Some counterexamples concerning maximal correlation and linear regression

Nickos Papadatos†

Section of Statistics and O.R., Department of Mathematics, University of Athens, Panepistemiopolis, 157 84 Athens, Greece

Abstract

A class of examples concerning the relationship of linear regression and maximal correlation is provided. More precisely, these examples show that if two random variables have (strictly) linear regression on each other, then their maximal correlation is not necessarily equal to their (absolute) correlation.

Key words and phrases: Maximal Correlation Coefficient; Linear Regression; Sarmanov Theorem.

1 Maximal correlation and linear regression

Let \((X,Y)\) be a bivariate random vector such that its Pearson correlation coefficient,

\[ \rho(X,Y) := \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}, \]  

is well defined. If \(W\) is a non-degenerate random variable then \(L^2_\ast(W)\) is defined to be the class of measurable functions \(g : \mathbb{R} \to \mathbb{R}\) such that \(0 < \text{Var}[g(W)] < \infty\). Under the present notation, the maximal correlation coefficient is defined as (Gebelein, 1941; Hirschfeld, 1935)

\[ R(X,Y) := \sup_{g_1 \in L^2_\ast(X), \ g_2 \in L^2_\ast(Y)} \rho(g_1(X),g_2(Y)). \]  

Due to results of Sarmanov (1958a, 1958b), it was believed for some time that if both \(X\) and \(Y\) have linear regression on each other, i.e., if for some constants \(a_0, a_1, b_0, b_1\),

\[ \mathbb{E}(X|Y) = a_1 Y + a_0 \ \text{(a.s.)}, \quad \mathbb{E}(Y|X) = b_1 X + b_0 \ \text{(a.s.)}, \]  

then

\[ R(X,Y) = |\rho(X,Y)|. \]  

*Work partially supported by the University of Athens Research Grant 70/4/5637.
† e-mail: npapadat@math.uoa.gr    url: http://users.uoa.gr/~npapadat/
The implication \((3) \Rightarrow (4)\) was cited in a number of subsequent works related to maximal correlation of order statistics and records, including Rohatgi and Székely (1992), Arnold, Balakrishnan and Nagaraja (1998, p. 101), Székely and Gupta (1998), David and Nagaraja (2003, p. 74), Ahsanullah (2004, p. 23) and Barakat (2012). However, as we shall show below, this implication is not valid even in the case of a strictly linear regression, \(a_1b_1 \neq 0\). Note that if \(R(X, Y) > 0\) then the converse implication, \((4) \Rightarrow (3)\), is valid; see Rényi (1959, p. 447) and Dembo, Kagan and Shepp (2001).

Examples of uncorrelated random variables \(X, Y\) with (trivial) linear regression

\[
E(X|Y) = E(Y|X) = 0 \quad \text{(a.s.)} \tag{5}
\]

and \(R(X, Y) > 0 = |\rho(X, Y)|\) are known for a long time. For instance, P. Bártfai has calculated \(R(X, Y) = 1/3\) for a uniform in the interior of the unit disc. This result was extended by P. Csáki and J. Fischer for the uniform distribution in the domain \(|x|^p + |y|^p < 1\) (\(p > 0\)), in which case \(R(X, Y) = (p + 1)^{-1}\); see Rényi (1959, p. 447) and Csáki and Fischer (1963). Furthermore, Székely and Móri (1985) extended this result to the multivariate case and with different exponents. Moreover, in response to a question asked by Sid Browne of Columbia University, Dembo, Kagan and Shepp (2001) constructed a pair \((X, Y)\) satisfying (5) and \(R(X, Y) = 1\). (Observe that the same is true for the uniform distribution in the four-point domain \(\{(0, \pm 1), (\pm 1, 0)\}\).)

Using characterizations of Vershik (1964) and Eaton (1986), they also showed that for any non-Gaussian spherically symmetric random vector \((U_1, \ldots, U_k)\), with covariance matrix of rank \(\geq 2\), there exists a pair of uncorrelated linear forms,

\[
X = a_1U_1 + \cdots + a_kU_k, \quad Y = b_1U_1 + \cdots + b_kU_k,
\]

such that (5) is fulfilled and \(R(X, Y) > |\rho(X, Y)| = 0\).

However, in the author’s opinion, it is important to definitely know that (3) does not imply (4) even in the non-trivial linear regression case. Indeed, if this implication were valid in the particular case where \(a_1b_1 \neq 0\), then several works concerning characterizations of distributions through maximal correlation of order statistics and records – including the papers by Terrell (1983), Székely and Móri (1985), Nevzorov (1992), López-Blázquez and Castaño-Martínez (2006), Castaño-Martínez, López-Blázquez and Salamanca-Miño (2007), Papadatos and Xifara (2012) – would be reduced to trivial consequences of this implication. The same is true for the main result in Dembo, Kagan and Shepp (2001), since it is easily checked that for the partial sums \(S_k = X_1 + \cdots + X_k\), based on an iid sequence with mean \(\mu\) and finite non-zero variance,

\[
E(S_{n+m}|S_n) = S_n + m\mu \quad \text{(a.s.)}, \quad E(S_n|S_{n+m}) = \frac{n}{n+m}S_{n+m} \quad \text{(a.s.)}.
\]

The purpose of the present note is to present a quite general class of random vectors \((X, Y)\), with \(X\) and \(Y\) possessing strictly linear regression on each other, and such that \(R(X, Y) > |\rho(X, Y)| > 0\). This class is elementary and it is defined in the next section.
2 Counterexamples

Let \( f_1 \) and \( f_2 \) be two univariate probability densities (with respect to Lebesgue measure on \( \mathbb{R} \)) with bounded supports, \( \text{supp}(f_i) \subseteq [\alpha_i, \omega_i] \), \(-\infty < \alpha_i < \omega_i < \infty \) \( (i = 1, 2) \). It is well known that there exists a uniquely defined orthonormal polynomial system \( \{\phi_n(x)\}_{n=0}^{\infty} \), standardized by \( \text{lead}(\phi_n) := p_n > 0 \), where \( \text{lead}(\phi_n) \) denotes the principal coefficient of \( \phi_n \). Also, there exists a uniquely defined orthonormal polynomial system \( \{\psi_n(x)\}_{n=0}^{\infty} \), standardized by \( \text{lead}(\psi_n) := q_n > 0 \). Each system is complete in the corresponding \( L_2 \)-space, since for any real \( t \),

\[
\int_{-\infty}^{\infty} e^{tx} f_1(x) dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} e^{ty} f_2(y) dy < \infty;
\]

see, e.g., Berg and Christensen (1981) or Afendras, Papadatos and Papathanasiou (2011). Since every polynomial is uniformly bounded in any finite interval, we can find constants \( c_n, d_n \) such that

\[
1 < \sup_{\alpha_1 \leq x \leq \omega_1} |\phi_n(x)| = c_n < \infty, \quad 1 < \sup_{\alpha_2 \leq y \leq \omega_2} |\psi_n(y)| = d_n < \infty, \quad n = 1, 2, \ldots.
\]

Consider an arbitrary real sequence \( \{\rho_n\}_{n=1}^{\infty} \) such that

\[
\sum_{n=1}^{\infty} |\rho_n| c_n d_n \leq 1, \quad (6)
\]

e.g., \( \rho_n = 6(\pi^2 n^2 c_n d_n)^{-1} \) \( (n = 1, 2, \ldots) \) or \( \rho_n = \lambda n \) \( (n = 1, \ldots, N) \) and \( \rho_n = 0 \), otherwise, where \( 0 < \lambda \leq \left(\sum_{n=1}^{N} n c_n d_n\right)^{-1} \). Then, the function

\[
f(x, y) := f_1(x) f_2(y) \left(1 + \sum_{n=1}^{\infty} \rho_n \phi_n(x) \psi_n(y)\right), \quad (x, y) \in [\alpha_1, \omega_1] \times [\alpha_2, \omega_2], \quad (7)
\]

and \( f := 0 \) outside \([\alpha_1, \omega_1] \times [\alpha_2, \omega_2]\), is a bivariate probability density with marginal densities \( f_1, f_2 \); this is so because, due to (6), the series in (7) converges, for each \((x, y)\) in the domain of definition, to a value greater than or equal to \(-1\). (Actually, the series converges uniformly and absolutely in \([\alpha_1, \omega_1] \times [\alpha_2, \omega_2]\).) Therefore, \( f(x, y) \) is nonnegative. Next, it is easily checked that its integral over \( \mathbb{R}^2 \) equals 1, due to the orthonormality of the polynomials. Finally, it is obvious that the marginal densities of \( f \) are \( f_1, f_2 \).

Assume now that the random vector \((X, Y)\) has density \( f \). Then \( X \) has density \( f_1 \) and \( Y \) has density \( f_2 \). Moreover, versions of the conditional densities are given by

\[
f_{X|Y}(x|y) = f_1(x) \left(1 + \sum_{n=1}^{\infty} \rho_n \phi_n(x) \psi_n(y)\right), \quad \alpha_1 \leq x \leq \omega_1 \quad (\text{for each } y \in \text{supp}(f_2)),
\]
\[
f_{Y|X}(y|x) = f_2(y) \left(1 + \sum_{n=1}^{\infty} \rho_n \phi_n(x) \psi_n(y)\right), \quad \alpha_2 \leq y \leq \omega_2 \quad (\text{for each } x \in \text{supp}(f_1)).
\]
Due to the orthonormality of the polynomials it follows that for all $n \geq 1$,
\[
E(\phi_n(X)|Y) = \rho_n \psi_n(Y) \quad \text{(a.s.)}, \quad E(\psi_n(Y)|X) = \rho_n \phi_n(X) \quad \text{(a.s.)}. \quad (8)
\]

Clearly, if $\rho_1 \neq 0$, (8) shows that $X$ and $Y$ have strictly linear regression on each other. In fact, it is easily checked, using (8) and induction on $n$, that
\[
E(X^n|Y) = \frac{\rho_0 q_n}{p_n} Y^n + P_{n-1}(Y) \quad \text{(a.s.)}, \quad E(Y^n|X) = \frac{\rho_0 p_n}{q_n} X^n + Q_{n-1}(X) \quad \text{(a.s.)}, \quad (9)
\]
where $P_{n-1}(t)$ and $Q_{n-1}(t)$ are polynomials of degree at most $n - 1$ in $t$. Using (9) and the main result of Papadatos and Xifara (2012), or directly from (8), it is a simple matter to conclude that $R(X,Y) = \sup_{n \geq 1} |\rho_n|$. Since the choice of $\{\rho_n\}_{n=1}^\infty$ is quite arbitrary (see (6)), it follows that
\[
R(X,Y) > |\rho(X,Y)| = |\rho_1| > 0 \quad \text{whenever} \quad 0 < |\rho_1| < \sup_{n \geq 2} |\rho_n|.
\]

**Remark.** (a) It is obvious that the construction (7) can be adapted to the discrete (lattice) case where $(X,Y) \in \{1, \ldots, N\}^2$, covering the characterizations (for finite populations) treated by López-Blázquez and Castaño-Martínez (2006) and Castaño-Martínez, López-Blázquez and Salamanca-Miño (2007).

(b) Distributions with densities of the form (7) are known as Lancaster distributions; see, e.g., Koudou (1998) or Diaconis and Griffiths (2012). They can be viewed as extensions of the Sarmanov-type distribution ($\rho_n = 0$ for $n \geq 2$) which, assuming standard uniform marginals, generalizes the so called Farlie-Gumbel-Morgenstern family.

**Acknowledgement.** I would like to thank H.N. Nagaraja and H.M. Barakat for bringing to my attention the papers by Rohatgi and Székely (1992) and Sarmanov (1958a, 1958b).

**REFERENCES**


