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EXACT BOUNDS FOR THE EXPECTATIONS OF ORDER STATISTICS FROM NON-NEGATIVE POPULATIONS

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Abstract. Some new exact bounds for the expected values of order statistics, under the assumption that the parent population is non-negative, are obtained in terms of the population mean. Similar bounds for the differences of any two order statistics are also given. It is shown that the existing bounds for the general case can be improved considerably under the above assumption.

Key words and phrases: Exact bounds, expectations, order statistics, non-negative populations.

1. Introduction

Let X be a rv with df $F_X(\cdot) \equiv F(\cdot)$ and consider the order statistics $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ corresponding to a random sample of size n drawn from F. The well known Hartley-David-Gumbel bound may be regarded as a special case $(i = n)$ of the general bound

$$|\mathrm{E}[X_{i:n}] - \mu| \le \sigma c_{i:n}, \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where μ and σ are the mean and the standard deviation, respectively, of X (see Moriguti (1953), Hartley and David (1954), Gumbel (1954) and David (1981)). For fixed i, $1 < i < n$, the constant $c_{i:n}$ in (1.1) can be calculated with the help of a specific value ρ_1 $(= \rho_1(i,n))$ which arises as the (unique) root of the polynomial equation

$$1 - G(x) = (1-x)g(x), \qquad 0 < x < 1, \tag{1.2}$$

where

$$g(x) = \frac{1}{B(i, n+1-i)} x^{i-1}(1-x)^{n-i}, \qquad G(x) = \int_0^x g(u)du \equiv I_x(i, n+1-i).$$

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Balakrishnan (1993) showed that the equation (1.2) can be rewritten in a much simpler form, namely,

$$(1.3)\qquad (n-1)\binom{n-1}{n-i}x^{i-1}=\sum_{j=0}^{i-2}\binom{n-i+j}{j}x^{j},$$

and, of course, as he pointed out, finding ρ_1 from (1.3) is a lot easier than determining it from (1.2) (the degree of the polynomial equation (1.2) is n, while the degree of (1.3) is $i-1$; see Balakrishnan (1993) for more details).

In the present paper, by using the further assumption that X is a non-negative rv (that is, $F(0_-)=0$) we improve the above results. Note that several improvements of the bounds (1.1) have been appeared in the literature the last few years (see for example Balakrishnan and Bendre (1993), Balakrishnan (1990)) but it seems that the assumption of non-negative parent population has never been imposed, though this is the usual case in most applications (e.g., when the X's represent life times in reliability etc.).

A main feature of the new bounds is that they do not involve the standard deviation σ. This fact leads to notable improvements of (1.1), especially when σ is large (see Table 1).

Since the assumption $X \geq 0$ alone fails to reach non-trivial lower bounds (see Theorem 2.2), we also give the modified sharp upper and lower bounds (see Theorem 2.3 and Corollaries 2.1 and 2.2) when it is further known that the X's lie in some bounded interval of the form $[0,\alpha]$, $0<\alpha<+\infty$.

In Section 3, the corresponding upper bounds for the expected value of the difference of any two order statistics (from non-negative populations) are derived.

2. Bounds for the expectations of order statistics

Suppose $X, X_1, \ldots, X_n$ are iid rv's with df F such that $X \geq 0$ and consider the ordered sample $0 \leq X_{1:n} \leq \cdots \leq X_{n:n}$. To avoid trivialities, we further assume that $\mathrm{E}[X]=\mu>0$. Since $F(0_-)=0$, we have the well-known relation

$$(2.1)\qquad \mu=\int_0^{+\infty}(1-F(x))dx,$$

and the analogue for $X_{i:n}$

$$(2.2)\qquad \mathrm{E}[X_{i:n}]=\int_0^{+\infty}(1-G(F(x)))dx$$

(since $G(F(x))$ is just the df of $X_{i:n}$), where $G(x)=I_x(i,n+1-i)$ is the incomplete beta function, defined as in the introduction.

Fix n and i with $1<i<n$. From Lemma 2.1 (i) in Papadatos (1995), the inequality

$$(2.3)\qquad \frac{1-G(x)}{1-x}\leq\frac{1-G(\rho_1)}{1-\rho_1}:=\mu_n(i)$$

holds for all x in $[0, 1)$, with equality iff (if and only if) $x = \rho_1$, where ρ_1 ($= \rho_1(i, n)$) is the unique root of $((1 - G(x))/(1 - x))' = 0$ in $(0, 1)$; i.e., ρ_1 is exactly the point mentioned in the introduction, which alternatively one founds by using Balakrishnan's result (1.3). Since it is readily verified that $\mu_n(i) = g(\rho_1)$ (g is the derivative of G) we have the following upper bounds.

THEOREM 2.1. *Suppose that* $X \geq 0$ *and* $\mathrm{E}[X] = \mu > 0$.

(i) *Fix* i *with* $1 < i < n$. *Then*

$$\mathrm{E}[X_{i:n}] \leq \mu_n(i)\mu, \tag{2.4}$$

and the equality holds iff $\mathrm{P}[X = 0] = \rho_1 = 1 - \mathrm{P}[X = \mu/(1 - \rho_1)]$.

(ii) $\mathrm{E}[X_{1:n}] \leq \mu$ *with equality iff* $\mathrm{P}[X = \mu] = 1$.

(iii) $\sup \mathrm{E}[X_{n:n}] = n\mu$, *where the supremum is taken over* $F \in \mathcal{F}$, *the family of all df's corresponding to non-negative rv's having mean* μ.

PROOF. (i) Since (2.3) can be rewritten in the form

$$1 - G(x) \leq (1 - x)\mu_n(i), \qquad 0 \leq x \leq 1, \tag{2.5}$$

with equality iff either $x = \rho_1$ or $x = 1$, we have

$$1 - G(F(x)) \leq (1 - F(x))\mu_n(i) \qquad \text{for all} \quad 0 \leq x < +\infty. \tag{2.6}$$

Integrating (2.6) over $[0, +\infty)$ and taking into account (2.1) and (2.2), we conclude (2.4). Observe that equality in (2.4) is equivalent to

$$\int_0^\infty [(1 - F(x))\mu_n(i) - (1 - G(F(x)))]dx = 0,$$

and this implies, in view of (2.6), that the integrand is null for almost all x in $[0, +\infty)$. Thus, from (2.5), $F(x) = 0, \rho_1, 1$ if $x < 0$, $0 \leq x < \mu/(1 - \rho_1)$, $\mu/(1 - \rho_1) \leq x$, respectively, and the proof of (i) is complete. (ii) is trivial since it holds for any df (not necessarily non-negative). For (iii) we simply have $X_{n:n} \leq \sum_{j=1}^n X_j$ and thus $\mathrm{E}[X_{n:n}] \leq n\mu$. Observe that $\mathrm{E}[X_{n:n}] = n\mu$ implies that $X_{1:n} = \cdots = X_{n-1:n} \equiv 0$ which in turn yields $\mu = 0$ (a contradiction). Thus, $\mathrm{E}[X_{n:n}] < n\mu$. However, this trivial bound is the best possible, since for the population with $\mathrm{P}[X = 0] = \alpha = 1 - \mathrm{P}[X = \mu/(1 - \alpha)]$, we have $\mathrm{E}[X] = \mu$ for all $\alpha \in (0, 1)$ and $\mathrm{E}[X_{n:n}] = [(1 - \alpha^n)/(1 - \alpha)]\mu \to n\mu$ as $\alpha \to 1$. This completes the proof of (iii).

In the simple cases $i = 2$ or $i = 3$ one can apply Balakrishnan's results for finding ρ_1 and thus $\mu_n(i)$. For example, when $i = 2$, (2.4) yields

$$\mathrm{E}[X_{2:n}] \leq \frac{n^{n-1}(n-2)^{n-2}}{(n-1)^{2n-3}}\mu, \qquad n = 3, 4, \ldots,$$

where the equality holds (for some n) only when

$$\mathrm{P}[X = 0] = \frac{1}{(n-1)^2} = 1 - P\left[X = \frac{(n-1)^2}{n(n-2)}\mu\right].$$

Similarly for $i = 3$ we have the upper bound

$$\mathrm{E}[X_{3:n}] \le \frac{n}{(n-1)^{2n-3}}[n^2 - n - 1 + \sqrt{n(n-2)(2n-3)}] \cdot \left[n^2 - 2n - \sqrt{\frac{n(2n-3)}{n-2}}\right]^{n-3} \mu$$

for $n = 4, 5, \dots$, where the equality holds for a specific two-valued df.

Since the bounds obtained by Theorem 2.1 do not depend upon σ, clearly they may be considerably lower than (1.1) at least whenever σ is large (of course these bounds continue to hold when $\sigma = +\infty$, provided μ is finite, and in this extremal case, (1.1) is not applicable, leading to the trivial bound $+\infty$). We give for illustration Table 1.

Table 1. Numerical comparison of upper bounds (1.1) and (2.4) for $\mathrm{E}[X_{2:n}]$, $\mathrm{E}[X_{3:n}]$ and $\mathrm{E}[X_{4:n}]^{(*)}$.

Upper bounds for $\mathrm{E}[X_{2:n}]$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
(2.4) for $\mu = 1$	1.125	1.053	1.030	1.019	1.013	1.010
(1.1) for $\mu = 1$, $\sigma = 1$	1.271	1.182	1.138	1.111	1.093	1.080
(1.1) for $\mu = 1$, $\sigma = 0.5$	1.135	1.091	1.069	1.056	1.046	1.040

Upper bounds for $\mathrm{E}[X_{3:n}]$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
(2.4) for $\mu = 1$	1.380	1.198	1.127	1.091	1.069
(1.1) for $\mu = 1$, $\sigma = 1$	1.506	1.376	1.306	1.260	1.228
(1.1) for $\mu = 1$, $\sigma = 0.5$	1.253	1.188	1.153	1.130	1.114

Upper bounds for $\mathrm{E}[X_{4:n}]$	$n=5$	$n=6$	$n=7$	$n=8$
(2.4) for $\mu = 1$	1.659	1.368	1.249	1.184
(1.1) for $\mu = 1$, $\sigma = 1$	1.683	1.527	1.439	1.381
(1.1) for $\mu = 1$, $\sigma = 0.5$	1.342	1.263	1.220	1.191

(*) The values have been truncated to three decimal points.

It should be noted that for a general non-negative population with mean $\mu > 0$ and standard deviation σ, the upper bound for $\mathrm{E}[X_{i:n}]$ provided by (2.4) is better (lower) than that provided by (1.1) iff $\sigma/\mu > (\mu_n(i) - 1)/c_{i:n}$, i.e., when the coefficient of variation of the population df is larger than a specific value depending only on i and n.

The following result shows that the obvious lower bounds given below can not be improved without using any further assumptions.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1,

(i) *If* $1 \le i < n$, *then* $\inf E[X_{i:n}] = 0$, *where the infimum is taken over* $\mathcal{F}$ *as in Theorem* 2.1(iii).

(ii) $E[X_{n:n}] \ge \mu$ *with equality iff* $P[X = \mu] = 1$.

PROOF. (i) Clearly $\mathrm{E}[X_{i:n}] \geq 0$. Equality never holds because it leads to the contradiction $F(x) = 1$ a.e. in $[0, +\infty)$ and $\mu > 0$. This bound is the best possible since for the special choice

$$\mathrm{P}[X = \mu/\epsilon] = \epsilon = 1 - \mathrm{P}[X = 0],$$

we have $\mathrm{E}[X] = \mu$ and $\mathrm{E}[X_{i:n}] \to 0$ as $\epsilon \to 0$. (ii) is obvious.

So far, the assumptions $X \geq 0$ and $\mathrm{E}[X] = \mu$ do not suffice alone for the derivation of reasonable lower bounds for the expectations of order statistics. It is, therefore, necessary to impose further conditions on X in order to obtain more interesting lower (and upper) bounds. A common natural condition which arises in the most practical situations is that the data are bounded above by some (finite) constant α. This is the subject of the following:

THEOREM 2.3. *Suppose that* $0 \leq X \leq 1$ *and* $\mathrm{E}[X] = \mu$, *where* $0 < \mu < 1$. *Then,*

(i) $\mathrm{E}[X_{n:n}] \leq 1 - (1-\mu)^n$ *with equality iff* $\mathrm{P}[X = 1] = \mu = 1 - \mathrm{P}[X = 0]$.

(ii) *Fix* i *with* $1 < i < n$. *Then, for* $\mu \leq 1 - \rho_1$, *the best upper bound for* $\mathrm{E}[X_{i:n}]$ *is given by* (2.4), *while, for* $\mu > 1 - \rho_1$ *we have*

$$\mathrm{E}[X_{i:n}] \leq 1 - G(1-\mu), \tag{2.7}$$

and the equality holds iff $\mathrm{P}[X = 1] = \mu = 1 - \mathrm{P}[X = 0]$.

PROOF. (i) Since $F(1) = 1$, we have from Holder's inequality

$$(1 - \mathrm{E}[X_{n:n}])^{1/n} = \left(\int_0^1 F^n(x)dx\right)^{1/n} \geq \int_0^1 F(x)dx = 1 - \mu,$$

which is equivalent to the desired result. Equality holds iff $F(x) \equiv c$ (constant) a.e. in $(0, 1)$ and thus $F(x) \equiv 1 - \mu$, $0 \leq x < 1$ and the proof is complete. As regards (ii), the interesting case is for $\mu > 1 - \rho_1$ (for $\mu \leq 1 - \rho_1$ the arguments of Theorem 2.1 (i) remain the same and (2.4) continuous to hold because the optimal population in this case already satisfies $F(1) = 1$). Suppose then that $1 - \mu < \rho_1$ and consider the function (see Moriguti (1953), David (1981), p. 63)

$$G_c(x) = \begin{cases} G(x), & 0 \leq x \leq \rho_1 \\ 1 - (1-x)g(\rho_1), & \rho_1 \leq x \leq 1 \end{cases}$$

(note that $1 - G(\rho_1) = (1-\rho_1)g(\rho_1)$). It is not hard to show that for all $x \in [0, 1]$,

$$1 - G(x) \leq 1 - G_c(x), \tag{2.8}$$

with equality only for $x \leq \rho_1$ or $x = 1$ (see (2.3)). Furthermore, $1 - G_c(x)$ is everywhere differentiable with non-increasing derivative

$$(1 - G_c(x))' = \begin{cases} -g(x), & 0 \leq x \leq \rho_1 \\ -g(\rho_1), & \rho_1 \leq x \leq 1 \end{cases}$$

and thus, $1 - G_c(x)$ is concave (note that $\rho_1 < (i-1)/(n-1)$; see Papadatos (1995) and Moriguti (1953)). Let U be a uniform $(0,1)$ rv. Then we have

$$\begin{aligned} \mathrm{E}[X_{i:n}] &= \int_0^1 (1 - G(F(x)))dx \\ &\le \int_0^1 (1 - G_c(F(x)))dx = \mathrm{E}[1 - G_c(F(U))] \\ &\le 1 - G_c[\mathrm{E}[F(U)]] = 1 - G_c\left[\int_0^1 F(x)dx\right] \\ &= 1 - G_c(1-\mu) = 1 - G(1-\mu), \end{aligned}$$

where the first inequality follows from (2.8), the second inequality is an application of Jensen's inequality to the concave function $1 - G_c(Z)$ of the rv $Z = F(U)$ and the last equality holds because $1 - \mu < \rho_1$.

It is evident that the equality holds if $\mathrm{P}[X = 1] = \mu = 1 - \mathrm{P}[X = 0]$. On the other hand, in order to have equality in the first of the preceding inequalities it is necessary that either $F(x) \le \rho_1$ or $F(x) = 1$ for all $x \in [0,1]$; as for the second, it is necessary that $F(x) \equiv c$ for those x's for which $F(x) \le \rho_1$ (because $1 - G_c(x)$ is strictly concave in $[0, \rho_1]$). Thus, the optimal solution must be of the form $\mathrm{P}[X = 0] = c = 1 - \mathrm{P}[X = \mu/(1-c)]$ for some $c \in [0, 1-\mu]$, and it is easily verified that the maximum is attained for $c = 1 - \mu$. This completes the proof.

Now the general case follows immediately.

COROLLARY 2.1. *Suppose that* $0 \le X \le \alpha$ *and* $\mathrm{E}[X] = \mu$, *where* $0 < \mu < \alpha$. *Then,*

(i) $\mathrm{E}[X_{n:n}] \le \alpha[1-(1-\mu/\alpha)^n]$ *with equality iff* $\mathrm{P}[X = \alpha] = \mu/\alpha = 1-\mathrm{P}[X = 0]$.

(ii) *Fix* i *with* $1 < i < n$. *Then, for* $\mu \le (1-\rho_1)\alpha$, *the best upper bound for* $\mathrm{E}[X_{i:n}]$ *is given by* (2.4), *while, for* $\mu > (1-\rho_1)\alpha$ *we have*

$$\mathrm{E}[X_{i:n}] \le \alpha(1 - G(1 - \mu/\alpha)),$$

and the equality holds iff $\mathrm{P}[X = \alpha] = \mu/\alpha = 1 - \mathrm{P}[X = 0]$.

PROOF. The result is evident if we apply Theorem 2.3 to $Y = X/\alpha \in [0,1]$ (observe that $Y_{i:n} = X_{i:n}/\alpha$).

For the lower-bound case the following result holds.

COROLLARY 2.2. *Under the conditions of Corollary* 2.1,

(i) $\mathrm{E}[X_{1:n}] \ge \mu^n/\alpha^{n-1}$ *with equality iff* $\mathrm{P}[X = \alpha] = \mu/\alpha = 1 - \mathrm{P}[X = 0]$.

(ii) *Fix* i *with* $1 < i < n$, *and let* $\rho_2 = \rho_2(i,n)$ *be the unique solution of* $G(x) = xg(x)$, $0 < x < 1$. *Then, for* $\mu \ge (1-\rho_2)\alpha$,

$$\mathrm{E}[X_{i:n}] \ge \alpha - g(\rho_2)(\alpha - \mu),$$

with equality iff $\mathrm{P}[X = \alpha] = 1 - \rho_2 = 1 - \mathrm{P}[X = \alpha - (\alpha - \mu)/\rho_2]$, *and for* $\mu < (1 - \rho_2)\alpha$,

$$\mathrm{E}[X_{i:n}] \geq \alpha(1 - G(1 - \mu/\alpha)),$$

where the equality holds iff $\mathrm{P}[X = \alpha] = \mu/\alpha = 1 - \mathrm{P}[X = 0]$.

PROOF. First note that the equation $G(x) = xg(x)$, $0 < x < 1$, has a unique root $\rho_2 > (i-1)/(n-1)$ which satisfies the inequality

$$\frac{G(x)}{x} < \frac{G(\rho_2)}{\rho_2}$$

for all $x \in (0,1)$, $x \neq \rho_2$ (see Papadatos (1995), Lemma 2.1 (i)). Furthermore, set $Y = \alpha - X \in [0, \alpha]$ and observe that $Y_{n+1-i:n} = \alpha - X_{i:n}$, $i = 1, \ldots, n$. Now the desired result follows by applying Corollary 2.1 to $Y_{n+1-i:n}$ and noting that the corresponding function $G^*(x) = I_x(n+1-i, i)$ satisfies the well-known relation $G^*(x) = 1 - G(1-x)$, and also, the corresponding value ρ_1^* (associated with G^*) is $\rho_1^* = \rho_1(n+1-i, n) = 1 - \rho_2(i, n)$.

It should be noted that letting $\alpha \to \infty$ in the above corollaries we get the results of Theorems 2.1 and 2.2. Note also that if $0 \leq b \leq X \leq \alpha$ for some $b \leq \alpha$, the lower bound provided by Corollary 2.2 is sharp only when $b = 0$, and it is very weak for $b > 0$. This, however, can be treated if we apply this bound to the rv $Y = X - b \in [0, \alpha - b]$.

3. Bounds for the expectation of the difference of two order statistics

In this section we discuss the upper bounds for the difference $\mathrm{E}[X_{j:n} - X_{i:n}]$, $i < j$ for the case of non-negative parent populations. Bounds of this form have been studied in the general case by Moriguti (1953) and Ludwig (1960); see also David (1981), Arnold and Balakrishnan (1989). The bounds obtained in the present section also have the advantages described in Section 2.

Fix $1 \leq i < j \leq n$ and consider the incomplete beta functions

$$G_i(x) = I_x(i, n+1-i), \qquad G_j(x) = I_x(j, n+1-j), \qquad 0 \leq x \leq 1.$$

Then we have from (2.2) the relation

$$\mathrm{E}[X_{j:n} - X_{i:n}] = \int_0^{+\infty} [G_i(F(x)) - G_j(F(x))]dx. \tag{3.1}$$

For $j = n$, the following theorem holds.

THEOREM 3.1. *Under the assumptions of Theorem* 2.1, $\sup \mathrm{E}[X_{n:n} - X_{i:n}] = n\mu$ $(i = 1, 2, \ldots, n-1)$, *where the supremum is taken over* $\mathcal{F}$ *as in Theorem* 2.1 (iii).

PROOF. Obviously,

$$\mathrm{E}[X_{n:n} - X_{n-1:n}] \le \mathrm{E}[X_{n:n} - X_{i:n}] \le \mathrm{E}[X_{n:n}] < n\mu$$

(the last inequality follows from Theorem 2.1 (iii)). However, for the population with $\mathrm{P}[X = 0] = \alpha = 1 - \mathrm{P}[X = \mu/(1-\alpha)]$, we have $\mathrm{E}[X] = \mu$ for all $\alpha \in (0,1)$ and $\mathrm{E}[X_{n:n} - X_{n-1:n}] = n\alpha^{n-1}\mu \to n\mu$ as $\alpha \to 1$. This completes the proof.

For the general case $j < n$, we need the following lemma.

LEMMA 3.1. *For $j < n$ the function $h(x) = (G_i(x) - G_j(x))/(1-x)$, $0 < x < 1$, is unimodal. Furthermore, there exists a unique point ρ $(= \rho(i,j;n))$ such that $h(x) < h(\rho)$ for all $x \ne \rho$. This unique mode ρ belongs to the interval $[i/(n-1), (j-1)/(n-1)]$.*

PROOF. By using the well-known relation between the incomplete beta function and the binomial sums, we have

$$h(x) = \sum_{s=i}^{j-1} \binom{n}{s} x^s (1-x)^{n-1-s} = \sum_{s=i}^{j-1} h_s(x),$$

where $h_s(x) = \binom{n}{s} x^s (1-x)^{n-1-s}$. Since $h_s(x)$ is unimodal with mode $s/(n-1)$ we conclude that the assertion holds when $j = i+1$ (in this case $\rho = i/(n-1)$). If $j > i+1$, observe that $h'(x) = \sum_{s=i}^{j-1} h_s'(x)$ which is positive if $x \le i/(n-1)$ and negative if $x \ge (j-1)/(n-1)$. Thus, from the continuity of h' follows that the equation $h'(x) = 0$ has at least one root ρ in $(i/(n-1), (j-1)/(n-1))$. Alternatively, we have

$$h'(x) = \frac{1}{(1-x)^2}[G_i(x) - G_j(x) + (1-x)(g_i(x) - g_j(x))]$$

(where $g_i(x) = G_i'(x)$, $g_j(x) = G_j'(x)$) and for $x \in (\frac{i-1}{n-1}, \frac{j-1}{n-1}]$,

$$\begin{aligned}&[G_i(x) - G_j(x) + (1-x)(g_i(x) - g_j(x))]' \\ &\quad = \frac{1}{x}[(i-1-(n-1)x)g_i(x) - (j-1-(n-1)x)g_j(x)] < 0.\end{aligned}$$

Hence, the function $(1-x)^2 h'(x)$ strictly decreases in the above interval. Since there exists at least one root of h' in this interval, it follows that this root must be unique and therefore h is unimodal with mode ρ. This completes the proof.

Remark. This point ρ arises as the unique solution of the equation

$$G_i(x) - G_j(x) = (1-x)(g_j(x) - g_i(x)), \qquad 0 < x < 1.$$

An immediate consequence of Lemma 3.1 is the following

COROLLARY 3.1. *For all x in $[0,1]$,*

$$G_i(x) - G_j(x) \le (1-x)(g_j(\rho) - g_i(\rho)),$$

with equality iff either $x = \rho$ or $x = 1$.

We now state the main result of this section.

THEOREM 3.2. *Fix $1 \le i < j < n$. Under the assumptions of Theorem 3.1, the inequality*

$$\mathrm{E}[X_{j:n} - X_{i:n}] \le (g_j(\rho) - g_i(\rho))\mu$$

holds true, with equality iff $\mathrm{P}[X=0] = \rho = 1 - \mathrm{P}[X = \mu/(1-\rho)]$, where ρ is as in Lemma 3.1.

PROOF. By using (2.1), (3.1) and Corollary 3.1 we have

$$\begin{aligned}\mathrm{E}[X_{j:n} - X_{i:n}] &= \int_0^{+\infty} [G_i(F(x)) - G_j(F(x))]dx \\ &\le \int_0^{+\infty} (1-F(x))(g_j(\rho) - g_i(\rho))dx = (g_j(\rho) - g_i(\rho))\mu.\end{aligned}$$

From Corollary 3.1, a necessary and sufficient condition for the equality to hold (c.f. the proof of Theorem 2.1) is either $F(x) = \rho$ or $F(x) = 1$ a.e. in $[0, +\infty)$, which completes the proof.

The corresponding lower bounds, as in Theorem 2.2, are again trivial, taking the form

$$\inf \mathrm{E}[X_{j:n} - X_{i:n}] = 0,$$

with equality iff $\mathrm{P}[X = \mu] = 1$ (note that we can not exclude the degenerate populations even if we assume that the data are bounded above).

An interesting application of the above theorem is given when $j = i+1$. In this case $\rho = i/(n-1)$ and it is easily showed, by using Stirling's formula, the following

COROLLARY 3.2. *If $n \to \infty$ and $i/n \to p$, $0 < p < 1$, we have the "asymptotic" bound*

$$\mathrm{E}[X_{i+1:n} - X_{i:n}] \le A_n(p)\mu,$$

where $A_n(p) \approx \frac{1}{\sqrt{n}}[2\pi p(1-p)^3]^{-1/2}$ ($a_n \approx b_n$ means here that $a_n/b_n \to 1$ as $n \to \infty$).

This corollary shows that for large n the expected difference of adjacent order statistics (from non-negative populations) is at most of order $1/\sqrt{n}$. For example, if $p = 1/2$ we have from Corollary 3.2 that for large n

$$\sqrt{n}\mathrm{E}[X_{i+1:n} - X_{i:n}] \le c(i,n)\mu \approx \frac{2\sqrt{2}}{\sqrt{\pi}}\mu.$$

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