$

The bound is best possible, since the equality is attained by the independent rv's X_i with

$$\begin{array}{ll} \mathbb{P}(X_i = \mu_k) = \frac{\mu_i}{\mu_k} = 1 - \mathbb{P}(X_i = 0), & i = 1, \ldots, k, \\ \mathbb{P}(X_i = \mu_i) = 1, & i = k+1, \ldots, n, \end{array} \tag{8}$$

where $k \in \{1, \ldots, n\}$ is the unique integer such that $k - 1 < \alpha \leqslant k$.

Proof: Since it is easily checked that the rv's in (8) attain the equality in (7), we proceed to verify the inequality (7). To this end, fix $\alpha \in (k-1, k]$ and consider the following deterministic inequalities, valid for $X_i \geqslant 0$:

$$\begin{array}{rcl} \min\{X_1, \ldots, X_n\} & \leqslant & X_1 \\ \min\{X_1, \ldots, X_n\} & \leqslant & X_2 \\ & \vdots & \\ \min\{X_1, \ldots, X_n\} & \leqslant & X_{k-1} \\ (\min\{X_1, \ldots, X_n\})^{\alpha-(k-1)} & \leqslant & (X_k)^{\alpha-(k-1)}. \end{array}$$

Multiplying, we get

$$(X_{1:n})^\alpha \leqslant X_1 \cdots X_{k-1}(X_k)^{\alpha-k+1}. \tag{9}$$

Hence, taking expectations in (9) and using independence, we deduce the inequality

$$\mathbb{E}(X_{1:n})^\alpha \leqslant \mu_1 \cdots \mu_{k-1}\mathbb{E}(X_k)^{\alpha-k+1}.$$

Finally, since $0 < \alpha - k + 1 \leqslant 1$, the function $x \mapsto x^{\alpha-k+1}$ is concave in $[0, \infty)$, and Jensen (or Lyapounov) inequality yields (7). $\square$

Notice that the inequality (9) shows that $(X_{1:n})^\alpha$ (for $\alpha \in (k-1, k]$) is integrable even if $\mu_{k+1} = \infty$. This is explained from the fact that $X_{1:n} \leqslant \min\{X_1, \ldots, X_k\}$ and, by Theorem 3, $X_{1:k}$ has finite k-th (hence α-th) moment. Note also that (7) yields Remark 4 for the iid case.

4 Moment bounds for the independent, identically distributed case

In this section we assume that $X_1, \ldots, X_n$ are iid non-negative rv's distributed like X, and $\mathbb{E}X = \mu$ is nonzero and finite. Our purpose is to derive the best possible upper bounds for the moments $\mathbb{E}(X_{k:n})^\alpha$, for $\alpha > 0$; however, due to Theorems 1 and 2, we see that the problem is meaningful only for $\alpha \in (0, n+1-k]$. Note that Papadatos (1997) treats the case $\alpha = 1$, which, as we shall see below, is a boundary case between $\alpha < 1$ and $\alpha > 1$. Also, we shall obtain the populations that attain the equality in the bounds.

We first prove some auxiliary results. In the following lemma we consider the usual Borel space

$$L^1(0,1) = \left\{ g : (0,1) \to \mathbb{R},\ g \text{ is Borel},\ \int_0^1 |g(t)| dt < \infty \right\},$$

where two functions that differ at a set of Lebesgue measure zero are considered as equal.

Lemma 1 *Let $\alpha > 1$. If a function $g : (0,1) \to [0,\infty)$ is nondecreasing and belongs to $L^1(0,1)$, then*

$$\alpha \int_0^1 (1-t)^{\alpha-1} g(t)^\alpha dt \leqslant \left(\int_0^1 g(t) dt \right)^\alpha, \tag{10}$$

and the equality holds if either g is constant or

$$g(t) = \begin{cases} 0, & 0 < t \leqslant t_0, \\ \theta, & t_0 < t < 1, \end{cases}$$

for some $t_0 \in (0,1)$ and some $\theta > 0$.

Proof: It is obvious that any constant function attains the equality in (10), and the same is true for the function $g(t) = \theta I_{(t_0,1)}(t)$, resulting in the identity $\theta^\alpha (1-t_0)^\alpha = (\theta(1-t_0))^\alpha$. To prove the inequality, assume first that g is simple nonnegative and nondecreasing, that is,

$$g(t) = \begin{cases} \delta_1, & t \in (0, s_1], \\ \delta_1 + \delta_2, & t \in (s_1, s_2], \\ \vdots & \\ \delta_1 + \delta_2 + \cdots + \delta_k, & t \in (s_{k-1}, 1), \end{cases}$$

where $\delta_i \geqslant 0$ and $0 < s_1 < \cdots < s_{k-1} < 1$. Note that the value of g at the end-points do not affect the value of the integrals, so we have assumed that g is left-continuous. With the notation $s_0 = 0$, $s_k = 1$, it is easily seen that

$$\int_0^1 g(t) dt = \sum_{j=1}^k (s_j - s_{j-1})(\delta_1 + \cdots + \delta_j) = \sum_{j=1}^k (1 - s_{j-1}) \delta_j.$$

Similarly,

$$\alpha\int_0^1 (1-t)^{\alpha-1}g(t)^\alpha dt = (\delta_1)^\alpha + \sum_{j=1}^{k-1}(1-s_j)^\alpha\left[(\delta_1+\cdots+\delta_{j+1})^\alpha - (\delta_1+\cdots+\delta_j)^\alpha\right].$$

Therefore, (10) for simple functions reduces to the inequality

$$\left(\sum_{j=1}^{k}(1-s_{j-1})\delta_j\right)^\alpha - (\delta_1)^\alpha - \sum_{j=1}^{k-1}(1-s_j)^\alpha\left[(\delta_1+\cdots+\delta_{j+1})^\alpha - (\delta_1+\cdots+\delta_j)^\alpha\right] \geqslant 0 \tag{11}$$

for $k \geqslant 2$, $0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ and $\delta_j \geqslant 0$ $(j = 1, \ldots, k)$. Note that $k = 1$ leads to the constant function $g \equiv \delta_1$, and in this case we have equality in (10). We shall show (11) using induction on k. For $k = 2$, (11) reads as

$$f(\delta_2) := [\delta_1 + (1-s_1)\delta_2]^\alpha - (\delta_1)^\alpha - (1-s_1)^\alpha\left[(\delta_1+\delta_2)^\alpha - (\delta_1)^\alpha\right] \geqslant 0.$$

However, this follows easily because $f(0) = 0$ and

$$f'(\delta_2) = \alpha(1-s_1)\left[(\delta_1 + (1-s_1)\delta_2)^{\alpha-1} - ((1-s_1)\delta_1 + (1-s_1)\delta_2)^{\alpha-1}\right] \geqslant 0,$$

since $\alpha > 1$ and $\delta_1 + (1-s_1)\delta_2 \geqslant (1-s_1)\delta_1 + (1-s_1)\delta_2$. Assuming that (11) holds for some $k \geqslant 2$, we shall verify it for $k+1$. Set

$$\begin{aligned} f(\delta_{k+1}) \;:=\; & \left(\sum_{j=1}^{k+1}(1-s_{j-1})\delta_j\right)^\alpha - (\delta_1)^\alpha \\ & - \sum_{j=1}^{k}(1-s_j)^\alpha\left[(\delta_1+\cdots+\delta_{j+1})^\alpha - (\delta_1+\cdots+\delta_j)^\alpha\right]. \end{aligned}$$

It is easily seen that $f(0) \geqslant 0$, due to the induction argument. Moreover,

$$f'(\delta_{k+1}) = \alpha(1-s_k)\left[\left(\sum_{j=1}^{k+1}(1-s_{j-1})\delta_j\right)^{\alpha-1} - \left(\sum_{j=1}^{k+1}(1-s_k)\delta_j\right)^{\alpha-1}\right] \geqslant 0,$$

since $\alpha > 1$ and $\sum_{j=1}^{k+1}(1-s_{j-1})\delta_j \geqslant \sum_{j=1}^{k+1}(1-s_k)\delta_j$. Hence, (11) is valid for simple functions. If $g \geqslant 0$ is an arbitrary nondecreasing right-continuous function, we can use standard arguments to find simple functions g_n such that $g_n \nearrow g$ pointwise. Then, $\alpha(1-t)^{\alpha-1}g_n(t) \nearrow \alpha(1-t)^{\alpha-1}g(t)$ and, by Lebesgue's monotone convergence theorem and (11) we get

$$\begin{aligned} & \alpha\int_0^1 (1-t)^{\alpha-1}g(t)^\alpha dt = \lim_n \left\{\alpha\int_0^1 (1-t)^{\alpha-1}g_n(t)^\alpha dt\right\} \\ & \leqslant \lim_n \left(\int_0^1 g_n(t)dt\right)^\alpha = \left(\lim_n \int_0^1 g_n(t)dt\right)^\alpha = \left(\int_0^1 g(t)dt\right)^\alpha, \end{aligned}$$

completing the proof. □

Corollary 4 *Let F be a distribution function of a nonnegative rv X with mean $\mu \in (0,\infty)$. Then, for all $\alpha > 1$,*

$$\alpha \int_0^\infty x^{\alpha-1}(1-F(x))^\alpha dx \leqslant \left(\int_0^\infty (1-F(x))dx\right)^\alpha,$$

and the equality is attained if X assumes two values, one of which is zero.

Proof: It is trivial to check that any distribution function (df) $F(x)$ that is constant in $[0,x_0)$ and equals to one in $[x_0,\infty)$ attains the equality. We now verify the inequality. Note that for integral values of $\alpha > 1$, say $\alpha = n$, it becomes obvious if we consider the rv $X_{1:n} = \min\{X_1,\ldots,X_n\}$, where $X_1,\ldots,X_n$ are iid with df equal F. Then,

$$\mathbb{E}(X_{1:n})^n \leqslant \mathbb{E}(X_{1:n}\cdots X_{n:n}) = \mathbb{E}(X_1\cdots X_n) = \mu^n,$$

and this inequality is equivalent to the desired one for $\alpha = n$. However, this simple argument is not sufficient to prove the result for non-integral values of $\alpha > 1$. In order to verify the inequality in its general form, let $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $0 < u < 1$, be the left-continuous inverse of F. Moreover, consider an rv Y_α with df $F_\alpha(x) = 1-(1-F(x))^\alpha$. It is easy to see that $F_\alpha^{-1}(u) = F^{-1}(1-(1-u)^{1/\alpha})$. Hence, from Lemma 1 with $g = F^{-1}$,

$$\mathbb{E}(Y_\alpha)^\alpha = \int_0^1 (F_\alpha^{-1}(u))^\alpha du = \alpha \int_0^1 (1-t)^{\alpha-1}(F^{-1}(t))^\alpha dt \leqslant \left(\int_0^1 F^{-1}(t)dt\right)^\alpha = \mu^\alpha,$$

where we used the substitution $t = 1-(1-u)^{1/\alpha}$. Moreover, since

$$\mathbb{E}(Y_\alpha)^\alpha = \alpha \int_0^\infty x^{\alpha-1}(1-F(x))^\alpha dx \quad \text{and} \quad \mu^\alpha = \left(\int_0^\infty (1-F(x))dx\right)^\alpha,$$

the result is proved. □

Lemma 2 *Let $n \geqslant 3$, $k \in \{2,\ldots,n-1\}$ and $\alpha \in [1, n+1-k)$. Let also*

$$G_{k:n}(x) = \sum_{j=k}^n \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leqslant x \leqslant 1, \tag{12}$$

be the df of $U_{k:n}$ from an iid sample $U_1,\ldots,U_n$ from the standard uniform df, and

$$g_{k:n}(x) = G'_{k:n}(x) = \frac{1}{B(k,n+1-k)} x^{k-1}(1-x)^{n-k}, \quad 0 < x < 1, \tag{13}$$

the corresponding Beta density function. Then,

$$1 - G_{k:n}(x) \leqslant A_{k:n}(\alpha)(1-x)^\alpha, \quad 0 \leqslant x \leqslant 1, \tag{14}$$

where

$$A_{k:n}(\alpha) = \frac{1 - G_{k:n}(\rho)}{(1-\rho)^\alpha} \tag{15}$$

and $\rho = \rho_{k:n}(\alpha)$ *is the unique solution to the equation*

$$\alpha(1 - G_{k:n}(\rho)) = (1-\rho) g_{k:n}(\rho), \quad 0 < \rho < 1. \tag{16}$$

The equality in (14) *is attained if and only if* $x = \rho$ *or* $x = 1$.

Proof: Define the function

$$h(x) = \frac{1 - G_{k:n}(x)}{(1-x)^\alpha}, \quad 0 \leqslant x \leqslant 1.$$

where the value at $x = 1$ is defined by continuity: $h(1) = 0$. We have $h(0) = 1$, $h(1) = 0$ and

$$h'(x) = (1-x)^{-\alpha-1}\Big(\alpha(1 - G_{k:n}(x)) - (1-x) g_{k:n}(x)\Big), \quad 0 < x < 1.$$

Setting $t(x) = \alpha(1 - G_{k:n}(x)) - (1-x) g_{k:n}(x)$, we calculate

$$t'(x) = \frac{g_{k:n}(x)}{x}\Big((n-\alpha)x - (k-1)\Big).$$

This shows that $t(x)$ is strictly decreasing in $(0, \frac{k-1}{n-\alpha}]$ and strictly increasing in $[\frac{k-1}{n-\alpha}, 1)$. Since $t(0) > 0$ and $t(1) = 0$, the function t has a global negative minimum at $\frac{k-1}{n-\alpha}$ and, therefore, there exists a $\rho \in (0, \frac{k-1}{n-\alpha})$ such that $t(x) > 0$ for $x \in (0, \rho)$ and $t(x) < 0$ for $x \in (\rho, 1)$. Since $h'(x) = t(x)/(1-x)^{\alpha+1}$, we see that the function h is strictly increasing in $(0, \rho)$ and strictly decreasing in $(\rho, 1)$, attaining its global maximum at $x = \rho$, where ρ is the unique root of (16). □

Remark 5 Due to (16), we can write $A_{k:n}(\alpha) = \dfrac{g_{k:n}(\rho)}{\alpha(1-\rho)^{\alpha-1}}$.

We can now state and prove the main result for the moments of the non-extreme order statistics.

Theorem 5 *Let* $X_1, \ldots, X_n$ $(n \geqslant 3)$ *be iid nonnegative rv's with mean* $\mu \in (0, \infty)$. *Then, for any* $k \in \{2, \ldots, n-1\}$ *and* $\alpha \in [1, n+1-k)$,

$$\mathbb{E}(X_{k:n})^\alpha \leqslant A_{k:n}(\alpha)\ \mu^\alpha, \tag{17}$$

where $A_{k:n}(\alpha)$ *is given by* (15). *The equality in* (17) *is attained if and only if* $\mathbb{P}(X_i = 0) = \rho$, $\mathbb{P}(X_i = \mu/(1-\rho)) = 1 - \rho$, *where* $\rho = \rho_{k:n}(\alpha)$ *is given by* (16).

Proof: If F is the df of the X_i's then $G_{k:n} \circ F$ is the df of $X_{k:n}$ (see David (1981), Arnold *et al* (2008), David and Nagaraja (2003)). Therefore,

$$\begin{aligned} \mathbb{E}(X_{k:n})^\alpha &= \alpha \int_0^\infty x^{\alpha-1}(1-G_{k:n}(F(x)))dx \\ &\leqslant A_{k:n}(\alpha)\alpha \int_0^\infty x^{\alpha-1}(1-F(x))^\alpha dx \\ &\leqslant A_{k:n}(\alpha)\left(\int_0^\infty (1-F(x))dx\right)^\alpha = A_{k:n}(\alpha)\ \mu^\alpha, \end{aligned}$$

where the first inequality follows from Lemma 2 and the second one from Corollary 4. In order to have equality in (17), it is necessary and sufficient that the set $\{F(x),\ 0 < x < \infty\}$ coincides with $\{\rho, 1\}$ (see (16) and Corollary 4). Therefore, X_1 assumes the value 0 w.p. ρ and a positive value x_0 w.p. $1-\rho$. Finally, the condition $\mathbb{E}X_1 = \mu$ shows that $x_0 = \mu/(1-\rho)$, completing the proof. □

Remark 6 Note that for $k = n$ the interval $[1, n+1-k)$ is empty, and so this case is not treated by Theorem 5. For $\alpha = 1$ the bounds coincide with the upper bounds given in Papadatos (1997), Theorem 2.1.

Example 1 For $k = 2$, $n \geqslant 3$, one finds

$$\rho = \frac{\alpha}{(n-1)(n-\alpha)} \quad \text{and} \quad A_{2:n}(\alpha) = \left(1 + \frac{\alpha}{n-\alpha}\right)^{n-\alpha}\left(1 - \frac{\alpha}{n-1}\right)^{n-1-\alpha},$$

$1 \leqslant \alpha < n-1$. It is easy to verify that

$$A_{2:n}(\alpha) = 1 + \frac{\alpha^2}{2n^2} + o(n^{-2}) \quad \text{as} \quad n \to \infty.$$

Closed forms can be found for $k = 3$ too; then (16) is reduced to a second degree polynomial equation (see Balakrishnan (1993)).

We now turn to the case $0 < \alpha < 1$, showing the following result.

Theorem 6 *Let $X_1, \ldots, X_n$ be iid nonnegative rv's with mean $\mu \in (0, \infty)$. If $n \geqslant 2$ and $k \in \{2, \ldots, n\}$, then*

$$\mathbb{E}(X_{k:n})^\alpha \leqslant A_{k:n}(\alpha)\ \mu^\alpha, \quad 0 < \alpha < 1, \tag{18}$$

where

$$A_{k:n}(\alpha) = \left(\int_0^1 \overline{g}_{k:n}(u)^{\frac{1}{1-\alpha}} du\right)^{1-\alpha} \tag{19}$$

and $\overline{g}_{k:n}$ is the derivative of $\overline{G}_{k:n}$, the greatest convex minorant of the function $G_{k:n}$ given in (12). *The equality in* (18) *is attained if and only if the inverse df of X_1 is given by*

$$F^{-1}(u) = \mu\ \overline{g}_{k:n}(u)^{\frac{1}{1-\alpha}} \Big/ \int_0^1 \overline{g}_{k:n}(t)^{\frac{1}{1-\alpha}} dt, \quad 0 < u < 1. \tag{20}$$

Proof: We shall apply a slight variation of the pioneer projection method due to Moriguti (1953). Since $X_{k:n} \stackrel{\mathrm{d}}{=} F^{-1}(U_{k:n})$ where $U_{k:n}$ is the k-th order statistic from the standard uniform df, we have (see Moriguti (1953), Rychlik (2001), Ahsanullah and Raqab (2006), Lemma 3.1.1)

$$\mathbb{E}(X_{k:n})^\alpha = \int_0^1 g_{k:n}(u)F^{-1}(u)^\alpha du \leqslant \int_0^1 \overline{g}_{k:n}(u)F^{-1}(u)^\alpha du,$$

by Moriguti's inequality (the function $(F^{-1})^\alpha$ is, clearly, non-decreasing). Applying Hölder's inequality,

$$\textstyle\int fg \leqslant \left(\int f^p\right)^{1/p}\left(\int g^q\right)^{1/q} \quad (p,q>1, \;\; 1/p+1/q=1),$$

to the last integral, with $f=\overline{g}_{k:n}$, $g=(F^{-1})^\alpha$, $p=1/(1-\alpha)>1$ and $q=1/\alpha>1$, we obtain the inequality

$$\int_0^1 \overline{g}_{k:n}(u)F^{-1}(u)^\alpha du \leqslant \left(\int_0^1 \overline{g}_{k:n}(u)^{\frac{1}{1-\alpha}}du\right)^{1-\alpha}\left(\int_0^1 F^{-1}(u)du\right)^{\alpha},$$

which verifies (18). We now examine the case of equality: it is well-known that for the Hölder inequality to hold as equality it is necessary and sufficient that $g^q = c\, f^p$ for some $c \geqslant 0$ (note that $f,g \geqslant 0$ in our case); that is, $F^{-1}(u) = c\;\overline{g}_{k:n}(u)^{\frac{1}{1-\alpha}}$. Taking into account the condition $\mathbb{E}X_1 = \mu$ we get

$$\mu = \int_0^1 F^{-1}(t)dt = c\int_0^1 \overline{g}_{k:n}(t)^{\frac{1}{1-\alpha}}dt.$$

Therefore, c is unique and, consequently, F is unique and its distribution inverse is given by (20). Finally, observe that with this choice of F^{-1}, the equality is also attained in Moriguti's inequality, because F^{-1} is constant in the interval where $\overline{G}_{k:n} < G_{k:n}$. □

Remark 7 It is known that for $n \geqslant 3$ and $k \in \{2,\ldots,n-1\}$,

$$\overline{g}_{k:n}(u) = g_{k:n}(\min\{u,\rho\}), \quad 0<u<1,$$

where $\rho = \rho_{k:n}$ is the unique root to the equation

$$1 - G_{k:n}(\rho) = (1-\rho)g_{k:n}(\rho), \quad 0<\rho<1,$$

and $g_{k:n}$, $G_{k:n}$ are given by (13) and (12), respectively (see, e.g., Rychlik (2001)).

Remark 8 For $k=n\geqslant 2$ the optimal bound (18) for the maximum reads as

$$\mathbb{E}(X_{n:n})^\alpha \leqslant n\Big(\frac{1-\alpha}{n-\alpha}\Big)^{1-\alpha}\mu^\alpha, \quad 0<\alpha<1.$$

This is so because $G_{n:n}(u) = \overline{G}_{n:n}(u) = u^n$ and $g_{n:n}(u) = \overline{g}_{n:n}(u) = nu^{n-1}$. Therefore, the optimal population is given by

$$F^{-1}(u) = \mu\Big(\frac{n-\alpha}{1-\alpha}\Big)u^{\frac{n-1}{1-\alpha}}, \quad 0 < u < 1,$$

and this corresponds to a power-type distribution function:

$$F(x) = \left(\frac{(1-\alpha)x}{(n-\alpha)\mu}\right)^{\frac{1-\alpha}{n-1}}, \quad 0 \leqslant x \leqslant \frac{n-\alpha}{1-\alpha}\mu.$$

It is worth pointing out that $\lim_{\alpha\nearrow 1} n\left(\frac{1-\alpha}{n-\alpha}\right)^{1-\alpha} = n$, yielding the best possible non-attainable bound $\mathbb{E}X_{n:n} \leqslant n\mu$ (see Corollary 1).

Example 2 Due to a result of Balakrishnan (1993), the value of $\rho_{2:n}$ can be calculated in a closed form. In fact, $\rho_{2:n} = 1/(n-1)^2$ and, consequently,

$$\overline{g}_{2:n}(u) = \begin{cases} n(n-1)u(1-u)^{n-2}, & 0 < u \leqslant \frac{1}{(n-1)^2}, \\ n(n-1)\rho_{2:n}(1-\rho_{2:n})^{n-2} = \frac{n^{n-1}(n-2)^{n-2}}{(n-1)^{2n-3}}, & \frac{1}{(n-1)^2} \leqslant u < 1. \end{cases}$$

Hence, for $n \geqslant 3$, (19) reads as

$$A_{2:n}(\alpha) = n(n-1)\left\{\rho_{2:n}^{\frac{1}{1-\alpha}}(1-\rho_{2:n})^{\frac{n-1-\alpha}{1-\alpha}} + \int_0^{\rho_{2:n}} u^{\frac{1}{1-\alpha}}(1-u)^{\frac{n-2}{1-\alpha}}du\right\}^{1-\alpha}, \quad 0 < \alpha < 1.$$

This expression should be compared to the corresponding one in Example 1, highlighting the different nature of the cases $\alpha < 1$, $\alpha \geqslant 1$.

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