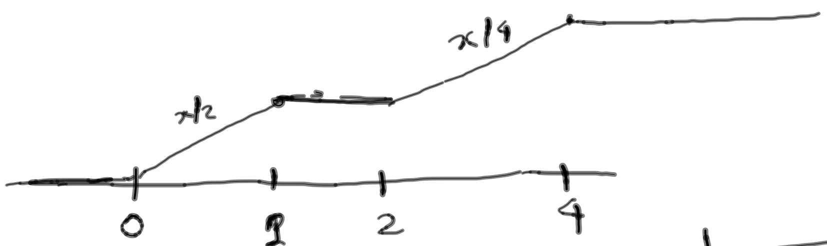
7ο 25/10/2017

<u>2.1)</u> X = χρόνος αναμονής, X συνεχής τ.μ. με σ.κ.

$$F(x) = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x < 1 \\ 1/2, & 1 \le x \le 2 \\ x/4, & 2 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

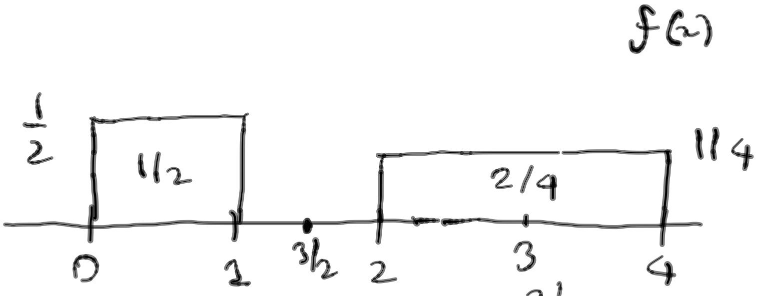
Βρείτε (β) την πυκνότητα $f(x)$ της τ.μ. X

(α) $P(X \le 3/2)$, $P(X > 3)$, $P(3 < X \le 5)$.



$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 < x < 1 \\ 0, & 1 < x < 2 \\ 1/4, & 2 < x < 4 \\ 0, & x > 4 \\ \nexists & x = 0, 1, 2, 4 \end{cases}$$

$$\boxed{f(x) = \begin{cases} 1/2, & 0 < x < 1 \\ 1/4, & 2 < x < 4 \\ 0, & \text{αλλού} \end{cases}}$$



(α) $P(X \leq 3/2) = F(3/2) = 1/2 = \int_{-\infty}^{3/2} f(x)\,dx$

(β) $P(X > 3) = 1 - F(3) = 1 - 3/4 = 1/4 = \int_{3}^{+\infty} f(x)\,dx$

$P(3 < X \leq 5) = F(5) - F(3) = 1 - \frac{3}{4} = 1/4 = \int_{3}^{5} f(x)\,dx.$

* $\forall$ F, $P(a < X \leq b) = F(b) - F(a)$

Βασική σχέση: $P(X \in B) = \int_B f(x)\,dx$, $\forall$ υποσύνολο B

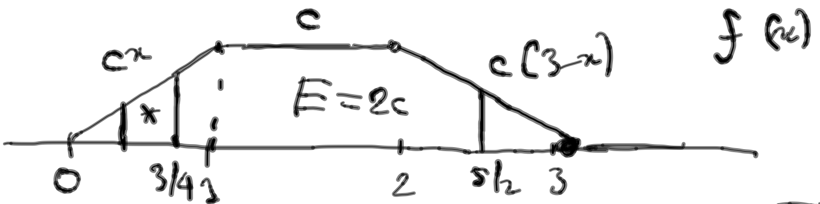
(f η πυκνότητα της X)

2.6| Η ποσότητα βενζίνης (σε χιλιόλιτρα) που πουλάει ένα πρατήριο σε μία μέρα είναι συνεχής τ.μ. με πυκνότητα

$$f(x) = \begin{cases} cx, & 0 \le x \le 1 \\ c, & 1 \le x \le 2 \\ c(3-x), & 2 \le x \le 3 \end{cases}$$

(0 αλλού).

Βρείτε το c, και τις $P(X \le 3/4)$, $P(\frac{1}{2} < X \le \frac{5}{2})$, $P(X > \frac{9}{4})$.



$$E=\int_{-\infty}^{\infty} f(x)\,dx = \frac{(3+1)\cdot c}{2} = 2c \qquad E=1 \Rightarrow \boxed{c=\frac{1}{2}}$$

$$P(X\le 3/4) = (*) = \int_{-\infty}^{3/4} f(x)\,dx = \cancel{\int_{-\infty}^{0} 0\,dx}^{\,0} + \int_{0}^{3/4} \frac{1}{2}x\,dx$$

$$= \frac{1}{2}\cdot \frac{x^2}{2}\Big|_0^{3/4} = \frac{(3/4)^2}{4} = \frac{9}{64}$$

$$P\left(\tfrac{1}{2} < X \le \tfrac{5}{2}\right) = \int_{1/2}^{5/2} f(x)\,dx = \int_{1/2}^{1} \tfrac{1}{2} x\,dx + \int_{1}^{2} \tfrac{1}{2}\,dx$$

$$+ \int_{2}^{5/2} \tfrac{1}{2}(3-x)\,dx = 1 - P(X \le 1/2) - P\left(X > \tfrac{5}{2}\right)$$

$$= 1 - 2P\left(X \le \tfrac{1}{2}\right) = 1 - 2\int_{0}^{1/2} \tfrac{1}{2} x\,dx = 1 - \int_{0}^{1/2} x\,dx$$

$$= 1 - (x^2/2)\Big|_{0}^{1/2} = 1 - \frac{(1/2)^2}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

$$P\left(X > \tfrac{9}{4}\right) = \int_{9/4}^{\infty} f(x)\,dx = \int_{9/4}^{3} \tfrac{1}{2}(3-x)\,dx = \dots = P(X \le 3/4)$$

$$= \frac{9}{64}.$$

2.10 X συνεχής, πυκνότητα $f_X(x) = \frac{2}{5}|x|$, $-1 < x < 2$.

$Y = X^2$. Βρείτε (α) $F_X(x)$, (β) $F_Y(y)$, (γ) $f_Y(y)$

(γ) $F_Y(y) \overset{\text{ορ}}{=} P(Y \le y) = P(X^2 \le y) = 0$ αν $y < 0$.

Αν $y \geq 0$ τότε $X^2 \le y \Leftrightarrow -\sqrt{y} \le X \le \sqrt{y}$

$$\Rightarrow F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) \overset{(\text{αν } X \text{ συνεχής})}{=} P(-\sqrt{y} < X \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad (y \geq 0)$$

$$f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] = F_X'(\sqrt{y})(\sqrt{y})' - F_X'(-\sqrt{y}) \cdot (-\sqrt{y})'$$

$(\sqrt{y})' = \frac{1}{2\sqrt{y}}$

$$\boxed{f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)} \quad y > 0 \quad (*)$$

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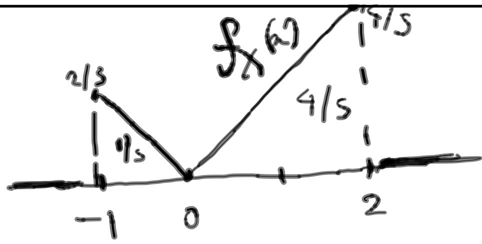
$f_X(x) = \frac{2}{5}|x|, \quad -1 \leq x \leq 2$



$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$ Για $y \geq 4$, προφανώς $= 0$

Για $\underline{0 < y < 4}$

(α) Αν $0 < y < 1 \Rightarrow \sqrt{y} \in (0,1)$ και $-\sqrt{y} \in (-1,0)$

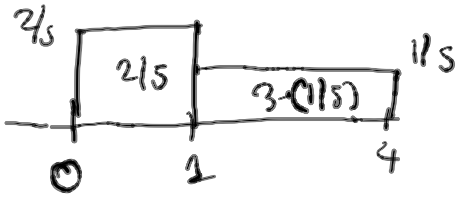
οπότε $f_X(\sqrt{y}) = \frac{2}{5}\sqrt{y}$ και $f_X(-\sqrt{y}) = \frac{2}{5}\sqrt{y}$

$\Rightarrow f_Y(y) = \frac{2}{5}$, για $0 < y < 1$

(β) $1 < y < 4 \Rightarrow \sqrt{y} \in (1,2)$, $-\sqrt{y} \in (-2,-1) \Rightarrow -\sqrt{y} < -1$

$\Rightarrow f_X(\sqrt{y}) = \frac{2}{5}\sqrt{y}$, $f_X(-\sqrt{y}) = 0 \Rightarrow f_Y(y) = \frac{1}{5}$, για $1 < y < 4$

(γ) $f_Y(y) = \begin{cases} 2/5, & 0 < y < 1 \\ 1/5, & 1 < y < 4 \end{cases}$ (0 αλλού)



(α) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ $f_X(x) = \frac{2}{5}|x|,\ -1 < x < 2$

Αν $x \leq -1$, $F_X(x) = 0$. Αν $-1 \leq x \leq 0$, $F_X(x) = \int_{-1}^{x} \frac{2}{5}|t|\,dt$

$= -\int_{-1}^{x} \frac{2}{5} t\,dt = \ldots = \frac{1-x^2}{5}$.

Αν $x \in [0,2]$

$$\Rightarrow F_X(x) = \int_{-1}^{0} f_X(t)\,dt + \int_0^x f_X(t)\,dt$$

$$= F_X(1) + \int_0^x \frac{2}{5}|t|\,dt$$

$$= \frac{1}{5} + \cdots = \frac{1+x^2}{5}, \quad x \in [0,2]$$

$F_X(x) = 1$ όταν $x \geq 2$

$$F_X(x) = \begin{cases} 0, & \text{αν } x \leq -1 \\ \frac{1-x^2}{5}, & \text{αν } -1 \leq x \leq 0 \\ \frac{1+x^2}{5}, & \text{αν } 0 \leq x \leq 2 \\ 1, & \text{αν } x \geq 2 \end{cases}$$

(β) $F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$$= P(-\sqrt{y} < X \le \sqrt{y}) + \underbrace{P(X = -\sqrt{y})}_{0} = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\boxed{F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0}$$

Αν $y \in (0,1) \Rightarrow \sqrt{y} \in (0,1)$ και $-\sqrt{y} \in (-1,0)$

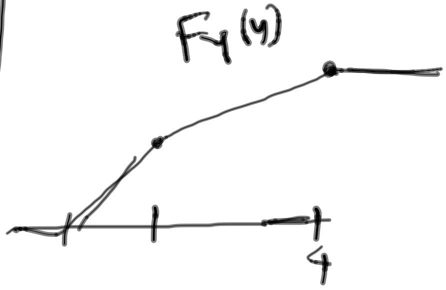
$$\Rightarrow F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1+(\sqrt{y})^2}{5} - \frac{1-(-\sqrt{y})^2}{5} = \frac{2}{5}y$$

Αν $y \in (1,4)$ $\quad \sqrt{y} \in (1,2)$, $\quad -\sqrt{y} \in (-2,-1) \Rightarrow -\sqrt{y} < -1$

$$\Rightarrow F_X(-\sqrt{y}) = 0$$

$$\Rightarrow F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1+(\sqrt{y})^2}{5} - 0 = \frac{1+y}{5}$$

$$F_Y(y)=\begin{cases}0, & y\le 0\\ \frac{2}{5}y, & 0\le y\le 1\\ \frac{y+1}{5}, & 1\le y\le 4\\ 1, & y\ge 4\end{cases}$$



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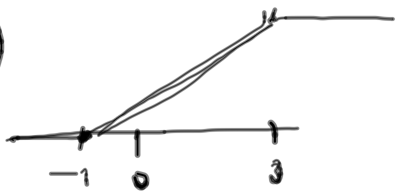
$$F_X(x)=\begin{cases}0, & \text{αν } x\le -1\\ \frac{x+1}{4}, & \text{αν } -1\le x\le 3\\ 1, & x\ge 3\end{cases}$$



$Y=|X|$. Βρείτε την $F_Y(y)$ και την $f_Y(y)$

Για $y \in (0,3)$ έχει ενδιαφέρον αφού

$F_Y(y) = 0$ για $y \le 0$ $\quad F_Y(y) = 1$ για $y \ge 3$.

Πάρε $y \in (0,3)$.

$$F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y)$$

$$= P(-y < X \le y) + \overset{X \text{ συνεχής}}{\underbrace{P(X = -y)}_{0}} = F_X(y) - F_X(-y)$$

Αν $y \in (0,1) \Rightarrow -y \in (-1,0)$ οπότε $F_X(y) = \frac{1+y}{4}$, $F_X(-y) = \frac{1-y}{4}$

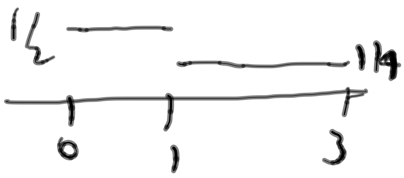
και $F_Y(y) = \frac{1+y}{4} - \frac{1-y}{4} = \frac{1}{2} y$.

Αν $y \in (1,3) \Rightarrow -y < -1 \Rightarrow F_X(-y) = 0$

$$F_Y(y) = \frac{1+y}{4} - 0 = \frac{1+y}{4}$$

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ y/2, & 0 \leq y \leq 1 \\ (1+y)/4, & 1 \leq y \leq 3 \\ 1, & y \geq 3 \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} 1/2, & y \in (0,1) \\ 1/4, & y \in (1,3) \end{cases}$$



<u>2.14</u> $f_X(x) = \frac{1}{3}, \quad -1 < x < 2$

και $Y = |X|$. Βρείτε την $f_Y(y)$

$$f_Y(y) = \begin{cases} 2/3, & 0 < y < 1 \\ 1/3, & 1 < y < 2 \end{cases}$$

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