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On matrix variance inequalities ☆

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ABSTRACT

Olkin and Shepp [2005, A matrix variance inequality. J. Statist. Plann. Inference 130, 351–358] presented a matrix form of Chernoff's inequality for Normal and Gamma (univariate) distributions. We extend and generalize this result, proving Poincaré-type and Bessel-type inequalities, for matrices of arbitrary order and for a large class of distributions.

1. Introduction

Let Z be a standard Normal random variable and assume that $g_1,\ldots,g_p$ are absolutely continuous, real-valued, functions of Z, each with finite variance (with respect to Z). Olkin and Shepp (2005) presented a matrix extension of Chernoff (1981) variance inequality, which reads as follows:

Olkin and Shepp's inequality. If $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{g})$ is the covariance matrix of the random vector $\boldsymbol{g}=(g_1(Z),\ldots,g_p(Z))^t$, where 't' denotes transpose, and if $\mathbb{E}[g_i'(Z)]^2<\infty$ for all $i=1,2,\ldots,p$, then (see Olkin and Shepp, 2005, p. 352)

$$\boldsymbol{D}\le\boldsymbol{H},$$

where $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{g})=(\mathbb{E}[g_i'(Z)g_j'(Z)])_{p\times p}$, and the inequality is considered in the sense of Loewner ordering, that is, the matrix $\boldsymbol{H}-\boldsymbol{D}$ is nonnegative definite.

In this note we extend and generalize this inequality for a large family of discrete and continuous random variables. Specifically, our results apply to any random variable X according to one of the following definitions (cf. Afendras et al., 2007).

Definition 1 (*Integrated Pearson family*). Let X be a random variable with density function f (w.r.t. Lebesgue measure on $\mathbb{R}$) and finite mean $\mu=\mathbb{E}(X)$. We say that X follows the Integrated Pearson distribution $\mathrm{IP}(\mu;\delta,\beta,\gamma)$, $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$, if there exists a quadratic polynomial $q(x)=\delta x^2+\beta x+\gamma$ (with $\beta,\delta,\gamma\in\mathbb{R}$, $|\delta|+|\beta|+|\gamma|>0$) such that

$$\int_{-\infty}^{x}(\mu-t)f(t)\,dt=q(x)f(x)\quad\text{for all } x\in\mathbb{R}. \tag{1}$$

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Definition 2 (*Cumulative Ord family*). Let X be an integer-valued random variable with finite mean μ and assume that $p(k)=\mathbb{P}(X=k)$, $k\in\mathbb{Z}$, is the probability function of X. We say that X follows the Cumulative Ord distribution $\mathrm{CO}(\mu;\delta,\beta,\gamma)$, $X\sim\mathrm{CO}(\mu;\delta,\beta,\gamma)$, if there exists a quadratic polynomial $q(j)=\delta j^2+\beta j+\gamma$ (with $\beta,\delta,\gamma\in\mathbb{R}$, $|\delta|+|\beta|+|\gamma|>0$) such that

$$\sum_{k\le j}(\mu-k)p(k)=q(j)p(j)\quad\text{for all } j\in\mathbb{Z}. \tag{2}$$

It is well known that the commonly used distributions are members of the above families, e.g. Normal, Gamma, Beta, F and t distributions belong to the Integrated Pearson family, while Poisson, Binomial, Pascal (Negative Binomial) and Hypergeometric distribution are members of the Cumulative Ord family. Therefore, the results of the present note also improve and unify the corresponding bounds for Beta random variables, given by Prakasa Rao (2006) and Wei and Zhang (2009). For multivariate variance inequalities involving the generalized Dirichlet distribution see, also, Chang and Richards (1999, Theorems 3.6 and 3.9).

2. Matrix variance inequalities of Poincaré-type and of Bessel-type

2.1. Continuous case

In this subsection we shall make use of the following notations.

Assume that $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ and denote by $q(x)=\delta x^2+\beta x+\gamma$ its quadratic polynomial. It is known that, under (1), the support $J=J(X)=\{x\in\mathbb{R}: f(x)>0\}$ is a (finite or infinite) open interval, say $J(X)=(\alpha,\omega)$—see Afendras et al. (2007, 2011).

For a fixed integer $n\in\{1,2,\ldots\}$ we shall denote by $\mathcal{H}^n(X)$ the class of functions $g:(\alpha,\omega)\to\mathbb{R}$ satisfying the following properties:

H_1: For each $k\in\{0,1,\ldots,n-1\}$, $g^{(k)}$ (with $g^{(0)}=g$) is an absolutely continuous function with derivative $g^{(k+1)}$.
H_2: For each $k\in\{0,1,\ldots,n\}$, $\mathbb{E}[q^k(X)(g^{(k)}(X))^2]<\infty$.

Also, for a fixed integer $n\in\{1,2,\ldots\}$ we shall denote by $\mathcal{B}^n(X)$ the class of functions $g:(\alpha,\omega)\to\mathbb{R}$ satisfying the following properties:

B_1: $\mathrm{Var}[g(X)]<\infty$.
B_2: For each $k\in\{0,1,\ldots,n-1\}$, $g^{(k)}$ (with $g^{(0)}=g$) is an absolutely continuous function with derivative $g^{(k+1)}$.
B_3: For each $k\in\{0,1,\ldots,n\}$, $\mathbb{E}[q^k(X)|g^{(k)}(X)|]<\infty$.

Since $\mathbb{E}^2[q^k(X)|g^{(k)}(X)|]\le\mathbb{E}[q^k(X)]\cdot\mathbb{E}[q^k(X)(g^{(k)}(X))^2]$ by the Cauchy–Schwarz inequality, it follows that $\mathcal{H}^n(X)\subseteq\mathcal{B}^n(X)$ whenever $\mathbb{E}|X|^{2n}<\infty$ (observe that H_2 (with $k=0$) yields B_1).

Consider now any p functions $g_1,\ldots,g_p\in\mathcal{H}^n(X)$ and set $\boldsymbol{g}=(g_1,g_2,\ldots,g_p)^{\mathrm t}$. Then, the following $p\times p$ matrices $\boldsymbol{H}_k=\boldsymbol{H}_k(\boldsymbol{g})$ are well-defined for $k=1,2,\ldots,n$:

$$\boldsymbol{H}_k=(h_{ij;k})\quad\text{where } h_{ij;k}:=\mathbb{E}[q^k(X)g_i^{(k)}(X)g_j^{(k)}(X)],\ i,j=1,2,\ldots,p. \tag{3}$$

Similarly, for any functions $g_1,\ldots,g_p\in\mathcal{B}^n(X)$, the following $p\times p$ matrices $\boldsymbol{B}_k=\boldsymbol{B}_k(\boldsymbol{g})$ are well-defined for $k=1,2,\ldots,n$:

$$\boldsymbol{B}_k=(b_{ij;k})\quad\text{where } b_{ij;k}:=\mathbb{E}[q^k(X)g_i^{(k)}(X)]\cdot\mathbb{E}[q^k(X)g_j^{(k)}(X)],\ i,j=1,2,\ldots,p. \tag{4}$$

Our first result concerns a class of Poincaré-type matrix variance bounds, as follows:

Theorem 1. *Let $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ and assume that $\mathbb{E}|X|^{2n}<\infty$ for some fixed integer $n\in\{1,2,\ldots\}$. Let $g_1,\ldots,g_p$ be arbitrary functions in $\mathcal{H}^n(X)$ and denote by $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{g})$ the dispersion matrix of the random vector $\boldsymbol{g}=\boldsymbol{g}(X)=(g_1(X),\ldots,g_p(X))^{\mathrm t}$. Also, denote by $\boldsymbol{S}_n=\boldsymbol{S}_n(\boldsymbol{g})$ the $p\times p$ matrix*

$$\boldsymbol{S}_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!\prod_{j=0}^{k-1}(1-j\delta)}\cdot\boldsymbol{H}_k,$$

where the matrices $\boldsymbol{H}_k$, $k=1,2,\ldots,n$, are defined by (3). *Then, the matrix*

$$\boldsymbol{A}_n=(-1)^n(\boldsymbol{D}-\boldsymbol{S}_n)$$

is nonnegative definite. Moreover, $\boldsymbol{A}_n$ is positive definite unless there exist constants $c_1,\ldots,c_p\in\mathbb{R}$, not all zero, such that the function $c_1g_1(x)+\cdots+c_pg_p(x)$ is a polynomial (*in x*) *of degree at most n.*

Proof. Fix $\boldsymbol{c}=(c_1,\ldots,c_p)^{\mathrm t}\in\mathbb{R}^p$ and define the function $h_{\boldsymbol{c}}(x)=\boldsymbol{c}^{\mathrm t}\cdot\boldsymbol{g}(x)=c_1g_1(x)+\cdots+c_pg_p(x)$. Since $g_1,\ldots,g_p\in\mathcal{H}^n(X)$ we see that the function $h_{\boldsymbol{c}}$ belongs to $\mathcal{H}^n(X)$ and, in particular, $h_{\boldsymbol{c}}(X)$ has finite variance and $\mathbb{E}[q^k(X)(h_{\boldsymbol{c}}^{(k)}(X))^2]<\infty$, $k=1,2\ldots,n$. Thus, we can make use of the inequality (see Afendras et al., 2007; Johnson, 1993; cf. Houdré and Kagan, 1995;

Papathanasiou, 1988)

$$(-1)^n[\mathrm{Var}h_{\boldsymbol c}(X)-S_n]\ge 0 \quad \text{where } S_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!\prod_{j=0}^{k-1}(1-j\delta)}\mathbb{E}[q^k(X)(h_{\boldsymbol c}^{(k)}(X))^2],$$

in which the equality holds if and only if $h_{\boldsymbol c}$ is a polynomial of degree at most n. It is well-known that Var $h_{\boldsymbol c}(X)=\boldsymbol c^t\boldsymbol D\boldsymbol c$ and it is easily seen that $\mathbb{E}[q^k(X)(h_{\boldsymbol c}^{(k)}(X))^2]=\boldsymbol c^t\boldsymbol H_k\boldsymbol c$ with $\boldsymbol H_k$ $(k=1,\ldots,n)$ as in (3). Thus, $S_n=\boldsymbol c^t\boldsymbol S_n\boldsymbol c$ and the preceding inequality takes the form

$$\boldsymbol c^t[(-1)^n(\boldsymbol D-\boldsymbol S_n)]\boldsymbol c\ge 0.$$

Since $\boldsymbol c\in\mathbb{R}^p$ is arbitrary it follows that the matrix $(-1)^n(\boldsymbol D-\boldsymbol S_n)$ is nonnegative definite. Clearly the inequality is strict for all $\boldsymbol c\in\mathbb{R}^p$ for which $h_{\boldsymbol c}\notin\mathrm{spam}[1,x,\ldots,x^n]$. □

Remark 1. (a) Olkin and Shepp's (2005) matrix inequalities are particular cases of Theorem 1 for $n=1$ and with X being a standard Normal or a Gamma random variable. For example, when $n=1$ and $X=Z\sim N(0,1)\equiv \mathrm{IP}(0;0,0,1)$ then $\boldsymbol S_1=\boldsymbol H_1=\boldsymbol H=(\mathbb{E}[g_i'(Z)g_j'(Z)])_{p\times p}$ and we get the inequality $\boldsymbol D\le\boldsymbol H$ (in the Loewner ordering). Moreover, for $X=Z$ and $n=2$ or 3, Theorem 1 yields the new matrix variance bounds $\boldsymbol H-\frac12\boldsymbol H_2\le\boldsymbol D$ and $\boldsymbol D\le\boldsymbol H-\frac12\boldsymbol H_2+\frac16\boldsymbol H_3$ where $\boldsymbol H_2=(\mathbb{E}[g_i''(Z)g_j''(Z)])_{p\times p}$ and $\boldsymbol H_3=(\mathbb{E}[g_i'''(Z)g_j'''(Z)])_{p\times p}$.

(b) Theorem 1 applies to Beta random variables. In particular, when $n=1$ and $X\sim$ Beta(a,b) (with density $f(x)\propto x^{a-1}(1-x)^{b-1}$, $0<x<1$, and parameters $a,b>0$) then $q(x)=x(1-x)/(a+b)$. Theorem 1 yields the inequality $\boldsymbol D\le\boldsymbol H$ (in the Loewner ordering) where $\boldsymbol H=(1/(a+b))(\mathbb{E}[X(1-X)g_i'(X)g_j'(X)])_{p\times p}$. This compares with Theorem 4.1 in Prakasa Rao (2006) and with Lemma 3.7 in Chang and Richards (1999); cf. Wei and Zhang (2009), Remark 1.

Next we show some similar Bessel-type matrix variance bounds. The particular case of a Beta random variable is covered by Theorem 1 of Wei and Zhang (2009).

Theorem 2. *Let $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ and assume that $\mathbb{E}|X|^{2n}<\infty$ for some fixed integer $n\in\{1,2,\ldots\}$. Let $g_1,\ldots,g_p$ be arbitrary functions in $\mathcal{B}^n(X)$ and denote by $\boldsymbol D=\boldsymbol D(\boldsymbol g)$ the dispersion matrix of the random vector $\boldsymbol g=\boldsymbol g(X)=(g_1(X),\ldots,g_p(X))^t$. Also, denote by $\boldsymbol L_n=\boldsymbol L_n(\boldsymbol g)$ the $p\times p$ matrix*

$$\boldsymbol L_n=\sum_{k=1}^{n}\frac{1}{k!\ \mathbb{E}[q^k(X)]\prod_{j=k-1}^{2k-2}(1-j\delta)}\cdot\boldsymbol B_k,$$

where the matrices $\boldsymbol B_k$, $k=1,2,\ldots,n$, are defined by (4). Then,

$$\boldsymbol L_n\le\boldsymbol D$$

in the Loewner ordering. Moreover, $\boldsymbol D-\boldsymbol L_n$ is positive definite unless there exist constants $c_1,\ldots,c_p\in\mathbb{R}$, not all zero, such that the function $c_1g_1(x)+\cdots+c_pg_p(x)$ is a polynomial (in x) of degree at most n.

Proof. Fix $\boldsymbol c=(c_1,\ldots,c_p)^t\in\mathbb{R}^p$ and, as in the previous proof, define the function $h_{\boldsymbol c}(x)=\boldsymbol c^t\cdot\boldsymbol g(x)=c_1g_1(x)+\cdots+c_pg_p(x)$. Since $g_1,\ldots,g_p\in\mathcal{B}^n(X)$ we see that the function $h_{\boldsymbol c}$ belongs to $\mathcal{B}^n(X)$. In particular, $h_{\boldsymbol c}(X)$ has finite variance and $\mathbb{E}[q^k(X)|h_{\boldsymbol c}^{(k)}(X)|]<\infty$, $k=1,2\ldots,n$. Thus, we can apply the inequality (see Afendras et al., 2011)

$$\mathrm{Var}h_{\boldsymbol c}(X)\ge L_n \quad \text{where } L_n=\sum_{k=1}^{n}\frac{\mathbb{E}^2[q^k(X)h_{\boldsymbol c}^{(k)}(X)]}{k!\ \mathbb{E}[q^k(X)]\prod_{j=k-1}^{2k-2}(1-j\delta)},$$

in which the equality holds if and only if $h_{\boldsymbol c}$ is a polynomial of degree at most n. Observe that Var $h_{\boldsymbol c}(X)=\boldsymbol c^t\boldsymbol D\boldsymbol c$ and $\mathbb{E}^2[q^k(X)|h_{\boldsymbol c}^{(k)}(X)|]=\boldsymbol c^t\boldsymbol B_k\boldsymbol c$ with $\boldsymbol B_k$ $(k=1,\ldots,n)$ as in (4). Thus, $L_n=\boldsymbol c^t\boldsymbol L_n\boldsymbol c$ and the preceding inequality takes the form

$$\boldsymbol c^t[\boldsymbol D-\boldsymbol L_n]\boldsymbol c\ge 0.$$

Since $\boldsymbol c\in\mathbb{R}^p$ is arbitrary it follows that the matrix $\boldsymbol D-\boldsymbol L_n$ is nonnegative definite. Clearly the inequality is strict for all $\boldsymbol c\in\mathbb{R}^p$ for which $h_{\boldsymbol c}\notin\mathrm{spam}[1,x,\ldots,x^n]$. □

2.2. Discrete case

In this subsection we shall make use of the following notations.

Assume that $X\sim\mathrm{CO}(\mu;\delta,\beta,\gamma)$. It is known (see Afendras et al., 2007, 2011) that, under (2), the support $J=J(X)=\{k\in\mathbb{Z}:p(k)>0\}$ is a (finite of infinite) interval of integers, say $J(X)=\{\alpha,\alpha+1,\ldots,\omega-1,\omega\}$. Write $q(x)=\delta x^2+\beta x+\gamma$ for the quadratic polynomial of X and let $q^{[k]}(x)=q(x)q(x+1)\cdots q(x+k-1)$ for $k=1,2,\ldots$ (with $q^{[0]}(x)\equiv 1$, $q^{[1]}(x)\equiv q(x)$). For any function $g:\mathbb{Z}\to\mathbb{R}$ we shall denote by $\Delta^k[g(x)]$ its k-th forward difference, i.e., $\Delta^k[g(x)]=\Delta[\Delta^{k-1}[g(x)]]$, $k=1,2,\ldots$, with $\Delta[g(x)]=g(x+1)-g(x)$ and $\Delta^0[g(x)]\equiv g(x)$.

For a fixed integer $n\in\{1,2,\ldots\}$ we shall denote by $\mathcal{H}_d^n(X)$ the class of functions $g:J(X)\to\mathbb{R}$ satisfying the following property:

HD_1: For each $k\in\{0,1,\ldots,n\}$, $\mathbb{E}[q^{[k]}(X)(\Delta^k[g(X)])^2]<\infty$.

Also, for a fixed integer $n \in \{1,2,\ldots\}$ we shall denote by $\mathcal{B}_d^n(X)$ the class of functions $g: J(X) \to \mathbb{R}$ satisfying the following properties:

BD_1: $\text{Var}[g(X)] < \infty$.
BD_2: For each $k \in \{0,1,\ldots,n\}$, $\mathbb{E}[q^k(X)|\Delta^k[g(X)]|] < \infty$.

Clearly, if $\mathbb{E}|X|^{2n} < \infty$ then $\mathcal{H}_d^n(X) \subseteq \mathcal{B}_d^n(X)$ (note that HD_1 (with $k=0$) yields BD_1). Indeed, since $\mathbb{P}[q^{[k]}(X) \geq 0] = 1$, the Cauchy–Schwarz inequality implies that $\mathbb{E}^2[q^{[k]}(X)|\Delta^k[g(X)]|] \leq \mathbb{E}[q^{[k]}(X)] \cdot \mathbb{E}[q^{[k]}(X)(\Delta^k[g(X)])^2]$.

Consider now any p functions $g_1,\ldots,g_p \in \mathcal{H}_d^n(X)$ and set $\boldsymbol{g} = (g_1,g_2,\ldots,g_p)^{\mathrm{t}}$. Then, the following $p \times p$ matrices $\boldsymbol{H}_k = \boldsymbol{H}_k(\boldsymbol{g})$ are well-defined for $k = 1,2,\ldots,n$:

$$\boldsymbol{H}_k = (h_{ij;k}) \quad \text{where } h_{ij;k} := \mathbb{E}[q^{[k]}(X)\Delta^k[g_i(X)]\Delta^k[g_j(X)]],\ i,j = 1,2,\ldots,p. \tag{5}$$

Similarly, for any functions $g_1,\ldots,g_p \in \mathcal{B}_d^n(X)$, the following $p \times p$ matrices $\boldsymbol{B}_k = \boldsymbol{B}_k(\boldsymbol{g})$ are well-defined for $k = 1,2,\ldots,n$:

$$\boldsymbol{B}_k = (b_{ij;k}) \quad \text{where} \quad b_{ij;k} := \mathbb{E}[q^{[k]}(X)\Delta^k[g_i(X)]] \cdot \mathbb{E}[q^{[k]}(X)\Delta^k[g_j(X)]], \quad i,j = 1,2,\ldots,p. \tag{6}$$

The matrix variance inequalities for the discrete case are summarized in the following theorem; its proof, being the same as in the continuous case, is omitted.

Theorem 3. *Let $X \sim \text{CO}(\mu;\delta,\beta,\gamma)$ and assume that $\mathbb{E}|X|^{2n} < \infty$ for some fixed integer $n \in \{1,2,\ldots\}$.*

(a) *Let $g_1,\ldots,g_p$ be arbitrary functions in $\mathcal{H}_d^n(X)$ and denote by $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{g})$ the dispersion matrix of the random vector $\boldsymbol{g} = \boldsymbol{g}(X) = (g_1(X),\ldots,g_p(X))^{\mathrm{t}}$. Also, denote by $\boldsymbol{S}_n = \boldsymbol{S}_n(\boldsymbol{g})$ the $p \times p$ matrix*

$$\boldsymbol{S}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!\prod_{j=0}^{k-1}(1-j\delta)} \cdot \boldsymbol{H}_k,$$

where the matrices $\boldsymbol{H}_k$, $k = 1,2,\ldots,n$, are defined by (5). *Then, the matrix*

$$\boldsymbol{A}_n = (-1)^n(\boldsymbol{D} - \boldsymbol{S}_n)$$

is nonnegative definite. Moreover, $\boldsymbol{A}_n$ is positive definite unless there exist constants $c_1,\ldots,c_p \in \mathbb{R}$, not all zero, and a polynomial $P_n: \mathbb{R} \to \mathbb{R}$, of degree at most n, such that $\mathbb{P}[c_1g_1(X) + \cdots + c_pg_p(X) = P_n(X)] = 1$.

(b) *Let $g_1,\ldots,g_p$ be arbitrary functions in $\mathcal{B}_d^n(X)$ and denote by $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{g})$ the dispersion matrix of the random vector $\boldsymbol{g} = \boldsymbol{g}(X) = (g_1(X),\ldots,g_p(X))^{\mathrm{t}}$. Also, denote by $\boldsymbol{L}_n = \boldsymbol{L}_n(\boldsymbol{g})$ the $p \times p$ matrix*

$$\boldsymbol{L}_n = \sum_{k=1}^{n} \frac{1}{k!\ \mathbb{E}[q^{[k]}(X)]\prod_{j=k-1}^{2k-2}(1-j\delta)} \cdot \boldsymbol{B}_k,$$

where the matrices $\boldsymbol{B}_k$, $k = 1,2,\ldots,n$, are defined by (6). *Then,*

$$\boldsymbol{L}_n \leq \boldsymbol{D}$$

in the Loewner ordering. Moreover, $\boldsymbol{D} - \boldsymbol{L}_n$ is positive definite unless there exist constants $c_1,\ldots,c_p \in \mathbb{R}$, not all zero, and a polynomial $P_n: \mathbb{R} \to \mathbb{R}$, of degree at most n, such that $\mathbb{P}[c_1g_1(X) + \cdots + c_pg_p(X) = P_n(X)] = 1$.

[*It should be noted that the k-th term in the sum defining the matrix $\boldsymbol{S}_n$ or the matrix $\boldsymbol{L}_n$, above, should be treated as the null matrix, $\boldsymbol{0}_{p\times p}$, whenever $\mathbb{E}[q^{[k]}(X)] = 0$.*]

As an example consider the case where $X \sim \text{Poisson}(\lambda)$ with probability function $p(k) = e^{-\lambda}\lambda^k/k!$, $k = 0,1,\ldots$ $(\lambda > 0)$. Then $X \sim \text{CO}(\lambda;0,0,\lambda)$ so that $q(x) \equiv \lambda$. It follows that $\boldsymbol{H}_k = \lambda^k(\mathbb{E}[\Delta^k[g_i(X)]\Delta^k[g_j(X)]])_{p\times p}$ and $\boldsymbol{B}_k = \lambda^{2k}(\mathbb{E}[\Delta^k[g_i(X)]] \cdot \mathbb{E}[\Delta^k[g_j(X)]])_{p\times p}$. Thus, for $n=1$ and $p=2$ Theorem 3(a) yields the matrix inequality

$$\begin{pmatrix} \text{Var}[g_1] & \text{Cov}[g_1,g_2] \\ \text{Cov}[g_1,g_2] & \text{Var}[g_2] \end{pmatrix} \leq \lambda \begin{pmatrix} \mathbb{E}[(\Delta[g_1])^2] & \mathbb{E}[\Delta[g_1]\Delta[g_2]] \\ \mathbb{E}[\Delta[g_1]\Delta[g_2]] & \mathbb{E}[(\Delta[g_2])^2] \end{pmatrix}.$$

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