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# Bounds on expectations of $L$-statistics from without replacement samples

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**Abstract**

Consider a simple random sample taken without replacement from a finite ordered population, where each element of the population has equal probability to be chosen in the sample. In the present paper, the best possible bounds for the expectation of any linear combination of order statistics based on the sample are derived in terms of the coefficients of the combination, the population mean and a central absolute moment. The results are specified for the trimmed means and the quasi-ranges.


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## 1. Introduction

Let $\Pi=\{x_1 \leqslant \cdots \leqslant x_N\}$ be a finite ordered population and consider a simple random sample $X_1,\ldots,X_n$, drawn without replacement from $\Pi$. Of course $n \leqslant N$ (usually, $n$ is much smaller than $N$), and the case $n = N$ leads to an exhaustive (trivial) sample from $\Pi$. Let us consider the ordered sample $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$, obtained from the simple random sample $X_1,\ldots,X_n$. Recently, Balakrishnan et al. (2003) derived the best possible bounds for $\mathbb{E}[X_{i:n}]$, $1 \leqslant i \leqslant n$, and for $\mathbb{E}[X_{n:n} - X_{1:n}]$, when the population mean $\mu$ and the population variance $\sigma_2^2$ were fixed.

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In Section 2 we extend the results of Balakrishnan et al. (2003) and derive sharp upper and lower expectation bounds for any $L$-statistic of the form

$$L = L(c_1, \ldots, c_n) = \sum_{i=1}^{n} c_i X_{i:n}, \tag{1.1}$$

with arbitrary real constants $c_i$. The bounds are expressed in terms of the coefficients $c_i$ and the population mean and the central absolute moments of various orders. A more detailed discussion of the cases of trimmed means $T(i,j) = (j - i + 1)^{-1} \sum_{r=i}^{j} X_{r:n}$, $1 \leqslant i \leqslant j \leqslant n$, (single order statistics, in particular), and quasi-ranges $Q(i,j) = X_{j:n} - X_{i:n}$, $1 \leqslant i < j \leqslant n$, (spacings, in particular) is presented in Section 3.

Bounds on expectations of $L$-statistics were studied in various models. In the classic i.i.d. case, the optimal mean-variance bounds for the sample maxima were established independently by Gumbel (1954), and Hartley and David (1954), and those for other order statistics by Moriguti (1953). Nagaraja (1981) analyzed extreme trimmed means (selection differentials), and the other ones were evaluated by Danielak and Rychlik (2003a). A general method of deriving bounds for arbitrary $L$-statistics was presented in Rychlik (1998). Some sharper bounds for the i.i.d. samples of restricted nonparametric families of distributions were considered in Danielak (2003), Danielak and Rychlik (2003b), Gajek and Rychlik (1998), López-Blázquez (1998, 2000), Papadatos (1997) and Rychlik (2002). For the dependent identically distributed samples, Arnold (1980, 1985) evaluated the sample maximum and range, whereas the general $L$-statistics were considered in Rychlik (1993b). The results are closely related to deterministic bounds considered by numerous authors, and reviewed in Arnold and Balakrishnan (1989) and Rychlik (1998). Sharper bounds for order statistics and trimmed means of restricted families were given in Gajek and Rychlik (1996) and Rychlik (2001a). A comprehensive presentation of bounds on expectations of $L$-statistics can be found in Rychlik (2001b). The importance of the without replacement drawing schemes from finite populations follows from the fact that this is the most natural sampling method in practical applications.

## 2. Expectation bounds for $L$-statistics

Let $\Pi = \{x_1 \leqslant \cdots \leqslant x_N\}$ be any ordered finite population of size $N$ and $X_1, \ldots, X_n$, $n \leqslant N$, be a simple random sample drawn without replacement from $\Pi$. Note that the observations are dependent identically distributed and the population mean $\mu = \mathbb{E}[X_1]$ and population central absolute moments $\sigma_p^p = \mathbb{E}|X_1 - \mu|^p$ are finite, and given by

$$\mu = \frac{1}{N} \sum_{k=1}^{N} x_k \quad \text{and} \quad \sigma_p^p = \frac{1}{N} \sum_{k=1}^{N} |x_k - \mu|^p, \quad 1 \leqslant p < \infty. \tag{2.1}$$

Natural scale units, denoted by $\sigma_p$, are defined as the $p$th roots of the $p$th moments $\sigma_p^p$. We also set

$$\sigma_\infty = \operatorname{ess\,sup} |X_1 - \mu| = \max\{\mu - x_1, x_N - \mu\}. \tag{2.2}$$

Let $\Pi_0=\{1,\dots,N\}$ be the standard uniform discrete population, and consider a simple random sample $U_1,\dots,U_n$, $n\leqslant N$, drawn from $\Pi_0$ without replacement. It is known (see Balakrishnan et al. (2003, Lemma 2.1)) that the ordered samples $X_{1:n}\leqslant\cdots\leqslant X_{n:n}$ from $\Pi$ and $U_{1:n}\leqslant\cdots\leqslant U_{n:n}$ from $\Pi_0$ are related through

$$(X_{1:n},\dots,X_{n:n})\overset{\mathrm{d}}{=}(g(U_{1:n}),\dots,g(U_{n:n})),$$

where $g:\Pi_0\to\Pi$ is a nondecreasing function given by $g(k)=x_k$, $k=1,\dots,N$. To avoid trivialities in the sequel, we assume that $x_1<x_N$. Since any $L$-statistic of the form (1.1) satisfies

$$L\overset{\mathrm{d}}{=}\sum_{i=1}^{n}c_i g(U_{i:n}),$$

and

$$p_i(k)=p_i(k;n,N)=\mathbb{P}(U_{i:n}=k)=\binom{k-1}{i-1}\binom{N-k}{n-i}\Big/\binom{N}{n} \tag{2.3}$$

(with the convention $\binom{a}{b}=0$ for $a<b$), we get

$$\mathbb{E}[L]=\sum_{i=1}^{n}c_i\sum_{k=1}^{N}x_k\,p_i(k)=\sum_{k=1}^{N}x_k\sum_{i=1}^{n}c_i\,p_i(k)=\sum_{k=1}^{N}C_kx_k,\quad \text{say.}$$

Note that

$$\sum_{k=1}^{N}C_k=\sum_{i=1}^{n}c_i\sum_{k=1}^{N}p_i(k)=\sum_{i=1}^{n}c_i,$$

so that the finite sequences $c_1,\dots,c_n$ and $C_1,\dots,C_N$ sum up to the same constant. For our purposes we need the following definition.

**Definition 2.1.** Define $(D_1,\dots,D_N)\in\mathbb{R}^N$ to be the $l^2$-projection of $(C_1,\dots,C_N)$ onto the convex cone of nondecreasing sequences in $\mathbb{R}^N$.

Note that the cone is a closed convex subset of $\mathbb{R}^N$, and the projection exists and is finite. The numbers $D_1,\dots,D_N$ are uniquely determined by $C_1,\dots,C_N$, and the latter depend on $c_1,\dots,c_n$. Two alternative constructions of projections are valid:

(I) Define $D:[0,N]\to\mathbb{R}$ to be the greatest convex function such that $D(0)=0$ and $D(k)\leqslant\sum_{i=1}^{k}C_i$ for $k=1,\dots,N$. Obviously, $D$ is a piecewise linear function. Then the numbers $D_k$ are defined as the 'slopes' of the function $D$ on the intervals $[k-1,k]$, i.e.,

$$D_k=\frac{D(k)-D(k-1)}{k-(k-1)}=D(k)-D(k-1),\quad k=1,\dots,N.$$

This construction has been used by Rychlik (1993a) and Papadatos (2001); see Rychlik (2001b) and Balakrishnan (1981) for the relevant elements of the Hilbert spaces theory.

(II) Define $k_0=0$ and

$$k_{i+1}=\max\left\{k\in\{k_i+1,\dots,N\}:\frac{1}{k-k_i}\sum_{i=k_i+1}^{k}C_i\ \text{is minimal}\right\}.$$

This procedure ends after $m$ steps, $1 \leqslant m \leqslant N$, and obviously,

$$k_0 = 0 < k_1 < \cdots < k_m = N.$$

Then, the sequence $D_1, \ldots, D_N$ is defined by

$$D_k = \frac{1}{k_i - k_{i-1}} \sum_{i=k_{i-1}+1}^{k_i} C_i, \quad k = k_{i-1}+1, \ldots, k_i, \; i = 1, \ldots, m.$$

In the first step, the algorithm determines the minimal average of the first elements $C_1, \ldots, C_{k_1}$, $1 \leqslant k_1 \leqslant N$, of the sequence $C_1 \ldots, C_N$, and replaces these elements by their average. In the next steps, the procedure is performed for the remaining parts $C_{k_i+1}, \ldots, C_N$ of the original sequence. It is obvious that we obtain the same result, once we calculate consecutive maximal averages starting from the right end of the sequence, or we mix both the methods. The above algorithm, called Pool-Adjacent-Violators-Algorithm (PAVA), has many other numerical modifications, and is extensively used in order restricted regression problems (see, e.g., Robertson et al., 1988).

From the former construction, it follows immediately that $D_1 \leqslant \cdots \leqslant D_N$. Using the latter one, we easily observe that $\sum_{k=1}^N D_k = \sum_{k=1}^N C_k$, and both the sequences have the same average $\bar{D} = N^{-1} \sum_{k=1}^N C_k$. It is also evident that $C_k = D_k$ for all $k$ iff $C_k$ is nondecreasing in $k$. In the other extremal case where $C_k$ is nonincreasing in $k$, the projection is constant: $D_k = \bar{D}$, $1 \leqslant k \leqslant N$. Note that this is not a necessary condition for a constant projection. E.g., even if $C_1, \ldots, C_{N-1}$ is increasing, but $C_N \leqslant (N-1)C_1 + \sum_{k=1}^{N-1} C_k$, then all $D_k$ are equal to $\bar{D}$.

In Theorem 2.1, we use the construction to establish mean-variance bounds on the expectations of arbitrary $L$-statistics. The bounds are sharp except for the cases of coefficients $c_1, \ldots, c_n$ which generate constant projections of respective sequences $C_1, \ldots, C_N$.

**Theorem 2.1.** *Consider a without replacement sample $X_1, \ldots, X_n$, $n \leqslant N$, from a finite ordered sample $\Pi = \{x_1, \ldots, x_N\}$. Let $\mu$ and $\sigma_2^2 > 0$ denote the population mean and variance, respectively, defined in* (2.1). *For arbitrary fixed reals $c_1, \ldots, c_n$, define the $L$-statistic by* (1.1). *Then*

$$\mathbb{E}[L] \leqslant \mu \sum_{i=1}^n c_i + \sigma_2 N^{1/2} \left( \sum_{k=1}^N D_k^2 - N\bar{D}^2 \right)^{1/2}. \tag{2.4}$$

*The bound is best possible if $D_1 < D_N$. If this is the case, then the equality holds only for the population with*

$$\frac{x_k - \mu}{\sigma_2} = \frac{N^{1/2}(D_k - \bar{D})}{(\sum_{r=1}^N D_r^2 - N\bar{D}^2)^{1/2}}, \quad k = 1, \ldots, N. \tag{2.5}$$

**Proof.** Set $y_k = (x_k - \mu)/\sigma_2$, $k = 1, \ldots, N$. Then $y_1 \leqslant \cdots \leqslant y_N$, and by (2.1), $\sum_{k=1}^N y_k = 0$ and $\sum_{k=1}^N y_k^2 = N$. By Proposition 1 in Rychlik (1992), we have

$$\sum_{k=1}^N C_k y_k \leqslant \sum_{k=1}^N D_k y_k,$$

and the equality holds iff

$$y_{k_{i-1}+1} = \cdots = y_{k_i}, \quad i = 1, \ldots, m. \tag{2.6}$$

Thus, applying first the above and next the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathbb{E}\left[\left(L - \mu\sum_{i=1}^{n} c_i\right)\Big/\sigma_2\right] &= \sum_{k=1}^{N} C_k y_k \\ &\leqslant \sum_{k=1}^{N} D_k y_k = \sum_{k=1}^{N} (D_k - \bar{D}) y_k \\ &\leqslant \left[\sum_{k=1}^{N} y_k^2\right]^{1/2} \left[\sum_{k=1}^{N} (D_k - \bar{D})^2\right]^{1/2} \\ &= N^{1/2}\left[\sum_{k=1}^{N} D_k^2 - N\bar{D}^2\right]^{1/2}, \end{aligned} \tag{2.7}$$

which proves (2.4). Regarding the case of equality, it follows easily that if $D_1 < D_N$, then the only sequence $y_1 \leqslant \cdots \leqslant y_N$ with $\sum_{k=1}^{N} y_k = 0$ and $\sum_{k=1}^{N} y_k^2 = N$ that attains both equalities in (2.7) is given by the RHS of (2.5), and the proof is complete. □

Using the Hölder inequality instead of the Cauchy-Schwarz one, we can obtain more general bounds in terms of the $p$th absolute central moments $\sigma_p^p$ for $p > 1$.

**Theorem 2.2.** *Let $p > 1$ and $q = p/(p-1)$. Then, under the notation of Theorem 2.1, for any population with mean $\mu$ and $p$th central absolute moment $\sigma_p^p > 0$, we have*

$$\mathbb{E}[L] \leqslant \mu \sum_{i=1}^{n} c_i + \sigma_p N^{1/p}\left(\sum_{k=1}^{N} |D_k - d|^q\right)^{1/q}, \tag{2.8}$$

*where $d \in [D_1, D_N]$ is the unique solution to the equation*

$$\sum_{k=1}^{N} |D_k - d|^{q/p}\,\mathrm{sgn}(D_k - d) = 0. \tag{2.9}$$

*The equality holds only for the population $\Pi = \{x_1 \leqslant \cdots \leqslant x_N\}$ with*

$$\frac{x_k - \mu}{\sigma_p} = \frac{N^{1/p}|D_k - d|^{q/p}\,\mathrm{sgn}(D_k - d)}{(\sum_{r=1}^{N} |D_r - d|^q)^{1/p}}, \quad k = 1, \ldots, N, \tag{2.10}$$

*provided that $D_1 < D_N$.*

**Proof.** The arguments are similar to those of the proof of Theorem 2.1, and so we merely outline the main points. Solution to (2.9) minimizes the strictly convex function

$x \mapsto \sum_{k=1}^{N} |D_k - x|^q$, whose derivative is the LHS of (2.9) multiplied by $q$. It is easy to see that the minimizing point $d$ cannot lie beyond the interval $[D_1, D_N]$. (In the case $p=2$, we have $d=\bar{D}$, but there is no simple analytic expression otherwise.) Replacing $\bar{D}$ by $d$, and applying the Hölder inequality instead of the Schwarz one in (2.7), we obtain the bound (2.8). Equality in the Hölder inequality holds if the sequence $y_k$, $1 \leqslant k \leqslant N$, is proportional to the RHS of (2.10). The proportion coefficient is chosen so that $\sum_{k=1}^{N} |y_k|^p = N$. By (2.9), $\sum_{k=1}^{N} y_k = 0$ holds then, and so both moment conditions are satisfied. Finally, we observe that (2.10) implies (2.6), and therefore the former inequality in (2.7) is attained as well. □

For completeness, we state without proofs analogous results for the extereme cases $p=1$ and $p=\infty$. In the former one, we have

**Theorem 2.3.** *Under the assumptions of Theorem* 2.1, *we have*

$$\mathbb{E}[L] \leqslant \mu \sum_{i=1}^{n} c_i + \sigma_1 N \frac{D_N - D_1}{2}. \tag{2.11}$$

*If* $D_1 < D_N$, *with the notation*

$$1 \leqslant r_1 = \#\{k\colon D_k = D_1\} \leqslant s_1 = \#\{k\colon D_k < D_N\} \leqslant N-1,$$

*the bound* (2.11) *is attained if*

$$\frac{x_k - \mu}{\sigma_1} = \begin{cases} -\dfrac{N}{2r_1}, & \text{for } k = 1, \ldots, r_1, \\ 0, & \text{for } k = r_1+1, \ldots, s_1, \\ \dfrac{N}{2(N-s_1)}, & \text{for } k = s_1+1, \ldots, N. \end{cases}$$

In fact, conditions of attainability are weaker if either $D_{r_1+1}$ or $D_{s_1}$ is equal to $d = \frac{D_N + D_1}{2}$. Then $x_k$, $1 \leqslant k \leqslant N$, may take on four values. Identical values are then assigned to the following sets of indices: $\{1, \ldots, r_1\}$, $\{s_1+1, \ldots, N\}$, $\{r_1+1 \leqslant k \leqslant s_1\colon D_k = d\}$, and $\{r_1+1 \leqslant k \leqslant s_1\colon D_k \neq d\}$. Only for the last one the value is strictly determined $x_k = \mu$. The other should be chosen so that monotonicity and both the moment conditions are satisfied.

**Theorem 2.4.** *With the notation of Theorem* 2.1, *and* (2.2),

$$d = \begin{cases} D_{(N+1)/2}, & \text{if } N \text{ is odd}, \\ (D_{N/2} + D_{N/2+1})/2, & \text{if } N \text{ is even}, \end{cases}$$

$$0 \leqslant r_\infty = \#\{k\colon D_k < d\} \leqslant \lfloor N/2 \rfloor \leqslant s_\infty = \#\{k\colon D_k \leqslant d\} \leqslant N, \tag{2.12}$$

*and* $\lfloor x \rfloor$ *standing for the floor of the number* $x$, *we have*

$$\mathbb{E}[L] \leqslant \mu \sum_{i=1}^{n} c_i + \sigma_\infty \left[ \sum_{k=s_\infty+1}^{N} C_k - \sum_{k=1}^{r_\infty} C_k + (r_\infty + s_\infty - N)d \right]. \tag{2.13}$$

*If* $D_1 < D_N$, *then the equality holds if*

$$\frac{x_k - \mu}{\sigma_\infty} = \begin{cases} -1, & \text{for } k = 1, \dots, r_\infty, \\ \dfrac{r_\infty + s_\infty - N}{s_\infty - r_\infty}, & \text{for } k = r_\infty + 1, \dots, s_\infty, \\ +1, & \text{for } k = s_\infty + 1, \dots, N. \end{cases} \tag{2.14}$$

Note that by (2.12), the sequence defined in the RHS of (2.14) is nondecreasing, and its elements are contained in $[-1, 1]$.

**Remark 2.1.** The above upper bounds can be easily transformed to the sharp lower ones. Indeed, if $Y_1, \dots, Y_n$ is a without replacement sample from $\Pi' = \{y_1 \leqslant \dots \leqslant y_N\}$ with $y_k = -x_{N+1-k}$, $k = 1, \dots, N$, then

$$\inf \mathbb{E} \left[ \sum_{i=1}^{n} c_i (X_{i:n} - \mu)/\sigma_p \right] = -\sup \mathbb{E} \left[ \sum_{i=1}^{n} c_{n+1-i} (Y_{i:n} + \mu)/\sigma_p \right],$$

and $\mathbb{E}[Y_i] = -\mu$, $\mathbb{E}|Y_i + \mu|^p = \sigma_p^p$. Thus, the sharp lower bounds can be determined by the corresponding upper bounds for $\Pi'$, using the constants $c_i' = c_{n+1-i}$. The optimal populations are also determined similarly.

**Remark 2.2.** Balakrishnan et al. (2003, Lemma 6.1) showed that every distribution of the i.i.d. sample of fixed size $n$ may be approximated with arbitrary desired accuracy by the distribution of the sample of the same size $n$ drawn without replacement from a finite population of size $N$, as $N$ becomes large. Theorem 5.1 of Balakrishnan et al. (2003) asserts that the upper bounds for the expected sample maximum of the without replacement models tend to the Hartley-David-Gumbel general bound for the maximum of arbitrary i.i.d. sample, as the population size increases. One can expect the same in the case of other $L$-statistics. On the other hand, Rychlik (1993b) proved that the optimal bounds on the expectations of arbitrary $L$-statistics based on arbitrarily dependent identically distributed samples with given marginal moments (as well as analogous deterministic bounds) are attained by the exhaustive (with $n = N$) drawing without replacement models. In this context, the without replacement schemes, except for their practical importance, can be treated as simple and natural transition models between the classical i.i.d. case and the arbitrarily dependent one.

**Remark 2.3.** If the projection $(D_1, \dots, D_N)$ of $(C_1, \dots, C_N)$ is constant with $D_k = \bar{D}$, $1 \leqslant k \leqslant N$, relations (2.4), (2.8), (2.11), and (2.13) simply state that

$$\mathbb{E}[L] \leqslant \mu \sum_{i=1}^{n} c_i.$$

In order to get better evaluations, one needs to use specific subtle tools. For instance, Balakrishnan et al. (2003, Theorem 4.1) proved that

$$\mathbb{E}[X_{1:n}] \leqslant \mu - \sigma_2(N-1)^{-1/2}$$

is the best possible upper bound for the expectation of the sample minimum.

## 3. Expectation bounds for the trimmed means and the quasi-ranges

In this section we apply the previous results to evaluate the best possible upper bounds for the expectation of the trimmed means $T(i,j) = (j-i+1)^{-1}\sum_{r=i}^{j} X_{r:n}$, $1 \leqslant i \leqslant j \leqslant n \leqslant N$, and quasi-ranges $Q(i,j) = X_{j:n} - X_{i:n}$, $1 \leqslant i < j \leqslant n \leqslant N$, when the population mean and a given absolute central moment are fixed.

We first consider the bounds for the expectations of trimmed means

$$\frac{1}{j-i+1}\,\mathbb{E}\sum_{r=i}^{j} X_{r:n} = \frac{1}{j-i+1}\sum_{k=1}^{N} x_k\, p_{i,j}(k), \qquad 1 \leqslant i \leqslant j \leqslant n,$$

with

$$p_{i,j}(k) = p_{i,j}(k;n,N) = \sum_{r=i}^{j} p_r(k) = \mathbb{P}(k \in \{U_{i:n},\ldots,U_{j:n}\}). \tag{3.1}$$

Therefore, in order to apply the results of the previous section, we have to evaluate the projection $D_k$ of the sequence

$$C_k = C_k(i,j,n,N) = \frac{1}{j-i+1}\,p_{i,j}(k), \qquad k = 1,\ldots,N. \tag{3.2}$$

(Note that here all $C_k$, $1 \leqslant k \leqslant N$, are nonnegative, and strictly positive iff $i \leqslant k \leqslant N-n+j$, and they sum up to 1.) First we show the following lemma.

**Lemma 3.1.** *For any integers $r$, $n$, $N$ and $k$ satisfying $1 \leqslant r \leqslant n \leqslant N$ and $1 \leqslant k \leqslant N$, we have the identity*

$$\sum_{s=1}^{k}\binom{s-1}{r-1}\binom{N-s}{n-r} = \sum_{w=r}^{n}\binom{k}{w}\binom{N-k}{n-w}. \tag{3.3}$$

(b) *The sequence $C_k = C_k(i,j,n,N)$ defined in* (3.2) *is unimodal, that is, there exists an integer $m \in \{1,\ldots,N\}$ (actually $m \in \{i,\ldots,N-n+j\}$) such that $C_k$ is nondecreasing in $\{1,\ldots,m\}$ and nonincreasing in $\{m,\ldots,N\}$.*

**Proof.** (a) We have

$$\binom{N}{n}\mathbb{P}(U_{r:n} \leqslant k) = \binom{N}{n}\sum_{s=1}^{k} p_r(s),$$

which is the LHS of (3.3). The proof now follows if we observe that

$$\begin{aligned}\binom{N}{n}\mathbb{P}(U_{r:n}\leqslant k) &= \binom{N}{n}\mathbb{P}(\text{at least } r \text{ among } U_1,\dots,U_n \text{ are } \leqslant k)\\ &= \binom{N}{n}\sum_{w=r}^{n}\mathbb{P}(\text{exactly } w \text{ among } U_1,\dots,U_n \text{ are } \leqslant k)\\ &= \sum_{w=r}^{n}\binom{k}{w}\binom{N-k}{n-w}\end{aligned}$$

is the RHS of (3.3).

(b) If $i=1$ and $j=n$ then, by Cauchy's formula, $C_k=1/N$ for all $k$. In the case where $2\leqslant i\leqslant j=n$, we have $C_1=0$ and, by (2.3) and (3.3),

$$C_k=\gamma\sum_{r=i}^{n}\binom{k-1}{r-1}\binom{N-k}{n-r}=\gamma\sum_{s=2}^{k}\binom{s-2}{i-2}\binom{N-s}{n-i},\quad k\geqslant 2,$$

with

$$\frac{1}{\gamma}=(j-i+1)\binom{N}{n}>0.$$

(Observe that $\gamma>0$ is independent of $k$.) Thus, $C_k$ is nondecreasing in this case. Similarly, if $1=i\leqslant j\leqslant n-1$, then, by definition,

$$C_1=\gamma\binom{N-1}{n-1},$$

while, by (3.3),

$$C_k=\gamma\binom{N-1}{n-1}-\gamma\sum_{s=2}^{k}\binom{s-2}{j-1}\binom{N-s}{n-j-1},\quad k\geqslant 2,$$

with $\gamma$ as above. Thus, $C_k$ is nonincreasing in this case. Finally, in the general case where $2\leqslant i\leqslant j\leqslant n-1$, we have $C_1=C_N=0$, while for $2\leqslant k\leqslant N-1$ we get by (3.3) (with $\gamma$ as above)

$$\begin{aligned}C_k&=\gamma\sum_{r=i}^{n}\binom{k-1}{r-1}\binom{N-k}{n-r}-\gamma\sum_{r=j+1}^{n}\binom{k-1}{r-1}\binom{N-k}{n-r}\\ &=\gamma\sum_{s=2}^{k}(a_s-b_s),\end{aligned}$$

where

$$a_s=\binom{s-2}{i-2}\binom{N-s}{n-i},\quad b_s=\binom{s-2}{j-1}\binom{N-s}{n-j-1},\quad s=2,\dots,N-1.$$

(Observe that $a_s$, $b_s$ are nonnegative and independent of $k$.) Since $a_s > 0$ iff $i \leqslant s \leqslant N-n+i$ and $b_s > 0$ iff $j+1 \leqslant s \leqslant N-n+j+1$, it follows immediately from the last expression that $C_k$ is unimodal if $N-n+i \leqslant j+1$. On the other hand, if $j+1 < N-n+i$, then $a_s$ and $b_s$ are both positive in the interval $j+1 \leqslant s \leqslant N-n+i$, which contains at least two points $s$. If this is the case, however, we may write for $s$ into this interval,

$$\frac{a_s}{b_s} = \delta \prod_{t=i}^{j} \theta_t(s), \quad \text{where } \delta = \frac{(j-1)!(n-j-1)!}{(i-1)!(n-i)!} > 0$$

is a positive constant (not depending on $s$) and

$$\theta_t(s) = \frac{N-n+1}{s-t} - 1 > 0, \quad t = i,\dots,j, \; s = j+1,\dots,N-n+i.$$

Since the positive function $\theta_t(s)$ is strictly decreasing in $s \in [j+1, N-n+i]$ for all fixed $t \in \{i,\dots,j\}$, it follows that the same is true for $a_s/b_s$. Therefore, into the above interval, $a_s < b_s$ only for the last elements of the interval (if any). Since $a_s - b_s = a_s \geqslant 0$ on the left of the interval and $a_s - b_s = -b_s \leqslant 0$ on the right, the desired result follows. □

Observe that the maximizing point $m$ defined in Lemma 3.1(b) may not be unique. In the sequel we shall use the notation $m_0 = m_0(i,j,n,N)$ for the minimal integer in $\{1,\dots,N\}$ that maximizes $C_k$, i.e.,

$$m_0 = \min\left\{k\colon\, C_k = \max_{1 \leqslant s \leqslant N} C_s\right\}. \tag{3.4}$$

Observe that $m_0 = 1$ iff $i = 1$. The above properties of the sequence are necessary for determining its projection.

**Lemma 3.2.** *Under the notation* (3.1), (3.2) *and* (3.4), *define* $t = t(i,j,n,N)$, $1 \leqslant i \leqslant j \leqslant n \leqslant N$, *as*

$$t(1,j,n,N) = 0, \quad \textit{for } j = 1,\dots,n-1,$$

$$t(i,n,n,N) = N-1, \quad \textit{for } i = 1,\dots,n,$$

*and as the greatest* $k \in \{i-1,\dots,m_0-1\}$ *such that*

$$p_{i,j}(k) < \frac{1}{N-k} \sum_{r=k+1}^{N} p_{i,j}(r), \tag{3.5}$$

*if* $2 \leqslant i \leqslant j \leqslant n-1$. *Then the projection of* $C_k = p_{i,j}(k)/(j+1-i)$, $1 \leqslant k \leqslant N$, *onto the set of nondecreasing sequences has the form*

$$D_k = \begin{cases} C_k, & \textit{for } k = 1,\dots,t, \\ \dfrac{1}{N-t} \displaystyle\sum_{r=t+1}^{N} C_r, & \textit{for } k = t+1,\dots,N. \end{cases}$$

**Proof.** If $1 \leqslant i \leqslant j = n$, the original sequence is nondecreasing and coincides with the projection. If $1 = i \leqslant j < n$, then the sequence is nonincreasing, and the projection is constant $D_k = \bar{D}$ for all $k$. In the remaining cases, we construct the projection using algorithm (II) of Section 2. In fact, we modify it looking for the consecutive greatest means of last elements of the sequence. We start with the observation that the mean of a set of numbers is a convex combination of means of its partition. Accordingly, including a new element to the mean, we increase (decrease) its value iff the element is greater (smaller) than the original mean. Therefore the consecutive means

$$\frac{1}{N-k} \sum_{r=k+1}^{N} C_r, \quad k = N-1, \ldots, 0,$$

are first equal to zero, and then positive increasing till the (either unique or smaller) maximizing point $m_0$. Next the decreasing elements are included in the mean. First they can still be greater than the average of the succeeding ones, which would result in further increase of the means. However, the increase process definitely ends before $k = i-1$, when we start including zeros in the positive means. We define $t = t(i,j,n,N)$ as the point at which the means start decreasing. This is the greatest $k$ for which

$$\frac{1}{N-k+1} \sum_{r=k}^{N} p_{i,j}(r) < \frac{1}{N-k} \sum_{r=k+1}^{N} p_{i,j}(r),$$

which can be rewritten as in (3.5). This is the globally maximal mean of last elements, and defines the $N-t$ greatest elements of the projection. We deduced above that $t+1$ lies necessarily between $i$ and $m_0$, which correspond with the first nonzero and (first) maximal elements of the original sequence, respectively. It suffices to notice now that $C_1, \ldots, C_t$ is a nondecreasing sequence, and completing the construction procedure results from setting $D_k = C_k$ for $k = 1, \ldots, t$. $\square$

Using Lemma 3.2, we specify the results of Section 2 for the trimmed means. We get the trivial bounds $\mathbb{E}T(i,j) \leqslant \mu$ if $i = 1$. The nontrivial bounds of Theorems 2.1 and 2.3 simplify here, because $\sum_{i=1}^{n} c_i = \sum_{k=1}^{N} C_k = 1 = N\bar{D}$, $r_1 = i-1$ and $s_1 = t$. For $2 \neq p \in (1, \infty)$, the formulae are more complicated, and we omit them. In case $p = +\infty$, for various choices of $i$ and $j$ the relations between $r_\infty$, $s_\infty$, and $t$ may be absolutely arbitrary, and the results of Theorem 2.4 cannot be much simplified.

**Corollary 3.1.** *If $2 \leqslant i \leqslant j \leqslant n$, with the notation of Lemma 3.2, we have*

$$\mathbb{E}\left[\frac{T(i,j) - \mu}{\sigma_2}\right] \leqslant A_2,$$

*where*

$$A_2 = \left[N \sum_{k=1}^{t} C_k^2 + \frac{N}{N-t}\left(\sum_{k=t+1}^{N} C_k\right)^2 - 1\right]^{1/2}. \tag{3.6}$$

*The bound is tight and becomes equality if*

$$\frac{x_k-\mu}{\sigma_2}=\begin{cases}\dfrac{N}{A_2}\left(C_k-\dfrac{1}{N}\right), & \textit{for } k=1,\dots,t,\\[2ex] \dfrac{N}{A_2(N-t)}\displaystyle\sum_{k=t+1}^{N}\left(C_k-\dfrac{1}{N}\right), & \textit{for } k=t+1,\dots,N.\end{cases}$$

*Moreover,*

$$\mathbb{E}\left[\frac{T(i,j)-\mu}{\sigma_1}\right]\leqslant A_1=\frac{N}{2(N-t)}\sum_{k=t+1}^{N}C_k, \tag{3.7}$$

*and the equality holds if*

$$\frac{x_k-\mu}{\sigma_1}=\begin{cases}-\dfrac{N}{2(i-1)}, & \textit{for } k=1,\dots,i-1,\\ 0, & \textit{for } k=i,\dots,t,\\ \dfrac{N}{2(N-t)}, & \textit{for } k=t+1,\dots,N.\end{cases}$$

**Remark 3.1.** Taking $i=j>1$ we get the single order statistics $T(i,i)=X_{i:n}$, and the first statement of Corollary 3.1 yields the results of Theorems 2.1 and 3.1 in Balakrishnan et al. (2003). We have also proved that the probability function of each order statistic $U_{i:n}$ from the standard uniform population is unimodal.

**Remark 3.2.** For the expectations of lower selection differentials $\mathbb{E}[T(1,j)-T(1,n)]=\mathbb{E}\,\frac{1}{j}\sum_{r=1}^{j}X_{r:n}-\mu$, all our bounds are trivial $A_p=0$ for $1\leqslant p\leqslant\infty$. If $j=n$, the bounds are clearly attained by any (arbitrary) population. Otherwise, the bounds can be improved.

**Remark 3.3.** For the upper selection differentials $T(i,n)-T(1,n)$, $i>1$, with all $D_k=C_k$, we have

$$A_2=\left(N\sum_{k=1}^{N}C_k^2-1\right)^{1/2}=\frac{1}{n-i+1}\left(N\sum_{k=i}^{N}p_{i,n}^2(k)-(n-i+1)^2\right)^{1/2}, \tag{3.8}$$

$$A_1=\frac{N}{2}C_N=\frac{n}{2(n-i+1)}. \tag{3.9}$$

If $N=n$, then $\sum_{k=i}^{n}p_{i,n}^2(k)=n-i+1=N-i+1$ and we get the deterministic bounds of Nagaraja (1981), and Rychlik (1993b) for $p=2$, and the other cases, respectively. Nagaraja (1981) also derived the best mean-variance bounds in the i.i.d. case.

**Remark 3.4.** Due to Remark 2.1, the negatives of the lower bounds on $\mathbb{E}[T(i,j)-\mu]/\sigma_p$ coincide with the upper bounds for $\mathbb{E}[T(n+1-j,n+1-i)-\mu]/\sigma_p$. In particular, they become 0 for the upper selection differentials, and we have (3.8) and (3.9) for the lower ones.

We now concentrate on the bounds for expectations of quasi-ranges

$$\mathbb{E}Q(i,j) = \sum_{k=1}^{N} x_k [p_j(k) - p_i(k)],$$

which depend on the sequence

$$C_k = \delta_{i,j}(k) = p_j(k) - p_i(k) = \left[\binom{k-1}{j-1}\binom{N-k}{n-j} - \binom{k-1}{i-1}\binom{N-k}{n-i}\right] \Big/ \binom{N}{n}, \quad 1 \leqslant k \leqslant N. \tag{3.10}$$

We easily check that $\sum_{k=1}^{N} C_k = \sum_{k=1}^{N} p_j(k) - \sum_{k=1}^{N} p_i(k) = 0$. Further properties of (3.10) are described in Lemma 3.3, and next used in Lemma 3.4 for constructing the projection.

**Lemma 3.3.** *The sequence* (3.10) *is first nonpositive and then nonnegative. Furthermore, this is first nonincreasing, then nondecreasing, and ultimately nonincreasing.*

**Proof.** Using identity (3.3) and notations (2.3) and (3.1), we obtain

$$\begin{aligned}
\binom{N}{n} \sum_{s=1}^{k} \delta_{i,j}(s) &= \sum_{s=1}^{k} \binom{s-1}{j-1}\binom{N-s}{n-j} - \sum_{s=1}^{k} \binom{s-1}{i-1}\binom{N-s}{n-i} \\
&= \sum_{w=j}^{n} \binom{k}{w}\binom{N-k}{n-w} - \sum_{w=i}^{n} \binom{k}{w}\binom{N-k}{n-w} \\
&= -\sum_{w=i}^{j-1} \binom{k}{w}\binom{N-k}{n-w} \\
&= -\binom{N+1}{n+1} \sum_{r=i+1}^{j} p_r(k+1; n+1, N+1) \\
&= -\binom{N+1}{n+1} (j-i) C_{k+1}(i+1, j, n+1, N+1),
\end{aligned}$$

where the sequence $C_{k+1}(i+1, j, n+1, N+1)$ is defined in (3.2). Therefore, for $k = 1, \ldots, N$, we have an alternative expression

$$\delta_{i,j}(k) = -\frac{N+1}{n+1}(j-i)[C_{k+1}(i+1, j, n+1, N+1) - C_k(i+1, j, n+1, N+1)],$$

and unimodality of $C_k(i+1, j, n+1, N+1)$ with respect to $k$ (see Lemma 3.1(b) above) implies that $\delta_{i,j}(k)$ is first nonpositive and then nonnegative.

We observe that, by (2.3), $p_i(k) = 0$ iff $k \notin \{i,\dots,N-n+i\}$. Consequently, $\delta_{i,j}(k) = p_j(k) - p_i(k) = 0$ if $k \notin \{i,\dots,N-n+i\} \cup \{j,\dots,N-n+j\}$. If $j > N-n+i$, the subsets of the union do not overlap. In this case we have

$$\delta_{i,j}(k) = \begin{cases} -p_i(k), & \text{for } k = i,\dots,N-n+i, \\ p_j(k), & \text{for } k = j,\dots,N-n+j, \\ 0, & \text{otherwise.} \end{cases}$$

In the first case, the sequence is negative nonincreasing and nondecreasing (cf. Remark 3.1). In the middle one, this is positive nondecreasing and nonincreasing. Accordingly, (3.10) satisfies the statement of the Lemma.

From now on we assume that $j \leqslant N-n+i$ (thus, $n < N$). We examine the differences

$$\begin{aligned} \Delta_{i,j}(k) &= \delta_{i,j}(k+1) - \delta_{i,j}(k) \\ &= [p_j(k+1) - p_j(k)] - [p_i(k+1) - p_i(k)], \\ &= \Delta[p_j(k)] - \Delta[p_i(k)], \quad k = 1,\dots,N-1. \end{aligned}$$

We readily see that

$$\Delta[p_j(k)] = 0 \quad \text{iff } k < j-1 \text{ or } k = \frac{j-1}{n-1}N \text{ or } k > N-n+j, \tag{3.11}$$

$$\Delta[p_j(k)] > 0 \quad \text{iff } j-1 \leqslant k < \frac{j-1}{n-1}N, \tag{3.12}$$

$$\Delta[p_j(k)] < 0 \quad \text{iff } \frac{j-1}{n-1}N < k \leqslant N-n+j, \tag{3.13}$$

$$\Delta[p_j(k)] \leqslant 0 \quad \text{iff } k < j-1 \text{ or } k \geqslant \frac{j-1}{n-1}N, \tag{3.14}$$

$$\Delta[p_j(k)] \geqslant 0 \quad \text{iff } k \leqslant \frac{j-1}{n-1}N \text{ or } k > N-n+j. \tag{3.15}$$

Similar relations hold for $\Delta[p_i(k)]$. Therefore, all the integers $k$ of $\{1,\dots,N-1\}$ (if any) belonging to the set

$$\begin{aligned} H_m &= \{k\colon \Delta[p_j(k)] > 0 \text{ and } \Delta[p_i(k)] \leqslant 0\} \\ &\quad \cup \{k\colon \Delta[p_j(k)] \geqslant 0 \text{ and } \Delta[p_i(k)] < 0\} \end{aligned}$$

satisfy $\Delta_{i,j}(k) > 0$. Applying (3.12) to (3.15), we check that

$$\{k\colon k_1 < k < k_2\} \subset H_m \subset \bar{H}_m = \{k\colon k_1 \leqslant k \leqslant k_2\}$$

for $k_1 = ((i-1)/(n-1))N$ and $k_2 = ((j-1)/(n-1))N$. Moreover, if $k_1$ is an integer, then $\Delta[p_i(k_1)] = 0$ (cf. (3.11) with $i$ in place of $j$) and $\Delta[p_j(k_1)] \geqslant 0$ (see (3.15)), imply $\Delta_{i,j}(k_1) = \Delta[p_j(k_1)] \geqslant 0$. Similarly, if $k_2$ is an integer, then (see (3.11)) $\Delta_{i,j}(k_2) = -\Delta[p_i(k_2)] \geqslant 0$, because $k_2 > k_1$ (cf. (3.14) with $i$ in place of $j$). Therefore, for any integer $k$ from $\bar{H}_m$, we have $\Delta_{i,j}(k) \geqslant 0$. On the other hand, we see that the set

$$H_s = \{k\colon\ \Delta[p_j(k)] \leqslant 0 \text{ and } \Delta[p_i(k)] \geqslant 0\}$$

can be written as $H_s = H_l \cup H_r$, where

$$H_l = \{k\colon\ k \leqslant \min\{j-2, k_1\}\}$$

and

$$H_r = \{k\colon\ k \geqslant \max\{N-n+i+1, k_2\}\}.$$

Therefore, since $\Delta_{i,j}(k) \leqslant 0$ for $k \in H_s$ and $\Delta_{i,j}(k) \geqslant 0$ for $k \in \bar{H}_m$, it suffices to investigate the signs of $\Delta_{i,j}(k)$ in the remaining cases $k \in H^+ \cup H^-$, where

$$H^+ = \{k\colon\ \Delta[p_j(k)] > 0 \text{ and } \Delta[p_i(k)] > 0\},$$

$$H^- = \{k\colon\ \Delta[p_j(k)] < 0 \text{ and } \Delta[p_i(k)] < 0\}.$$

Using (3.12) and (3.13), we verify that

$$H^+ = \{k\colon\ j-1 \leqslant k < k_1\},$$

$$H^- = \{k\colon\ k_2 < k \leqslant N-n+i\}.$$

Of course, $H^+$ or $H^-$ may be empty in some cases, and otherwise $H^+$ lies on the left of $\bar{H}_m$ and on the right of $H_l$, while $H^-$ lies on the right of $\bar{H}_m$ and on the left of $H_r$. If both $H^+$ and $H^-$ contain at most one point, then the desired result follows. Otherwise, for $k \in H^+ \cup H^-$ we may write

$$\begin{aligned}\varrho_{i,j}(k) &= \frac{p_j(k+1) - p_j(k)}{p_i(k+1) - p_i(k)} \\ &= \frac{(n-j+1)\ldots(n-i)}{i\ldots(j-1)} \frac{(k-j+2)\ldots(k+1-i)}{(N-n+i+1-k)\ldots(N-n+j-k)} \frac{k_2-k}{k_1-k}.\end{aligned}$$

The first fraction of the last representation is positive constant, and the second one is positive increasing. If $k \in H^+$ (thus, $k < k_1$), then the last factor is positive increasing, and so is the product $\varrho_{i,j}(k)$. Observe that in this case the condition $\Delta_{i,j}(k) > 0$ is equivalent to $\varrho_{i,j}(k) > 1$. Therefore either $\Delta_{i,j}(k)$ changes the sign once in $H^+$, and this is from $-$ to $+$, or there are no sign changes there. If $k \in H^-$ (thus, $k > k_2$), then the last factor is also positive increasing as well as the whole product. In this case, however, positivity of $\Delta_{i,j}(k)$ coincides with the relation $\varrho_{i,j}(k) < 1$. This implies that $\Delta_{i,j}(k)$ may have at most one sign change, from $+$ to $-$, for $k \in H^-$. Summarizing the results for all subcases (note that $\{1,\ldots,N-1\} = H_l \cup H^+ \cup H_m \cup H^- \cup H_r$, where each subset (if nonempty) of this disjoint union lies on the right of the preceding one), we conclude that the sequence $\Delta_{i,j}(k)$, $1 \leqslant k \leqslant N-1$, satisfies the following: If there exist two integers $1 \leqslant t_1 < t_2 \leqslant N-1$ such that $\Delta_{i,j}(t_r) > 0$ for $r = 1, 2$, then $\Delta_{i,j}(k) \geqslant 0$ for all integers $k \in [t_1, t_2]$. This completes the proof. $\square$

In order to construct the projection of the sequence $C_k$ defined in (3.10), it will be convenient to know the locations for both extremes of the sequence. We also need to consider the locations of the sign change of the sequence. Thus, we define the integers $w_1$, $w_2$, $z_1$ and $z_2$ as follows:

$$z_1 = z_1(i,j,n,N) = \max\{k\colon\ C_k < 0\}, \tag{3.16}$$

$$z_2 = z_2(i,j,n,N) = \min\{k\colon\ C_k > 0\}, \tag{3.17}$$

$$w_1 = w_1(i,j,n,N) = \max\left\{k\colon\ C_k = \min_{1\leqslant s\leqslant N} C_s\right\}, \tag{3.18}$$

$$w_2 = w_2(i,j,n,N) = \min\left\{k\colon\ C_k = \max_{1\leqslant s\leqslant N} C_s\right\}. \tag{3.19}$$

By Lemma 3.3, it follows that $i \leqslant w_1 \leqslant z_1 < z_2 \leqslant w_2 \leqslant N-n+j$.

**Lemma 3.4.** *Under the notation* (3.10) *and* (3.16) *to* (3.19), *we define* $q_1 = q_1(i,j,n,N)$ *as the smallest* $k \in \{w_1,\dots,z_1\}$ *such that*

$$\frac{1}{k}\sum_{r=1}^{k} \delta_{i,j}(r) < \delta_{i,j}(k+1), \tag{3.20}$$

*and* $q_2 = q_2(i,j,n,N)$ *as the greatest* $k \in \{z_2-1,\dots,w_2-1\}$ *such that*

$$\delta_{i,j}(k) < \frac{1}{N-k}\sum_{r=k+1}^{N} \delta_{i,j}(r). \tag{3.21}$$

*Then the projection of* (3.10) *onto the set of nondecreasing sequences has the form*

$$D_k = \begin{cases} \dfrac{1}{q_1}\displaystyle\sum_{r=1}^{q_1} C_r, & \text{for } k = 1,\dots,q_1, \\ C_k, & \text{for } k = q_1+1,\dots,q_2, \\ \dfrac{1}{N-q_2}\displaystyle\sum_{r=q_2+1}^{N} C_r, & \text{for } k = q_2+1,\dots,N. \end{cases} \tag{3.22}$$

**Proof.** We use algorithm (II) and reasoning similar to that of the proof of Lemma 3.2. We start with establishing the minimal mean $D_1$ of the first elements. By Lemma 3.3, the first means are nonpositive and nonincreasing at least till the $C_k = p_{i,j}(k)$'s reach their global negative minimum. The means may continue to decrease if the next $C_k$'s are less than the previous means. The decrease stops before the sequence changes sign from $-$ to $+$. From that moment, they only include either increasing negative or nonnegative elements, and hence the further means are greater. Therefore the minimal mean with $q_1$ elements, say, contains the (last) negative minimum at least, and the last negative element of the sequence at most. The condition determining the index $q_1$ is given by (3.20). Next we look for the maximal mean of the last elements. We similarly deduce that the positive maximal mean $D_N$ of $N-q_2$ last elements, say, contains no more than all positive elements with the possible zeros at the end of the sequence, and no less than its ultimate nondecreasing part. The number $q_2$ is precisely defined in (3.21).

Replacing the terms of both the extreme means by the respective mean values, we obtain $q_1$ negative constants $D_1$ and $N-q_2$ positive constants $D_N$ at the left and right ends of the sequence, respectively. Besides, there are possibly some elements $C_{q_1+1},\dots,C_{q_2}$ satisfying $D_1 < C_{q_1+1} \leqslant \cdots \leqslant C_{q_2} < D_N$ (cf. (3.22)). We see that the modified sequence is nondecreasing, and completing the procedure (II) does not change it. □

Combining the results of Theorems 2.1, 2.3 with Lemma 3.4, and observing that $\bar{D}=0$, $r_1=q_1$ and $s_1=q_2$, we are in a position to formulate Corollary 3.2.

**Corollary 3.2.** *Under the notation of Lemma* 3.4, *we have*

$$\mathbb{E}\left[\frac{X_{j:n}-X_{i:n}}{\sigma_2}\right] \leqslant B_2,$$

*where*

$$B_2 = N^{1/2}\left[\frac{1}{q_1}\left(\sum_{k=1}^{q_1} C_k\right)^2 + \sum_{k=q_1+1}^{q_2} C_k^2 + \frac{1}{N-q_2}\left(\sum_{k=q_2+1}^{N} C_k\right)^2\right]^{1/2} \tag{3.23}$$

(a sum of the form $\sum_{r=a}^{b}$ for $a>b$ should be treated as 0). *The equality holds if*

$$\frac{x_k-\mu}{\sigma_2} = \begin{cases} \dfrac{N}{B_2 q_1}\displaystyle\sum_{r=1}^{q_1} C_r, & \text{for } k=1,\dots,q_1,\\[2ex] \dfrac{N}{B_2}\,C_k, & \text{for } k=q_1+1,\dots,q_2,\\[2ex] \dfrac{N}{B_2(N-q_2)}\displaystyle\sum_{r=q_2+1}^{N} C_r, & \text{for } k=q_2+1,\dots,N. \end{cases}$$

*Furthermore, we have the inequality*

$$\mathbb{E}\left[\frac{X_{j:n}-X_{i:n}}{\sigma_1}\right] \leqslant B_1 = \frac{N}{2}\left[\frac{1}{N-q_2}\sum_{k=q_2+1}^{N} C_k - \frac{1}{q_1}\sum_{k=1}^{q_1} C_k\right], \tag{3.24}$$

*which becomes equality if*

$$\frac{x_k-\mu}{\sigma_1} = \begin{cases} -\dfrac{N}{2q_1}, & \text{for } k=1,\dots,q_1,\\[2ex] 0, & \text{for } k=q_1+1,\dots,q_2,\\[2ex] \dfrac{N}{2(N-q_2)}, & \text{for } k=q_2+1,\dots,N. \end{cases}$$

Table 1
Numerical values of upper bounds (3.6), (3.7), (3.23) and (3.24) on expectations of trimmed means and quasi-ranges for varying $i, j, n$ and $N$

| $i$ | $j$ | $n$ | $N$ | $t$ | $A_2$ | $A_1$ | $q_1$ | $q_2$ | $B_2$ | $B_1$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 | 0.00000 | 0.00000 | 1 | 1 | 2.00000 | 2.00000 |
| 1 | 2 | 2 | 5 | 4 | 0.00000 | 0.00000 | 1 | 4 | 1.41421 | 2.00000 |
| 1 | 2 | 2 | 10 | 9 | 0.00000 | 0.00000 | 1 | 9 | 1.27657 | 2.00000 |
| 1 | 2 | 5 | 5 | 0 | 0.00000 | 0.00000 | 1 | 1 | 2.50000 | 3.12500 |
| 1 | 2 | 5 | 10 | 0 | 0.00000 | 0.00000 | 1 | 2 | 1.70783 | 2.84722 |
| 1 | 2 | 10 | 10 | 0 | 0.00000 | 0.00000 | 1 | 1 | 3.33333 | 5.55556 |
| 3 | 5 | 5 | 5 | 4 | 0.81650 | 0.83333 | 3 | 4 | 2.58199 | 3.33333 |
| 3 | 5 | 5 | 6 | 5 | 0.74536 | 0.83333 | 4 | 5 | 2.41523 | 3.25000 |
| 3 | 5 | 5 | 7 | 6 | 0.71270 | 0.83333 | 5 | 6 | 2.28869 | 3.16667 |
| 3 | 5 | 5 | 8 | 7 | 0.69253 | 0.83333 | 5 | 7 | 2.18996 | 3.14286 |
| 3 | 5 | 5 | 9 | 8 | 0.67847 | 0.83333 | 6 | 8 | 2.12459 | 3.12500 |
| 3 | 5 | 5 | 10 | 9 | 0.66799 | 0.83333 | 7 | 9 | 2.07233 | 3.09524 |
| 3 | 5 | 5 | 100 | 99 | 0.59772 | 0.83333 | 61 | 99 | 1.71492 | 3.01193 |
| 3 | 5 | 6 | 6 | 2 | 0.70711 | 0.75000 | 3 | 4 | 2.23607 | 2.50000 |
| 3 | 5 | 6 | 7 | 3 | 0.63308 | 0.70833 | 4 | 5 | 2.02661 | 2.12500 |
| 3 | 5 | 6 | 8 | 3 | 0.59590 | 0.70476 | 4 | 5 | 1.85806 | 1.97619 |
| 3 | 5 | 6 | 9 | 3 | 0.57275 | 0.69048 | 5 | 6 | 1.75449 | 1.91071 |
| 3 | 5 | 6 | 10 | 4 | 0.55503 | 0.68783 | 5 | 7 | 1.67233 | 1.82540 |
| 3 | 5 | 6 | 100 | 39 | 0.45292 | 0.64889 | 49 | 73 | 1.28111 | 1.43579 |
| 3 | 5 | 7 | 7 | 2 | 0.63246 | 0.70000 | 3 | 4 | 2.16025 | 2.33333 |
| 3 | 5 | 7 | 8 | 2 | 0.57735 | 0.66667 | 4 | 4 | 2.00000 | 2.00000 |
| 3 | 5 | 7 | 9 | 3 | 0.53468 | 0.64583 | 4 | 5 | 1.76777 | 1.87500 |
| 3 | 5 | 7 | 10 | 3 | 0.50874 | 0.64484 | 5 | 5 | 1.66667 | 1.66667 |
| 3 | 5 | 7 | 100 | 27 | 0.37804 | 0.59835 | 41 | 59 | 1.16251 | 1.24085 |
| 3 | 5 | 8 | 8 | 2 | 0.57735 | 0.66667 | 3 | 4 | 2.16025 | 2.33333 |
| 3 | 5 | 8 | 9 | 2 | 0.53452 | 0.64286 | 4 | 4 | 2.01246 | 2.02500 |
| 3 | 5 | 8 | 10 | 2 | 0.50000 | 0.62500 | 4 | 5 | 1.77951 | 1.86111 |
| 3 | 5 | 8 | 100 | 21 | 0.32943 | 0.57247 | 35 | 50 | 1.11863 | 1.19534 |
| 3 | 5 | 9 | 9 | 2 | 0.53452 | 0.64286 | 3 | 4 | 2.19089 | 2.40000 |
| 3 | 5 | 9 | 10 | 2 | 0.50000 | 0.62500 | 4 | 4 | 2.04124 | 2.08333 |
| 3 | 5 | 9 | 100 | 16 | 0.29484 | 0.55689 | 30 | 44 | 1.10548 | 1.20224 |
| 3 | 5 | 10 | 10 | 2 | 0.50000 | 0.62500 | 3 | 4 | 2.23607 | 2.50000 |
| 3 | 5 | 10 | 11 | 2 | 0.47140 | 0.61111 | 4 | 4 | 2.07880 | 2.16071 |
| 3 | 5 | 10 | 12 | 2 | 0.44721 | 0.60000 | 4 | 4 | 1.92847 | 2.04545 |
| 3 | 5 | 10 | 13 | 2 | 0.42640 | 0.59091 | 4 | 5 | 1.74275 | 1.92045 |
| 3 | 5 | 10 | 14 | 2 | 0.40825 | 0.58333 | 4 | 5 | 1.64946 | 1.81235 |
| 3 | 5 | 10 | 15 | 2 | 0.39223 | 0.57692 | 5 | 5 | 1.58940 | 1.68581 |
| 3 | 5 | 10 | 20 | 3 | 0.34639 | 0.56760 | 6 | 7 | 1.39164 | 1.52566 |
| 3 | 5 | 10 | 50 | 7 | 0.28616 | 0.55087 | 14 | 19 | 1.16333 | 1.29124 |
| 3 | 5 | 10 | 100 | 13 | 0.26882 | 0.54651 | 27 | 39 | 1.10793 | 1.23451 |
| 3 | 8 | 10 | 10 | 2 | 0.50000 | 0.62500 | 3 | 7 | 2.58199 | 3.33333 |
| 3 | 8 | 10 | 20 | 5 | 0.38746 | 0.59245 | 7 | 13 | 2.07627 | 2.35736 |
| 3 | 8 | 10 | 30 | 7 | 0.36543 | 0.58697 | 10 | 20 | 2.00501 | 2.24695 |
| 3 | 8 | 10 | 40 | 10 | 0.35536 | 0.58448 | 14 | 26 | 1.97101 | 2.20914 |
| 3 | 8 | 10 | 50 | 12 | 0.34960 | 0.58306 | 17 | 33 | 1.95100 | 2.18239 |
| 3 | 8 | 10 | 100 | 25 | 0.33852 | 0.58032 | 35 | 65 | 1.91189 | 2.13837 |

**Remark 3.5.** Observe that all the upper bounds for quasi-ranges are strictly positive. On the other hand, our projection method applied for calculating lower bounds for quasi-ranges provides merely obvious zero evaluations.

**Remark 3.6.** We have $q_1 = 1$ and $q_2 = N - 1$ for $i = 1$ and $j = n$, respectively. For the sample range $Q(1, n) = X_{n:n} - X_{1:n}$ in particular, the original sequence (3.10) and its projection (3.22) coincide, and Corollary 3.2 yields the result of Theorem 2.2 in Balakrishnan et al. (2003).

Exemplary numerical results are presented in Table 1.

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