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Strengthened Chernoff-type variance bounds

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Let X be an absolutely continuous random variable from the integrated Pearson family and assume that X has finite moments of any order. Using some properties of the associated orthonormal polynomial system, we provide a class of strengthened Chernoff-type variance bounds.

Keywords: Chernoff-type variance bounds; derivatives; integrated Pearson family; Rodrigues polynomials

1. Introduction

Let Z be a standard normal random variable and $g:\mathbb{R}\to\mathbb{R}$ any absolutely continuous function with derivative g' such that $\mathbb{E}(g'(X))^2<\infty$. Chernoff [14], using Hermite polynomials, proved that

$$\operatorname{Var} g(Z)\le \mathbb{E}\big(g'(Z)\big)^2; \tag{1.1}$$

see, also, Nash [20] and Brascamp and Lieb [9]. In (1.1), the equality holds if and only if g is a polynomial of degree at most one – a linear function. This inequality plays an important role in the isoperimetric problem, as well as to several areas in probability and statistics. It has been extended and generalized by many authors, including [1,8,10,11,13,17–19,21–25]. On the other hand, Cacoullos [10] showed the inequality

$$\operatorname{Var} g(Z)\ge \mathbb{E}^2 g'(Z), \tag{1.2}$$

in which the equality again holds if and only if g is linear.

In this article, we provide improvements on Chernoff's bound. In particular, an application of the main result (Theorem 3.1, $n=1$) to Z yields the inequality

$$\operatorname{Var} g(Z)\le \tfrac{1}{2}\mathbb{E}^2 g'(Z)+\tfrac{1}{2}\mathbb{E}\big(g'(Z)\big)^2, \tag{1.3}$$

in which the equality holds if and only if g is a polynomial of degree at most two. In view of (1.2), it is clear that the upper bound in (1.3) improves the one given in (1.1) and, in fact, it is strictly better, unless g is linear. The difference in right-hand sides (1.1) minus (1.3) is equal to $\frac{1}{2}\operatorname{Var} g'(Z)$, indicating the magnitude of this improvement.

Similar bounds are valid for all distributions that will be studied in the sequel, namely, the Beta, Gamma and Normal. The main result applies to any Pearson (more precisely, integrated Pearson) random variable possessing moments of any order. Hence, Theorem 3.1 also improves the bounds for Beta random variables, given by [24,25]. The integrated Pearson distributions are defined as follows, see [1–3,18]:

Definition 1.1 (Integrated Pearson family). *Let X be an absolutely continuous random variable with density f and finite mean $\mu = \mathbb{E}X$. We say that X (or its density f) belongs to the integrated Pearson family if there exists a quadratic polynomial $q(x) = \delta x^2 + \beta x + \gamma$ with $\delta, \beta, \gamma \in \mathbb{R}$, $|\delta| + |\beta| + |\gamma| > 0$, such that*

$$\int_{-\infty}^{x} (\mu - t) f(t)\,\mathrm{d}t = q(x) f(x) \qquad \text{for all } x \in \mathbb{R}. \tag{1.4}$$

This fact will be denoted by

$$X \sim \mathrm{IP}(\mu; q) \text{ or } f \sim \mathrm{IP}(\mu; q) \quad \text{or, more explicitly,} \quad X \text{ or } f \sim \mathrm{IP}(\mu; \delta, \beta, \gamma). \tag{1.5}$$

In the sequel, whenever we claim that X or $f \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$, it will be understood that the density f has been chosen in $C^\infty(\alpha, \omega)$ and is vanishing outside (α, ω), where $(\alpha, \omega) := (\operatorname{ess\,inf}(X), \operatorname{ess\,sup}(X))$ is the interval support of X; see [2], Proposition 2.1. Consider an arbitrary real polynomial q with $\deg(q) \le 2$ such that the set $S^+(q) := \{x : q(x) > 0\}$ is nonempty. It can be shown that for any $\mu \in S^+(q)$ (i.e., with $q(\mu) > 0$), there exists a unique (up to equality in distribution) random variable X with mean μ such that its density f satisfies (1.4); see [2], Section 2.

Many commonly used continuous distributions are members of the integrated Pearson family, for example, Normal, Beta, Gamma and Negative Gamma. This list also includes Pareto (with density $f(x) = a(x+1)^{-a-1}$, $x > 0$, and parameter $a > 1$), Reciprocal Gamma (with density $f(x) = \lambda^a x^{-a-1} \mathrm{e}^{-\lambda/x} / \Gamma(a)$, $x > 0$, and parameters $a > 1$ and $\lambda > 0$), $F_{n,m}$ (with $m > 2$) and t_n (with $n > 1$) distributions, their location-scale families and their negatives – see Table 2.1 in [2] for a complete description. The proof of the main result is based on specific properties of the associated orthogonal polynomials that can be found in [2]. For easy reference, all required results are reviewed in Appendix.

2. Preliminaries

The following definition will be used in the sequel.

Definition 2.1 (Cf. [1], page 3629). *Assume that $X \sim \mathrm{IP}(\mu; q)$ and denote by $q(x) = \delta x^2 + \beta x + \gamma$ its quadratic polynomial. Let (α, ω) be the support of X and fix an integer $n \in \{1, 2, \ldots\}$. We shall denote by $\mathcal{H}^n(X)$ the class of functions $g : (\alpha, \omega) \to \mathbb{R}$ satisfying the following two properties*:

H_1: *For each $k \in \{0, 1, \ldots, n-1\}$, $g^{(k)}$ (with $g^{(0)} = g$) is an absolutely continuous function with a.s. derivative $g^{(k+1)}$. That is, $g \in C^{n-1}(\alpha, \omega)$ and the function $g^{(n-1)} : (\alpha, \omega) \to \mathbb{R}$, with*

$$g^{(n-1)}(x) := \frac{\mathrm{d}^{n-1} g(x)}{\mathrm{d}x^{n-1}}, \qquad \alpha < x < \omega,$$

is absolutely continuous in (α,ω) *with a.s. derivative* $g^{(n)}$ *such that*

$$g^{(n-1)}(y)-g^{(n-1)}(x)=\int_x^y g^{(n)}(t)\,\mathrm{d}t \qquad \textit{for every compact interval } [x,y]\subseteq(\alpha,\omega).$$

H_2: $\mathbb{E}q^n(X)(g^{(n)}(X))^2<\infty$.

Also, we denote by $\mathcal{H}^0(X)$ and $\mathcal{H}^\infty(X)$ the following classes of functions:

$$\mathcal{H}^0(X):=L^2(\mathbb{R},X)\equiv\big\{g:(\alpha,\omega)\to\mathbb{R}, \text{Borel measurable, such that } \operatorname{Var} g(X)<\infty\big\};$$

$$\mathcal{H}^\infty(X):=\bigcap_{n=0}^{\infty}\mathcal{H}^n(X)=\big\{g\in C^\infty(\alpha,\omega):\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2<\infty \text{ for all } n=0,1,\dots\big\}.$$

It is clear that $\mathbb{E}^2q^n(X)|g^{(n)}(X)|\le\mathbb{E}q^n(X)\mathbb{E}q^n(X)(g^{(n)}(X))^2<\infty$, provided $\mathbb{E}|X|^{2n}<\infty$ (equivalently, $\delta<1/(2n-1)$; see Lemma A.1). On the other hand, under suitable moment conditions on X, the assumption H_2 implies that $\mathbb{E}q^i(X)(g^{(i)}(X))^2<\infty$ for all $i\in\{0,1,\dots,n\}$. In particular, if all moments exist (equivalently, if $\delta\le 0$), then

$$L^2(\mathbb{R},X)=\mathcal{H}^0(X)\supseteq\mathcal{H}^1(X)\supseteq\mathcal{H}^2(X)\supseteq\cdots\supseteq\mathcal{H}^\infty(X),$$

that is, $\mathcal{H}^n(X)=\bigcap_{i=0}^n\mathcal{H}^i(X)$ for all n. In order to verify this fact we first show a lemma.

Lemma 2.1. *If* $X\sim\mathrm{IP}(\mu;q)$ *with support* (α,ω) *and* $g:(\alpha,\omega)\to\mathbb{R}$ *is an absolutely continuous function with a.s. derivative* g' *such that* $\mathbb{E}q(X)(g'(X))^2<\infty$ *then* $\mathbb{E}g^2(X)<\infty$.

Proof. Observe that $g^2(X)\le 2g^2(\mu)+2(g(X)-g(\mu))^2$. Since $\mu\in(\alpha,\omega)$,

$$\begin{aligned}\mathbb{E}\big(g(X)-g(\mu)\big)^2&=\int_\alpha^\mu f(x)\bigg(\int_x^\mu g'(t)\,\mathrm{d}t\bigg)^2\mathrm{d}x+\int_\mu^\omega f(x)\bigg(\int_\mu^x g'(t)\,\mathrm{d}t\bigg)^2\mathrm{d}x\\&\le\int_\alpha^\mu f(x)(\mu-x)\int_x^\mu\big(g'(t)\big)^2\mathrm{d}t\,\mathrm{d}x+\int_\mu^\omega f(x)(x-\mu)\int_\mu^x\big(g'(t)\big)^2\mathrm{d}t\,\mathrm{d}x\\&=\mathbb{E}q(X)\big(g'(X)\big)^2,\end{aligned}$$

by the Cauchy–Schwarz inequality and Tonelli's theorem; cf. Lemma 3.1 in [22]. □

Corollary 2.1. *If* $X\sim\mathrm{IP}(\mu;q)$, $\mathbb{E}|X|^{2n-1}<\infty$ *and* $g\in\mathcal{H}^n(X)$ *for some fixed* $n\in\{1,2,\dots\}$ *then* $\mathbb{E}q^i(X)(g^{(i)}(X))^2<\infty$ *for all* $i\in\{0,1,\dots,n\}$. *In particular,* $\operatorname{Var} g(X)<\infty$, *that is,* $g\in L^2(\mathbb{R},X)$.

Proof. According to Theorem A.3, the assumptions on X enable us to define the random variables X_k with densities

$$f_k(x)=\frac{q^k(x)f(x)}{\mathbb{E}q^k(X)},\qquad \alpha<x<\omega, k=0,1,\dots,n-1,$$

where (α,ω) is the support of X (and of each X_k). If $q(x)=\delta x^2+\beta x+\gamma$ is the quadratic of X, then $X_k\sim \mathrm{IP}(\mu_k;q_k)$ with mean μ_k and quadratic q_k given by

$$\mu_k=\frac{\mu+k\beta}{1-2k\delta},\qquad q_k(x)=\frac{\delta x^2+\beta x+\gamma}{1-2k\delta}=\delta_k x^2+\beta_k x+\gamma_k,\qquad k=0,1,\ldots,n-1.$$

Set $\widetilde{g}=g^{(n-1)}$, $\widetilde{\mu}=\mu_{n-1}$, $\widetilde{q}=q_{n-1}$, $\widetilde{X}=X_{n-1}$ and observe that $\widetilde{X}\sim \mathrm{IP}(\widetilde{\mu};\widetilde{q})$ and

$$\mathbb{E}\widetilde{q}(\widetilde{X})\big(\widetilde{g}'(\widetilde{X})\big)^2=\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2}{(1-(2n-2)\delta)\mathbb{E}q^{n-1}(X)}<\infty,$$

because $g\in\mathcal{H}^n(X)$ so that the numerator is finite. [In view of Lemma A.1, $\mathbb{E}|X|^{2n-1}<\infty$ implies the inequality $(2n-2)\delta<1$; moreover, $\deg(q^{n-1})\le 2n-2$ shows that $0<\mathbb{E}q^{n-1}(X)<\infty$.] An application of Lemma 2.1 to $\widetilde{g}$, $\widetilde{X}$ shows that $\mathbb{E}\widetilde{g}^2(\widetilde{X})<\infty$, and thus,

$$\mathbb{E}q^{n-1}(X)\big(g^{(n-1)}(X)\big)^2=\mathbb{E}\widetilde{g}^2(\widetilde{X})\mathbb{E}q^{n-1}(X)<\infty.$$

Hence, $g\in\mathcal{H}^{n-1}(X)$. Continuing inductively the result follows. □

Turn now to the case where $X\sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\le 0$. It follows that all moments exist and, moreover, the moment generating function of X is finite in a neighborhood of zero (see [2], Table 2.1, types 1–3). Then, it is well known that the orthonormalized polynomial system $\{\phi_k\}_{k=0}^{\infty}$, given by (A.6) (with $n=\infty$), is complete in $L^2(\mathbb{R},X)$; see, for example, [3,7]; see also Remark A.3, below. Consider a function $g\in\mathcal{H}^n(X)$ for some fixed $n\in\{1,2,\ldots\}$. Since $\mathcal{H}^n(X)\subseteq L^2(\mathbb{R},X)$, g can be expanded as

$$g(x)\sim\sum_{k=0}^{\infty}\alpha_k\phi_k(x), \tag{2.1}$$

where $\alpha_k=\mathbb{E}\phi_k(X)g(X)$ are the Fourier coefficients of g. The series converges in the norm of $L^2(\mathbb{R},X)$, that is, $\mathbb{E}[g(X)-\sum_{k=0}^{N}\alpha_k\phi_k(X)]^2\to 0$ as $N\to\infty$. Parseval's identity shows that

$$\operatorname{Var} g(X)=\sum_{k=1}^{\infty}\alpha_k^2,\qquad g\in L^2(\mathbb{R},X). \tag{2.2}$$

On the other hand, since $g\in\mathcal{H}^n(X)$, (A.8) yields the expression

$$\alpha_k=\frac{\mathbb{E}q^k(X)g^{(k)}(X)}{\sqrt{k!c_k(\delta)\mathbb{E}q^k(X)}}\qquad\text{for } k=1,2,\ldots,n,$$

where $c_k(\delta)=\prod_{j=k-1}^{2k-2}(1-j\delta)$, see (A.3), and $\mathbb{E}q^k(X)$ is given explicitly in (A.9). Thus, in the particular case where $g\in\mathcal{H}^n(X)$, (2.2) produces the equivalent formula

$$\operatorname{Var} g(X)=\sum_{k=1}^{n}\frac{\mathbb{E}^2q^k(X)g^{(k)}(X)}{k!c_k(\delta)\mathbb{E}q^k(X)}+\sum_{k=n+1}^{\infty}\alpha_k^2,\qquad g\in\mathcal{H}^n(X). \tag{2.3}$$

Now, consider the following heuristic derivation: Formally, we differentiate term by term (n times) the series (2.1) to get, in view of Theorem A.5, the expansion

$$g^{(n)}(x) \sim \sum_{k=0}^{\infty} \alpha_{k+n} \phi_{k+n}^{(n)}(x) = \sum_{k=0}^{\infty} \nu_k^{(n)} \alpha_{k+n} \phi_{k,n}(x). \tag{2.4}$$

Let $\text{lead}(P)$ be the leading coefficient of a polynomial P. The constants $\nu_k^{(n)} = \nu_k^{(n)}(\mu; q)$ are given by (A.18) and $\{\phi_{k,n}(x)\}_{k=0}^{\infty}$ (with $\text{lead}(\phi_{k,n}) > 0$) is the orthonormal polynomial system corresponding to X_n with density $f_n = q^n f / \mathbb{E} q^n(X)$; $\phi_{k,n}$ is a (positive) scalar multiple of the polynomial $P_{k,n}$ given in (A.16). Now, if the expansion (2.4) was indeed correct in the $L^2(\mathbb{R}, X_n)$-sense, then the completeness of the system $\{\phi_{k,n}\}_{k=0}^{\infty}$ in $L^2(\mathbb{R}, X_n)$ would result to the corresponding Parseval identity:

$$\frac{\mathbb{E} q^n(X)(g^{(n)}(X))^2}{\mathbb{E} q^n(X)} = \mathbb{E}\big(g^{(n)}(X_n)\big)^2 = \sum_{k=0}^{\infty} \big(\nu_k^{(n)}\big)^2 \alpha_{k+n}^2, \qquad g \in \mathcal{H}^n(X). \tag{2.5}$$

Finally, from (A.18) we have

$$\big(\nu_k^{(n)}\big)^2 = \frac{(k+n)!}{k! \mathbb{E} q^n(X)} \prod_{j=k+n-1}^{k+2n-2} (1 - j\delta).$$

A combination of the last equation with (2.5) yields the identity

$$\begin{aligned} \mathbb{E} q^n(X)\big(g^{(n)}(X)\big)^2 &= \sum_{k=0}^{\infty} \frac{(k+n)! \prod_{j=k+n-1}^{k+2n-2} (1 - j\delta)}{k!} \alpha_{k+n}^2 \\ &= \sum_{k=n}^{\infty} \frac{k! \prod_{j=k-1}^{k+n-2} (1 - j\delta)}{(k-n)!} \alpha_k^2. \end{aligned} \tag{2.6}$$

This must be correct for all $g \in \mathcal{H}^n(X)$, provided that expansion (2.4) is valid. However, the above arguments are heuristic; they are not sufficient even to conclude convergence of the series (2.6) or (2.5). Notice that the same technicality appeared in Chernoff's [14] proof, although in this case the polynomials are the well-known Hermite polynomials (with derivatives again Hermite, i.e., orthogonal to the same weight function, the normal density). Chernoff overcame this difficulty by applying Weierstrass (uniform) approximations to g in compact intervals.

In the sequel, we shall make the above arguments rigorous by applying a different technique, in the spirit of Sturm–Liouville theory. In fact, we shall show more, namely, that an initial segment of the Fourier coefficients for the nth derivative of g, suggested by (2.4), can be derived for any $X \sim \text{IP}(\mu; \delta, \beta, \gamma)$ having a sufficient number of moments. This result holds even if $\delta > 0$, noting that if $\delta > 0$ then X possesses only a finite number of moments. Specifically, the following result, which may have some interest in itself, holds true.

Lemma 2.2. *Assume that X has density f, support (α, ω), $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ and $\mathbb{E}|X|^{2N} < \infty$ for some $N \geq 1$, that is, $\delta < \frac{1}{2N-1}$. Let $\{\phi_k\}_{k=0}^{N} \subseteq L^2(\mathbb{R}, X)$ be the orthonormal polynomial system associated with X (where, to be specific, assume that $\mathrm{lead}(\phi_k) > 0$). Then, for every $x \in (\alpha, \omega)$,*

$$\begin{aligned} q(x) f(x) \phi_k'(x) &= -\lambda_k(\delta) \int_\alpha^x \phi_k(y) f(y)\,\mathrm{d}y \\ &= \lambda_k(\delta) \int_x^\omega \phi_k(y) f(y)\,\mathrm{d}y, \qquad k = 1, 2, \ldots, N, \end{aligned} \tag{2.7}$$

where $\lambda_k(\delta) := k(1 - (k-1)\delta)$. Moreover, if $g \in \mathcal{H}^n(X)$ for some $n \in \{1, 2, \ldots, N\}$ then

$$\mathbb{E}\phi_{k,n}(X_n) g^{(n)}(X_n) = \nu_k^{(n)} \mathbb{E}\phi_{k+n}(X) g(X), \qquad k = 0, 1, \ldots, N-n, \tag{2.8}$$

where X_n has density $f_n = q^n f / \mathbb{E}q^n(X)$,

$$\nu_k^{(n)} = \sqrt{\frac{(k+n)!}{k!} \frac{\prod_{j=k+n-1}^{k+2n-2}(1 - j\delta)}{\mathbb{E}q^n(X)}}$$

is given by (A.18) and $\{\phi_{k,n}\}_{k=0}^{N-n} \subseteq L^2(\mathbb{R}, X_n)$ is the orthonormal polynomial system corresponding to X_n, with $\mathrm{lead}(\phi_{k,n}) > 0$.

Proof. From (1.4) it follows that

$$\frac{f'(x)}{f(x)} = \frac{\mu - x - q'(x)}{q(x)} = \frac{-(1 + 2\delta)x + (\mu - \beta)}{\delta x^2 + \beta x + \gamma}, \qquad \alpha < x < \omega.$$

Consider the polynomials P_k defined in (A.2). By (A.6), each ϕ_k is a scalar multiple of the Rodrigues-type polynomial $h_k = D^k[q^k f]/f = (-1)^k P_k$. Hence, Theorem 1 of Diaconis and Zabell [15] (see, also, equation (4.4) in [2]) implies that

$$\left[q(x) f(x) \phi_k'(x)\right]' = -\lambda_k(\delta) \phi_k(x) f(x), \qquad \alpha < x < \omega, k = 1, 2, \ldots, N. \tag{2.9}$$

Fix t and x with $\alpha < t < x < \omega$ and integrate (2.9) over the interval $[t, x]$ to get

$$-\lambda_k(\delta) \int_t^x \phi_k(y) f(y)\,\mathrm{d}y = q(x) f(x) \phi_k'(x) - q(t) f(t) \phi_k'(t);$$

thus, taking limits as $t \searrow \alpha$ we see that the l.h.s. converges to $-\lambda_k(\delta) \int_\alpha^x \phi_k(y) f(y)\,\mathrm{d}y$, by dominated convergence, while the r.h.s. tends to $q(x) f(x) \phi_k'(x)$ because, by Lemma A.2, $\lim_{t \searrow \alpha} q(t) f(t) h(t) = 0$ for any polynomial h with $\deg(h) \leq 2N - 1$. This verifies the first equality in (2.7), while the second one is obvious since $\mathbb{E}\phi_k(X) = 0$ (because ϕ_k is orthogonal to $\phi_0 \equiv 1$).

Fix now an integer $k \in \{0, 1, \ldots, N-1\}$. Observing that $\deg(q(x) x^{2k}) \leq 2k + 2 \leq 2N$ we have $\mathbb{E}(X_1^k)^2 = \mathbb{E}q(X) X^{2k} / \mathbb{E}q(X) < \infty$. Thus, the Rodrigues-type polynomial $P_{k,1}$ (see (A.16) with $m = 1$) belongs to $L^2(\mathbb{R}, X_1)$. By Corollary 2.1, $\mathbb{E}(g'(X_1))^2$ is also finite. Indeed, $n \le N$ implies that $\mathbb{E}|X|^{2n-1} < \infty$ so that $g \in \mathcal{H}^n(X) \subseteq \mathcal{H}^1(X)$ and, therefore,

$$\mathbb{E}\big(g'(X_1)\big)^2 = \frac{1}{\mathbb{E}q(X)} \mathbb{E}q(X)\big(g'(X)\big)^2 < \infty.$$

Hence, the Fourier coefficient of g' with respect to $\phi_{k,1}$, $\mathbb{E}\phi_{k,1}(X_1)g'(X_1)$, is well-defined (and finite):

$$\mathbb{E}^2\big|\phi_{k,1}(X_1)g'(X_1)\big| \le \mathbb{E}\big(\phi_{k,1}(X_1)\big)^2 \mathbb{E}\big(g'(X_1)\big)^2 = \mathbb{E}\big(g'(X_1)\big)^2 < \infty.$$

Let $\rho_1 < \rho_2 < \cdots < \rho_m$ be the distinct roots of ϕ_{k+1} that lie into the interval (α, ω). Clearly, $1 \le m \le k+1$ because $\mathbb{E}\phi_{k+1}(X) = 0$ and $\deg(\phi_{k+1}) = k+1$. Fix now a number $\rho \in [\rho_1, \rho_m] \subseteq (\alpha, \omega)$. From (A.19), we see that $\phi_{k,1}(x) = \phi'_{k+1}(x)/\nu_k^{(1)}$ where $\nu_k^{(1)} = \sqrt{(k+1)(1-k\delta)/\mathbb{E}q(X)}$. Therefore, using (2.7), we have

$$\begin{aligned}
\mathbb{E}\phi_{k,1}(X_1)g'(X_1) &= \frac{1}{\mathbb{E}q(X)} \int_\alpha^\omega g'(x)q(x)f(x)\phi_{k,1}(x)\,\mathrm{d}x \\
&= \frac{1}{\nu_k^{(1)}\mathbb{E}q(X)} \int_\alpha^\omega g'(x)q(x)f(x)\phi'_{k+1}(x)\,\mathrm{d}x \\
&= \frac{-\lambda_{k+1}(\delta)}{\nu_k^{(1)}\mathbb{E}q(X)} \int_\alpha^\rho g'(x) \int_\alpha^x f(y)\phi_{k+1}(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \frac{\lambda_{k+1}(\delta)}{\nu_k^{(1)}\mathbb{E}q(X)} \int_\rho^\omega g'(x) \int_x^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Observing that

$$\frac{\lambda_{k+1}(\delta)}{\nu_k^{(1)}\mathbb{E}q(X)} = \frac{(k+1)(1-k\delta)}{\mathbb{E}q(X)\sqrt{(k+1)(1-k\delta)/\mathbb{E}q(X)}} = \nu_k^{(1)},$$

the preceding equation can be rewritten as

$$\mathbb{E}\phi_{k,1}(X_1)g'(X_1) = \nu_k^{(1)}(I_2 - I_1), \tag{2.10}$$

where

$$I_1 := \int_\alpha^\rho g'(x) \int_\alpha^x f(y)\phi_{k+1}(y)\,\mathrm{d}y\,\mathrm{d}x, \qquad I_2 := \int_\rho^\omega g'(x) \int_x^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y\,\mathrm{d}x. \tag{2.11}$$

Now, we wish to change the order of integration to both integrals I_1 and I_2. To this end, for I_2 it suffices to show that

$$I_2^* := \int_\rho^\omega \big|g'(x)\big| \int_x^\omega f(y)\big|\phi_{k+1}(y)\big|\,\mathrm{d}y\,\mathrm{d}x < \infty. \tag{2.12}$$

Similarly, for I_1 it suffices to show that $I_1^* := \int_\alpha^\rho |g'(x)| \int_\alpha^x f(y)|\phi_{k+1}(y)|\,\mathrm{d}y\,\mathrm{d}x < \infty$. We now proceed to verify (2.12). Write $I_2^* = I_{21}^* + I_{22}^*$ where

$$I_{21}^* := \int_\rho^{\rho_m} \big|g'(x)\big| \int_x^\omega f(y)\big|\phi_{k+1}(y)\big|\,\mathrm{d}y\,\mathrm{d}x,$$
$$I_{22}^* := \int_{\rho_m}^\omega \big|g'(x)\big| \int_x^\omega f(y)\big|\phi_{k+1}(y)\big|\,\mathrm{d}y\,\mathrm{d}x.$$

Since the polynomial ϕ_{k+1} does not change sign in the interval (ρ_m, ω), we can define the constant π as

$$\pi := \operatorname{sign}\big(\phi_{k+1}(x)\big) \in \{-1, 1\}, \qquad \rho_m < x < \omega.$$

Then, $\pi\phi_{k+1}(x) = |\phi_{k+1}(x)|$ holds for all $x \in (\rho_m, \omega)$ and from (2.7) we get

$$\begin{aligned}
I_{22}^* &= \pi \int_{\rho_m}^\omega \big|g'(x)\big| \int_x^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y\,\mathrm{d}x = \frac{\pi}{\lambda_{k+1}(\delta)} \int_{\rho_m}^\omega \big|g'(x)\big| q(x)f(x)\phi'_{k+1}(x)\,\mathrm{d}x \\
&\le \frac{1}{\lambda_{k+1}(\delta)} \int_{\rho_m}^\omega \big|g'(x)\big| q(x)f(x)\big|\phi'_{k+1}(x)\big|\,\mathrm{d}x \\
&\le \frac{1}{\lambda_{k+1}(\delta)} \int_\alpha^\omega \big|g'(x)\big| q(x)f(x)\big|\phi'_{k+1}(x)\big|\,\mathrm{d}x = \frac{1}{\lambda_{k+1}(\delta)} \mathbb{E}q(X)\big|\phi'_{k+1}(X)g'(X)\big| \\
&= \frac{\nu_k^{(1)}}{\lambda_{k+1}(\delta)} \mathbb{E}q(X)\big|\phi_{k,1}(X)g'(X)\big| = \frac{1}{\nu_k^{(1)}} \mathbb{E}\big|\phi_{k,1}(X_1)g'(X_1)\big| < \infty.
\end{aligned}$$

This shows that $I_{22}^* < \infty$. On the other hand, the function $x \mapsto q(x)f(x)$ is strictly positive and continuous for x in the compact interval $[\rho, \rho_m] \subseteq (\alpha, \omega)$, so that, $\theta := \min\{q(x)f(x) : \rho \le x \le \rho_m\} > 0$. Then, from the fact that $g \in \mathcal{H}^1(X)$, we get

$$\begin{aligned}
\int_\rho^{\rho_m} \big|g'(x)\big|\,\mathrm{d}x &\le \frac{1}{\theta} \int_\rho^{\rho_m} q(x)f(x)\big|g'(x)\big|\,\mathrm{d}x \le \frac{1}{\theta} \mathbb{E}q(X)\big|g'(X)\big| \\
&\le \frac{1}{\theta} \sqrt{\mathbb{E}q(X)\mathbb{E}q(X)\big(g'(X)\big)^2} < \infty.
\end{aligned}$$

Moreover, for any u_1, u_2 with $\alpha \le u_1 \le u_2 \le \omega$ it is readily seen that

$$\int_{u_1}^{u_2} \big|\phi_{k+1}(y)\big| f(y)\,\mathrm{d}y \le \int_\alpha^\omega \big|\phi_{k+1}(y)\big| f(y)\,\mathrm{d}y = \mathbb{E}\big|\phi_{k+1}(X)\big| := M_{k+1} < \infty.$$

Combining the above, we conclude that

$$I_{21}^* = \int_\rho^{\rho_m} \big|g'(x)\big| \int_x^\omega f(y)\big|\phi_{k+1}(y)\big|\,\mathrm{d}y\,\mathrm{d}x \le M_{k+1} \int_\rho^{\rho_m} \big|g'(x)\big|\,\mathrm{d}x < \infty.$$

Therefore, $I_2^* = I_{21}^* + I_{22}^* < \infty$ and (2.12) follows. Using similar arguments it is shown that $I_1^* < \infty$. Thus, we can indeed interchange the order of integration to both integrals I_1 and I_2 of (2.11). It follows that

$$I_2 = \int_\rho^\omega f(y)\phi_{k+1}(y) \int_\rho^y g'(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_\rho^\omega f(y)\phi_{k+1}(y)g(y)\,\mathrm{d}y - g(\rho)\int_\rho^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y$$

and, similarly,

$$I_1 = g(\rho)\int_\alpha^\rho f(y)\phi_{k+1}(y)\,\mathrm{d}y - \int_\alpha^\rho f(y)\phi_{k+1}(y)g(y)\,\mathrm{d}y.$$

Taking into account the fact that $\int_\alpha^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y = \mathbb{E}\phi_{k+1}(X) = 0$, we get

$$I_2 - I_1 = \int_\alpha^\omega f(y)\phi_{k+1}(y)g(y)\,\mathrm{d}y - g(\rho)\int_\alpha^\omega f(y)\phi_{k+1}(y)\,\mathrm{d}y = \mathbb{E}\phi_{k+1}(X)g(X).$$

Finally, from (2.10), we conclude that

$$\mathbb{E}\phi_{k,1}(X_1)g'(X_1) = \sqrt{\frac{(k+1)(1-k\delta)}{\mathbb{E}q(X)}}\mathbb{E}\phi_{k+1}(X)g(X), \qquad k = 0, 1, \dots, N-1. \tag{2.13}$$

So far we have shown that $g \in \mathcal{H}^n(X)$ and $\mathbb{E}|X|^{2N} < \infty$ for some $N \ge n$ implies that $g \in \mathcal{H}^1(X)$ and (2.13) is fulfilled. Assume now that for some $i \in \{1, 2, \dots, n-1\}$ we have shown that $g \in \mathcal{H}^i(X)$ and that for every $k \in \{0, 1, \dots, N-i\}$,

$$\mathbb{E}\phi_{k,i}(X_i)g^{(i)}(X_i) = \sqrt{\frac{(k+i)!}{k!}\frac{\prod_{j=k+i-1}^{k+2i-2}(1-j\delta)}{\mathbb{E}q^i(X)}}\mathbb{E}\phi_{k+i}(X)g(X). \tag{2.14}$$

Clearly, we can apply (2.13) for $g = g^{(i)}$, $X = X_i$ and for $k = 0, 1, \dots, \widetilde{N}-1$, provided that $\mathbb{E}|X_i|^{2\widetilde{N}} < \infty$. Observing that $\mathbb{E}|X_i|^{2\widetilde{N}} = \frac{\mathbb{E}q^i(X)|X|^{2\widetilde{N}}}{\mathbb{E}q^i(X)}$ it follows that $\widetilde{N} = N - i$ is a suitable choice. Therefore, for $k = 0, 1, \dots, N-i-1$, (2.13) yields

$$\mathbb{E}\phi_{k,i+1}(X_{i+1})g^{(i+1)}(X_{i+1}) = \sqrt{\frac{(k+1)(1-k\delta_i)}{\mathbb{E}q_i(X_i)}}\mathbb{E}\phi_{k+1,i}(X_i)g^{(i)}(X_i),$$

where $\delta_i = \frac{\delta}{1-2i\delta}$, $q_i(x) = \frac{q(x)}{1-2i\delta}$ (see Theorem A.3) and, thus,

$$\mathbb{E}q_i(X_i) = \frac{\mathbb{E}q(X_i)}{1-2i\delta} = \frac{\mathbb{E}q^{i+1}(X)}{(1-2i\delta)\mathbb{E}q^i(X)}.$$

Finally, calculating $\mathbb{E}\phi_{k+1,i}(X_i)g^{(i)}(X_i)$ from (2.14) (for $k=0,1,\ldots,N-i-1$) we see that

$$\begin{aligned}
&\mathbb{E}\phi_{k,i+1}(X_{i+1})g^{(i+1)}(X_{i+1})\\
&\quad=\sqrt{\frac{(k+1)(1-k\delta/(1-2i\delta))}{\mathbb{E}q^{i+1}(X)/((1-2i\delta)\mathbb{E}q^{i}(X))}}\sqrt{\frac{(k+i+1)!}{(k+1)!}\frac{\prod_{j=k+i}^{k+2i-1}(1-j\delta)}{\mathbb{E}q^{i}(X)}}\mathbb{E}\phi_{k+i+1}(X)g(X)\\
&\quad=\sqrt{\frac{(k+i+1)!}{k!}\frac{\prod_{j=k+i}^{k+2i-1}(1-j\delta)}{\mathbb{E}q^{i+1}(X)}}\mathbb{E}\phi_{k+i+1}(X)g(X),\qquad k=0,1,\ldots,N-i-1,
\end{aligned}$$

which verifies the inductional step and shows that (2.14) holds for all $i\in\{1,2,\ldots,n\}$. Letting $i=n$ in (2.14) completes the proof. □

3. The strengthened inequality

In the present section, we assume that $X\sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\le 0$. The well-known Normal, Gamma and Beta random variables and their affine transformations are of this form – see [2], Table 2.1. In this case the orthonormal polynomial system $\{\phi_k\}_{k=0}^{\infty}$ is complete in $L^2(\mathbb{R},X)$ and, therefore, the following result holds.

Lemma 3.1. *If $X\sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\le 0$, then*

$$\operatorname{Var} g(X)=\sum_{k=1}^{\infty}\alpha_k^2\qquad \text{for any } g\in L^2(\mathbb{R},X),\tag{3.1}$$

where

$$\alpha_k=\mathbb{E}\phi_k(X)g(X),\qquad k=0,1,2,\ldots,\tag{3.2}$$

are the Fourier coefficients of g with respect to the orthonormal polynomial system $\{\phi_k\}_{k=0}^{\infty}$. If, furthermore, $g\in\mathcal{H}^n(X)$ for some $n\in\{1,2,\ldots\}$, then

$$\alpha_k=\mathbb{E}\phi_k(X)g(X)=\frac{\mathbb{E}q^k(X)g^{(k)}(X)}{\sqrt{k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)}},\qquad k=1,2,\ldots,n\tag{3.3}$$

and

$$\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2=\sum_{k=n}^{\infty}\frac{k!\prod_{j=k-1}^{k+n-2}(1-j\delta)}{(k-n)!}\alpha_k^2,\tag{3.4}$$

with α_k given by (3.2).

Proof. (3.1) is the well-known Parseval's identity. Also, if $g\in\mathcal{H}^n(X)$ then, by Corollary 2.1, $g\in\mathcal{H}^k(X)$ for all $k\in\{0,1,\ldots,n\}$. Therefore, the Cauchy–Schwarz inequality shows that

$\mathbb{E}q^k(X)|g^{(k)}(X)| \le \mathbb{E}q^k(X)\mathbb{E}q^k(X)(g^{(k)}(X))^2 < \infty$. Hence, (3.3) follows from (A.4) – see Theorem A.2 – and the fact that the polynomials $P_k(x) := (-1)^k D^k[q^k(x)f(x)]/f(x)$ are related to ϕ_k by $P_k(x) = \phi_k(x)\sqrt{k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)}$ for all $k \in \{1, 2, \dots\}$. Moreover, by Lemma 2.2 we have that for any $g \in \mathcal{H}^n(X)$, the Fourier coefficients $\alpha_k = \mathbb{E}\phi_k(X)g(X)$ (of g with respect to X) and the Fourier coefficients $\alpha_k^{(n)} := \mathbb{E}\phi_{k,n}(X_n)g^{(n)}(X_n)$ of $g^{(n)}$ with respect to X_n are related through

$$\alpha_k^{(n)} = \sqrt{\frac{(k+n)!}{k!}\frac{\prod_{j=k+n-1}^{k+2n-2}(1-j\delta)}{\mathbb{E}q^n(X)}}\alpha_{k+n}, \qquad k = 0, 1, 2, \dots,$$

where $\mathbb{E}q^n(X)$ is given explicitly by (A.9). Finally, Theorem A.3 asserts that

$$X_n \sim \mathrm{IP}(\mu_n; \delta_n, \beta_n, \gamma_n) \qquad \text{with } \delta_n = \frac{\delta}{1-2n\delta} \le 0.$$

Hence, $\delta_n \le 0$ guarantees that the corresponding orthonormal polynomial system $\{\phi_{k,n}\}_{k=0}^{\infty}$ is complete in $L^2(\mathbb{R}, X_n)$. Since $g \in \mathcal{H}^n(X)$, $g^{(n)} \in L^2(\mathbb{R}, X_n)$ and, by Parseval's identity,

$$\mathbb{E}\big(g^{(n)}(X_n)\big)^2 = \sum_{k=0}^{\infty}\big(\alpha_k^{(n)}\big)^2 = \frac{1}{\mathbb{E}q^n(X)}\sum_{k=0}^{\infty}\frac{(k+n)!\prod_{j=k+n-1}^{k+2n-2}(1-j\delta)}{k!}\alpha_{k+n}^2$$

(thus, the series converges). Observing that

$$\mathbb{E}\big(g^{(n)}(X_n)\big)^2 = \frac{1}{\mathbb{E}q^n(X)}\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2,$$

(3.4) is deduced and the proof is complete. □

We are now in a position to state and prove the main result of the paper.

Theorem 3.1. *If $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ with $\delta \le 0$ and if $g \in \mathcal{H}^n(X)$ for some $n \in \{1, 2, \dots\}$ then*

$$\begin{aligned}\operatorname{Var} g(X) \le \sum_{k=1}^{n}&\frac{\mathbb{E}^2 q^k(X)g^{(k)}(X)}{k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)}\\ &+\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2 - (1/\mathbb{E}q^n(X))\mathbb{E}^2q^n(X)g^{(n)}(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)},\end{aligned} \tag{3.5}$$

with equality if and only if g is a polynomial of degree at most $n+1$.

In particular, if $\sigma^2 = \operatorname{Var} X$ and g is absolutely continuous with a.s. derivative g' such that $\mathbb{E}q(X)(g'(X))^2 < \infty$ (i.e., $g \in \mathcal{H}^1(X)$) then

$$\operatorname{Var} g(X) \le \left(1 - \frac{1}{2(1-\delta)}\right)\frac{1}{\sigma^2}\mathbb{E}^2 q(X)g'(X) + \frac{1}{2(1-\delta)}\mathbb{E}q(X)\big(g'(X)\big)^2, \tag{3.6}$$

with equality if and only if g is a polynomial of degree at most two.

Three examples of (3.6) are as follows:

Example 3.1. If $X \sim N(\mu, \sigma^2) \equiv \mathrm{IP}(\mu; 0, 0, \sigma^2)$ then $\delta = 0$, $q(x) \equiv \sigma^2$ and we obtain the inequality

$$\operatorname{Var} g(X) \le \tfrac{1}{2}\sigma^2 \mathbb{E}^2 g'(X) + \tfrac{1}{2}\sigma^2 \mathbb{E}\big(g'(X)\big)^2, \tag{3.7}$$

in which the equality holds if and only if g is a polynomial of degree at most two. Chernoff's upper bound, $\operatorname{Var} g(X) \le \sigma^2 \mathbb{E}(g'(X))^2$, is strictly weaker than (3.7) since, obviously, $\mathbb{E}^2 g'(X) \le \mathbb{E}(g'(X))^2$, and the equality holds if and only if g is linear. It should be noted that $\sigma^2 \mathbb{E}^2 g'(X)$ is, actually, a lower bound for $\operatorname{Var} g(X)$; see, for example, [10].

Example 3.2. If $X \sim \Gamma(a, \lambda) \equiv \mathrm{IP}(a/\lambda; 0, 1/\lambda, 0)$ so that $f(x) = \lambda^a x^{a-1} \mathrm{e}^{-\lambda x}/\Gamma(a)$, $x > 0$, then $\delta = 0$, $q(x) = x/\lambda$, $\sigma^2 = a/\lambda^2$ and we obtain the inequality

$$\operatorname{Var} g(X) \le \frac{1}{2a} \mathbb{E}^2 X g'(X) + \frac{1}{2\lambda} \mathbb{E} X \big(g'(X)\big)^2, \tag{3.8}$$

in which the equality holds if and only if g is a polynomial of degree at most two.

Example 3.3. If $X \sim B(a, b) \equiv \mathrm{IP}(\frac{a}{a+b}; \frac{-1}{a+b}, \frac{1}{a+b}, 0)$ then $\delta = \frac{-1}{a+b}$, $q(x) = \frac{x(1-x)}{a+b}$, $\sigma^2 = \frac{ab}{(a+b)^2(a+b+1)}$ and we obtain the inequality

$$\operatorname{Var} g(X) \le \frac{a+b+2}{2ab} \mathbb{E}^2 X(1-X) g'(X) + \frac{1}{2(a+b+1)} \mathbb{E} X(1-X) \big(g'(X)\big)^2, \tag{3.9}$$

in which the equality holds if and only if g is a polynomial of degree at most two. In the particular case where $a = b = 1$, $X = U$ is uniformly distributed over the interval (0, 1) and (3.9) yields an improvement of Polya's inequality (see, e.g., [4]),

$$\int_0^1 g^2(x)\,\mathrm{d}x - \left(\int_0^1 g(x)\,\mathrm{d}x\right)^2 \le \frac{1}{2} \int_0^1 x(1-x)\big(g'(x)\big)^2\,\mathrm{d}x.$$

Indeed, for $a = b = 1$, (3.9) yields

$$\int_0^1 g^2(x)\,\mathrm{d}x - \left(\int_0^1 g(x)\,\mathrm{d}x\right)^2 \le 2\left(\int_0^1 x(1-x) g'(x)\,\mathrm{d}x\right)^2 + \frac{1}{6} \int_0^1 x(1-x)\big(g'(x)\big)^2\,\mathrm{d}x,$$

and the upper bound is smaller than Polya's bound because, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \left(\int_0^1 x(1-x) g'(x)\,\mathrm{d}x\right)^2 &\le \int_0^1 x(1-x)\,\mathrm{d}x \int_0^1 x(1-x)\big(g'(x)\big)^2\,\mathrm{d}x \\ &= \frac{1}{6} \int_0^1 x(1-x)\big(g'(x)\big)^2\,\mathrm{d}x. \end{aligned}$$

Remark 3.1. In [11,18,22] it was shown that $\operatorname{Var} g(X) \le \mathbb{E}q(X)(g'(X))^2$; the equality in this Chernoff-type variance bound is attained only by linear functions g. Also, in [10,12,18,22] it was shown that $\operatorname{Var} g(X) \ge \frac{1}{\sigma^2}\mathbb{E}^2 q(X)g'(X)$, in which the equality characterizes again the linear functions. We observe that the upper bound in (3.6) is a convex combination of the preceding lower and upper bounds and, thus, smaller than the Chernoff-type upper bound, $\mathbb{E}q(X)(g'(X))^2$. Also, the last term in the upper bound (3.5) can be rewritten as

$$\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2 - (1/\mathbb{E}q^n(X))\mathbb{E}^2 q^n(X)g^{(n)}(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)} = \frac{\mathbb{E}q^n(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)} \operatorname{Var} g^{(n)}(X_n).$$

Thus, we can apply the Chernoff-type upper bound to $\operatorname{Var} g^{(n)}(X_n)$, provided that $g^{(n)} \in \mathcal{H}^1(X_n)$. Recall that $g^{(n)} \in \mathcal{H}^1(X_n)$ means that $g^{(n)}$ is absolutely continuous with a.s. derivative $g^{(n+1)}$ such that $\mathbb{E}q_n(X_n)(g^{(n+1)}(X_n))^2 < \infty$. Since $X_n \sim f_n = q^n f/\mathbb{E}q^n(X)$, $\delta \le 0$ and $q_n(x) = q(x)/(1-2n\delta)$, the preceding requirement is equivalent to

$$\frac{1}{(1-2n\delta)\mathbb{E}q^n(X)}\mathbb{E}q^{n+1}(X)\big(g^{(n+1)}(X)\big)^2 < \infty;$$

thus, $g^{(n)} \in \mathcal{H}^1(X_n)$ if and only if $g \in \mathcal{H}^{n+1}(X)$. Therefore, if $g \in \mathcal{H}^{n+1}(X)$ then we have

$$\operatorname{Var} g^{(n)}(X_n) \le \mathbb{E}q_n(X_n)\big(g^{(n+1)}(X_n)\big)^2 = \frac{\mathbb{E}q^{n+1}(X)(g^{(n+1)}(X))^2}{(1-2n\delta)\mathbb{E}q^n(X)},$$

with equality if and only if $g^{(n)}$ is linear, that is, g is a polynomial of degree at most $n+1$. The preceding inequality shows that for any $g \in \mathcal{H}^{n+1}(X)$,

$$\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2 - (1/\mathbb{E}q^n(X))\mathbb{E}^2 q^n(X)g^{(n)}(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)} \le \frac{\mathbb{E}q^{n+1}(X)(g^{(n+1)}(X))^2}{(n+1)!\prod_{j=n}^{2n}(1-j\delta)},$$

with equality only for polynomial g of degree at most $n+1$. Combining the upper bound in (3.5) with the last displayed inequality, we obtain the weaker bound

$$\operatorname{Var} g(X) \le \sum_{k=1}^{n-1} \frac{\mathbb{E}^2 q^k(X)g^{(k)}(X)}{k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)} + \frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2}{n!\prod_{j=n-1}^{2n-2}(1-j\delta)}, \tag{3.10}$$

which holds for any $g \in \mathcal{H}^n(X)$, and the equality is attained if and only if g is a polynomial of degree at most n. For $n=1$ this is the Chernoff-type variance bound. Also, for $X \sim B(a,b)$, (3.10) has been shown by Wei and Zhang [25], using Jacobi polynomials.

Proof of Theorem 3.1. From (3.1) and (3.3),

$$\operatorname{Var} g(X) - \sum_{k=1}^{n} \frac{\mathbb{E}^2 q^k(X)g^{(k)}(X)}{k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta)} = \alpha_{n+1}^2 + \alpha_{n+2}^2 + \cdots, \tag{3.11}$$

with α_k given by (3.2). Also, from (3.3) with $k=n$,

$$\frac{1}{\mathbb{E}q^n(X)}\mathbb{E}^2 q^n(X)g^{(n)}(X) = n!\left(\prod_{j=n-1}^{2n-2}(1-j\delta)\right)\alpha_n^2.$$

Thus, in view of (3.4),

$$\begin{aligned}&\mathbb{E}q^n(X)\big(g^{(n)}(X)\big)^2 - \frac{1}{\mathbb{E}q^n(X)}\mathbb{E}^2 q^n(X)g^{(n)}(X)\\ &\quad=\sum_{k=n}^{\infty}\frac{k!\prod_{j=k-1}^{k+n-2}(1-j\delta)}{(k-n)!}\alpha_k^2 - n!\left(\prod_{j=n-1}^{2n-2}(1-j\delta)\right)\alpha_n^2 = \sum_{k=n+1}^{\infty}\frac{k!\prod_{j=k-1}^{k+n-2}(1-j\delta)}{(k-n)!}\alpha_k^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2 - (1/\mathbb{E}q^n(X))\mathbb{E}^2 q^n(X)g^{(n)}(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)}\\ &\quad=\sum_{k=n+1}^{\infty}\frac{k!\prod_{j=k-1}^{k+n-2}(1-j\delta)}{(k-n)!(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)}\alpha_k^2 = \alpha_{n+1}^2 + \sum_{k=n+2}^{\infty}\lambda_k\alpha_k^2,\end{aligned}$$

where

$$\lambda_k := \frac{1}{n+1}\binom{k}{n}\frac{\prod_{j=k-1}^{k+n-2}(1-j\delta)}{\prod_{j=n}^{2n-1}(1-j\delta)}, \qquad k=n+2,n+3,\ldots.$$

The sequence $\{\lambda_k\}_{k=n+2}^{\infty}$ is nondecreasing in k. Indeed, since $\delta\le 0$, we have

$$1\le 1-\delta\le 1-2\delta\le 1-3\delta\le\cdots$$

and thus, $k\mapsto\prod_{j=k-1}^{k+n-2}(1-j\delta)$ is nondecreasing in k and positive (for each k the product contains n positive factors). Also,

$$k\mapsto\binom{k}{n}$$

is, obviously, positive and nondecreasing in k. Thus, for every $k\ge n+2$,

$$\lambda_k\ge\lambda_{n+2}=\left(1+\frac{n}{2}\right)\left(1-\frac{n\delta}{1-n\delta}\right)>1,$$

because $1+n/2>1$ and $1-n\delta/(1-n\delta)\ge 1$ (since $\delta\le 0$). It follows that

$$\frac{\mathbb{E}q^n(X)(g^{(n)}(X))^2 - (1/\mathbb{E}q^n(X))\mathbb{E}^2 q^n(X)g^{(n)}(X)}{(n+1)!\prod_{j=n}^{2n-1}(1-j\delta)}\ge\alpha_{n+1}^2+\alpha_{n+2}^2+\cdots, \tag{3.12}$$

with equality if and only if $\alpha_{n+2} = \alpha_{n+3} = \cdots = 0$, that is, if and only if g is a polynomial of degree at most $n+1$. A combination of (3.11) and (3.12) completes the proof. □

Remark 3.2. The upper bound in (3.5) is meaningful (it is nonnegative and makes sense) even for $0 < \delta < \frac{1}{2n-1}$, in which case $\mathbb{E}|X|^{2n} < \infty$. Also, since $x^{n+1} \in L^2(\mathbb{R}, X)$ if and only if $\delta < \frac{1}{2n+1}$, it would be desirable to show the validity of (3.5) at least when $0 < \delta < \frac{1}{2n+1}$. For example, we have tried, without success, to prove (3.6) when $0 < \delta < \frac{1}{3}$. In contrast to the corresponding Chernoff-type bound, which can be shown directly (without Fourier expansions – see, e.g., [13]; cf. Lemma 2.1, above), it seems that the completeness of the corresponding orthonormal polynomial system in $L^2(\mathbb{R}, X)$ plays a crucial role in proving (3.6).

Appendix

Proposition A.1 ([2], Proposition 2.1). *Let* $X \sim \mathrm{IP}(\mu; q)$ *and set* $(\alpha, \omega) := (\operatorname{ess\,inf}(X), \operatorname{ess\,sup}(X))$. *Then, there is a version* f *of the density of* X *such that*

(i) $f(x)$ *is strictly positive for* x *in* (α, ω) *and zero otherwise, that is,* $\{x : f(x) > 0\} = (\alpha, \omega)$;
(ii) $f \in C^{\infty}(\alpha, \omega)$, *that is,* f *has derivatives of any order in* (α, ω);
(iii) X *is a (usual) Pearson random variable supported in* (α, ω), *that is,* $f'(x)/f(x) = p_1(x)/q(x)$, $x \in (\alpha, \omega)$, *where* $p_1(x) = \mu - x - q'(x)$ *is a polynomial of degree at most one*;
(iv) $q(x) = \delta x^2 + \beta x + \gamma > 0$ *for all* $x \in (\alpha, \omega)$;
(v) *if* $\alpha > -\infty$ *then* $q(\alpha) = 0$ *and, similarly, if* $\omega < +\infty$ *then* $q(\omega) = 0$;
(vi) *for any* $\theta, c \in \mathbb{R}$ *with* $\theta \neq 0$, *the random variable* $\widetilde{X} := \theta X + c \sim \mathrm{IP}(\widetilde{\mu}; \widetilde{q})$ *with* $\widetilde{\mu} = \theta\mu + c$ *and* $\widetilde{q}(x) = \theta^2 q((x-c)/\theta)$.

Lemma A.1 ([2], Corollary 2.2). *Assume that* $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$.

(i) *If* $\delta \le 0$, *then* $\mathbb{E}|X|^{\theta} < \infty$ *for any* $\theta \in [0, \infty)$.
(ii) *If* $\delta > 0$, *then* $\mathbb{E}|X|^{\theta} < \infty$ *for any* $\theta \in [0, 1 + 1/\delta)$, *while* $\mathbb{E}|X|^{1+1/\delta} = \infty$.

Lemma A.2 ([2], Lemma 2.1). *If* $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma) \equiv \mathrm{IP}(\mu; q)$ *has support* (α, ω) *and* $\mathbb{E}|X|^n < \infty$ *for some* $n \ge 1$ *(equivalently,* $\delta < 1/(n-1)$*), then for any polynomial* Q_{n-1} *of degree at most* $n-1$,

$$\lim_{x \nearrow \omega} q(x) f(x) Q_{n-1}(x) = \lim_{x \searrow \alpha} q(x) f(x) Q_{n-1}(x) = 0. \tag{A.1}$$

Theorem A.1 ([16], page 401; [6], pages 99–100; [15], page 295; [2], Theorem 4.1). *Assume that* f *is the density of a random variable* $X \sim \mathrm{IP}(\mu; q) \equiv \mathrm{IP}(\mu; \delta, \beta, \gamma)$ *with support* (α, ω). *Then, the functions* $P_k : (\alpha, \omega) \to \mathbb{R}$ *with*

$$P_k(x) := \frac{(-1)^k}{f(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\big[q^k(x) f(x)\big], \qquad \alpha < x < \omega, k = 0, 1, 2, \ldots \tag{A.2}$$

are (*Rodrigues-type*) *polynomials with*

$$\deg(P_k) \le k \quad \textit{and} \quad \operatorname{lead}(P_k) = \prod_{j=k-1}^{2k-2} (1 - j\delta) := c_k(\delta), \qquad k = 0, 1, 2, \ldots, \tag{A.3}$$

where $\operatorname{lead}(P_k)$ *is the coefficient of* x^k *in* $P_k(x)$. *Here* $c_0(\delta) := 1$, *that is, an empty product should be treated as one.*

Theorem A.2 ([3], pages 515–516; [2], Theorem 5.1). *Let* $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma) \equiv \mathrm{IP}(\mu; q)$ *with density* f *and support* (α, ω). *Assume that* X *has* $2k$ *finite moments for some fixed* $k \in \{1, 2, \ldots\}$. *Let* $g : (\alpha, \omega) \to \mathbb{R}$ *be any function such that* $g \in C^{k-1}(\alpha, \omega)$, *and assume that the function*

$$g^{(k-1)}(x) := \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} g(x)$$

is absolutely continuous in (α, ω) *with a.s. derivative* $g^{(k)}$. *If* $\mathbb{E}q^k(X)|g^{(k)}(X)| < \infty$ *then* $\mathbb{E}|P_k(X)g(X)| < \infty$, *where* P_k *is the polynomial defined by* (A.2) *of Theorem* A.1, *and the following covariance identity holds*:

$$\mathbb{E} P_k(X) g(X) = \mathbb{E} q^k(X) g^{(k)}(X). \tag{A.4}$$

It should be noted that when we claim that $h : (\alpha, \omega) \to \mathbb{R}$ is an absolutely continuous function with a.s. derivative h' we mean that there exists a Borel measurable function $h' : (\alpha, \omega) \to \mathbb{R}$ such that h' is integrable in every finite subinterval $[x, y]$ of (α, ω), and

$$\int_x^y h'(t)\,\mathrm{d}t = h(y) - h(x) \qquad \text{for all compact intervals } [x, y] \subseteq (\alpha, \omega).$$

Corollary A.1 ([3], equation (3.5), page 516; [2], Corollary 5.1). *Let* $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma) \equiv \mathrm{IP}(\mu; q)$. *Assume that for some* $n \in \{1, 2, \ldots\}$, $\mathbb{E}|X|^{2n} < \infty$ *or, equivalently,* $\delta < 1/(2n - 1)$. *Then, the polynomials defined by* (A.2) *of Theorem* A.1 *satisfy the orthogonality condition*

$$\begin{aligned} \mathbb{E}\big[P_k(X) P_m(X)\big] &= \delta_{k,m} k! \mathbb{E} q^k(X) \prod_{j=k-1}^{2k-2} (1 - j\delta) \\ &= \delta_{k,m} k! c_k(\delta) \mathbb{E} q^k(X), \qquad k, m \in \{0, 1, \ldots, n\}, \end{aligned} \tag{A.5}$$

where $\delta_{k,m}$ *is Kronecker's delta and where an empty product should be treated as one.*

Remark A.1. The orthogonality of P_k and P_m, $k \neq m$, $k, m \in \{0, 1, \ldots, n\}$, remains valid even if $\delta \in [\frac{1}{2n-1}, \frac{1}{2n-2})$; in this case, however, $P_n \notin L^2(\mathbb{R}, X)$ since $\operatorname{lead}(P_n) > 0$ and $\mathbb{E}|X|^{2n} = \infty$.

Remark A.2. In view of Lemma A.1, the assumption $\mathbb{E}|X|^{2n} < \infty$ is equivalent to the condition $\delta < \frac{1}{2n-1}$. Therefore, for each $k \in \{1, \ldots, n\}$ and for all $j \in \{k-1, \ldots, 2k-2\}$ we have $1 - j\delta >$

0 because

$$\{k-1,\dots,2k-2\}\subseteq\{0,1,\dots,2n-2\}.$$

Thus, $c_k(\delta)>0$. Since $\mathbb{P}[q(X)>0]=1$, $\deg(q)\le 2$ and $\mathbb{E}|X|^{2n}<\infty$ we conclude that $0<\mathbb{E}q^k(X)<\infty$ for all $k\in\{0,1,\dots,n\}$. It follows that the set $\{\phi_0,\phi_1,\dots,\phi_n\}\subset L^2(\mathbb{R},X)$, where

$$\begin{aligned}\phi_k(x) &:= \frac{P_k(x)}{(k!c_k(\delta)\mathbb{E}q^k(X))^{1/2}}\\ &= \frac{((-1)^k/f(x))(\mathrm{d}^k/\mathrm{d}x^k)[q^k(x)f(x)]}{(k!\mathbb{E}q^k(X)\prod_{j=k-1}^{2k-2}(1-j\delta))^{1/2}}, \qquad k=0,1,\dots,n,\end{aligned} \tag{A.6}$$

is an orthonormal basis of all polynomials with degree at most n. By (A.3), the leading coefficient of ϕ_k is

$$\mathrm{lead}(\phi_k)=\left(\frac{\prod_{j=k-1}^{2k-2}(1-j\delta)}{k!\mathbb{E}q^k(X)}\right)^{1/2}=\left(\frac{c_k(\delta)}{k!\mathbb{E}q^k(X)}\right)^{1/2}>0, \qquad k=0,1,\dots,n. \tag{A.7}$$

The orthonormal system $\{\phi_k\}_{k=0}^n$ is characterized by the fact that $\deg(\phi_k)=k$ and $\mathrm{lead}(\phi_k)>0$ for each k.

Remark A.3. The identity (A.4) enables a convenient calculation of the Fourier coefficients of any (smooth enough) function g with $\operatorname{Var} g(X)<\infty$. More precisely, if $X\sim \mathrm{IP}(\mu;\delta,\beta,\gamma)\equiv \mathrm{IP}(\mu;q)$ and $\mathbb{E}|X|^{2n}<\infty$ for some $n\ge 1$ then the Fourier coefficients of g, $\alpha_k=\mathbb{E}\phi_k(X)g(X)$, are given by $\alpha_0=\mathbb{E}g(X)$ and

$$\alpha_k=\frac{\mathbb{E}q^k(X)g^{(k)}(X)}{(k!c_k(\delta)\mathbb{E}q^k(X))^{1/2}}, \qquad k=1,2,\dots,n, \tag{A.8}$$

provided that g is smooth enough so that $\mathbb{E}q^k(X)|g^{(k)}(X)|<\infty$ for $k\in\{1,2,\dots,n\}$; cf. [3], Theorem 5.1(a). Here $c_k(\delta)$ is given by (A.3) and for any $k\in\{1,\dots,n\}$ (see [2], Corollary 5.3)

$$\mathbb{E}q^k(X)=\frac{\prod_{j=0}^{k-1}(1-2j\delta)}{\prod_{j=0}^{k-1}(1-(2j+1)\delta)}\prod_{j=0}^{k-1}q\left(\frac{\mu+j\beta}{1-2j\delta}\right). \tag{A.9}$$

In the particular case where $X\sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ and $\delta\le 0$ (i.e., if X is of Normal, Gamma or Beta-type), it follows that $\mathbb{E}|X|^n<\infty$ for all n. Moreover, there exists an $\varepsilon>0$ such that $\mathbb{E}e^{tX}<\infty$ for $|t|<\varepsilon$ (see types 1–3 of Table 2.1 in [2]). Hence, the polynomials $\{\phi_k\}_{k=0}^{\infty}$, given by (A.6) (with $n=\infty$), form a complete orthonormal system in $L^2(\mathbb{R},X)$; see, for example, [3,7]. Therefore, the Fourier coefficients are easily obtained for any smooth enough function g such that $\operatorname{Var} g(X)<\infty$ and $\mathbb{E}q^k(X)|g^{(k)}(X)|<\infty$ for all $k\ge 1$. Indeed, in this case we have

$$\alpha_k=\mathbb{E}\phi_k(X)g(X)=\frac{\mathbb{E}q^k(X)g^{(k)}(X)}{(k!c_k(\delta)\mathbb{E}q^k(X))^{1/2}}, \qquad k=0,1,2,\dots, \tag{A.10}$$

where $\mathbb{E}q^k(X)$ is as in (A.9). Thus, by Parseval's identity, the variance of g equals to ([3], Theorem 5.1(a))

$$\operatorname{Var} g(X) = \sum_{k=1}^{\infty} \frac{\mathbb{E}^2 q^k(X) g^{(k)}(X)}{k! c_k(\delta) \mathbb{E}q^k(X)}, \tag{A.11}$$

with $\mathbb{E}q^k(X)$ given by (A.9) and $c_k(\delta)$ by (A.3).

Theorem A.3 ([2], Theorem 5.2). *Let X be a random variable with density $f \sim \mathrm{IP}(\mu; q) \equiv \mathrm{IP}(\mu; \delta, \beta, \gamma)$, supported in (α, ω). Furthermore, assume that $\mathbb{E}|X|^{2n+1} < \infty$ (i.e., $\delta < \frac{1}{2n}$) for some $n \in \{0, 1, \ldots\}$. Define the random variable X_k with density f_k given by*

$$f_k(x) := \frac{q^k(x) f(x)}{\mathbb{E}q^k(X)}, \qquad \alpha < x < \omega, k = 0, 1, \ldots, n. \tag{A.12}$$

Then, $f_k \sim \mathrm{IP}(\mu_k; q_k)$ with (the same) support (α, ω),

$$\mu_k = \frac{\mu + k\beta}{1 - 2k\delta} \quad \text{and} \quad q_k(x) = \frac{q(x)}{1 - 2k\delta}, \qquad \alpha < x < \omega, k = 0, 1, \ldots, n. \tag{A.13}$$

Theorem A.4 ([2], Theorem 5.3; cf. [5], page 207). *If $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ with support (α, ω) and $\mathbb{E}|X|^{2n} < \infty$ for some $n \ge 1$ (i.e., $\delta < \frac{1}{2n-1}$), then for any $m \in \{1, 2, \ldots, n\}$,*

$$P_{k+m}^{(m)}(x) = C_k^{(m)}(\delta) P_{k,m}(x), \qquad \alpha < x < \omega, k = 0, 1, \ldots, n - m, \tag{A.14}$$

where

$$C_k^{(m)}(\delta) := \frac{(k+m)!}{k!} (1 - 2m\delta)^k \prod_{j=k+m-1}^{k+2m-2} (1 - j\delta). \tag{A.15}$$

Here, P_k are the polynomials given by (A.2) associated with f, and $P_{k,m}$ are the corresponding Rodrigues polynomials of (A.2), associated with the density $f_m(x) = \frac{q^m(x) f(x)}{\mathbb{E}q^m(X)}$, $\alpha < x < \omega$, of the random variable $X_m \sim \mathrm{IP}(\mu_m; q_m)$ defined in Theorem A.3, that is,

$$\begin{aligned} P_{k,m}(x) &:= \frac{(-1)^k}{f_m(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \bigl[q_m^k(x) f_m(x) \bigr] \\ &= \frac{(-1)^k}{(1 - 2m\delta)^k q^m(x) f(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \bigl[q^{k+m}(x) f(x) \bigr], \qquad \alpha < x < \omega, k = 0, 1, \ldots, n - m. \end{aligned} \tag{A.16}$$

Theorem A.5 ([2], Corollary 5.4). *Let $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma) \equiv \mathrm{IP}(\mu; q)$ and assume that $\mathbb{E}|X|^{2n} < \infty$ for some fixed $n \ge 1$ (i.e., $\delta < \frac{1}{2n-1}$). Let $\{\phi_k\}_{k=0}^n$ be the orthonormal polynomials associated with X, with $\operatorname{lead}(\phi_k) > 0$; see (A.6), (A.7). Fix a number $m \in \{0, 1, \ldots, n\}$, and consider the corresponding orthonormal polynomials $\{\phi_{k,m}\}_{k=0}^{n-m}$, with $\operatorname{lead}(\phi_{k,m}) > 0$, associated with $X_m \sim f_m = q^m f / \mathbb{E}q^m(X)$. Then,*

$$\phi_{k+m}^{(m)}(x) = \nu_k^{(m)} \phi_{k,m}(x), \qquad k = 0, 1, \ldots, n - m, \tag{A.17}$$

where the constants $\nu_k^{(m)} = \nu_k^{(m)}(\mu; q) > 0$ *are given by*

$$\nu_k^{(m)} = \nu_k^{(m)}(\mu; q) := \left\{ \frac{((k+m)!/k!) \prod_{j=k+m-1}^{k+2m-2} (1 - j\delta)}{\mathbb{E} q^m(X)} \right\}^{1/2}, \tag{A.18}$$

with $\mathbb{E} q^m(X)$ *as in* (A.9) *with* m *in place of* k. *In particular, setting* $\sigma^2 = \operatorname{Var} X = \mathbb{E} q(X)$ *we have*

$$\begin{aligned} \phi'_{k+1}(x) &= \frac{\sqrt{(k+1)(1-k\delta)}}{\sigma} \phi_{k,1}(x) \\ &= \sqrt{\frac{(k+1)(1-\delta)(1-k\delta)}{q(\mu)}} \phi_{k,1}(x), \qquad k = 0, 1, \ldots, n-1. \end{aligned} \tag{A.19}$$

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