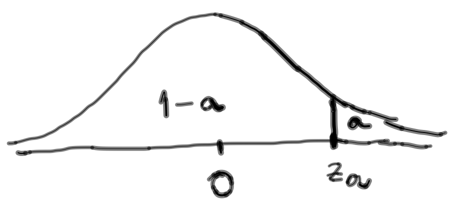
13o 22/11/2017

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



$a \in (0,1)$

$\Phi(z_a) = 1 - a$

$z_a = \Phi^{-1}(1-a)$

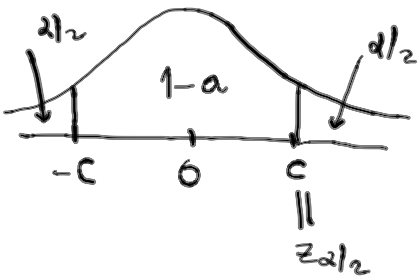
$z_{0.50} = 0 \qquad \Phi(z_{0.50}) = 1 - 0.50 = 0.50$

$z_{0.80} \qquad \Phi(z_{0.80}) = 1 - 0.80 = 0.20 \qquad ?$

$z_{0.80} = -z_{0.20} \qquad z_a = -z_{1-a}$

$z_{0.20}: \ \Phi(z_{0.20}) = 0.80 \qquad z_{0.20} = 0.84 \Rightarrow z_{0.80} = -0.84$

4.18) Αν η $Z \sim N(0,1)$ να θρεθεί $c>0$ $(c=c(\alpha))$ ώστε $P(-c<Z<c)=1-\alpha$. $(\alpha \in (0,1))$



$$1-\alpha = P(-c<Z<c) = \Phi(c)-\Phi(-c) = \Phi(c)-(1-\Phi(c))$$

$$= 2\Phi(c)-1 \Rightarrow 2\Phi(c)-1 = 1-\alpha$$

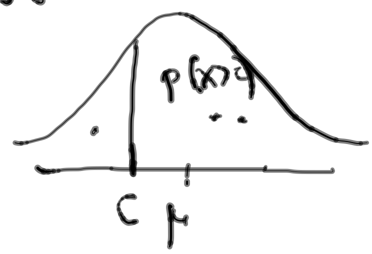
$$2\Phi(c) = 2-\alpha$$

$$\Phi(c) = 1-\alpha/2 \Rightarrow c = z_{\alpha/2}$$

4.17) $X \sim N(\mu, \sigma^2)$. Να δειχθεί ότι η σχέση

$$P(X > c) = 2P(X \le c)$$ συνεπάγεται

$$c = \mu - 0.43 \cdot \sigma$$



$$P(X > c) = 2P(X \le c)$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{c-\mu}{\sigma}\right) = 2P\left(\frac{X-\mu}{\sigma} \le \frac{c-\mu}{\sigma}\right)$$

$$\Rightarrow P\left(Z > \frac{c-\mu}{\sigma}\right) = 2P\left(Z \le \frac{c-\mu}{\sigma}\right) \Rightarrow 1 - \Phi\left(\frac{c-\mu}{\sigma}\right) = 2\Phi\left(\frac{c-\mu}{\sigma}\right)$$

$$3\Phi\left(\frac{c-\mu}{\sigma}\right) = 1 \qquad \Phi\left(\frac{c-\mu}{\sigma}\right) = \frac{1}{3} = 1 - \frac{2}{3} \qquad (= 1-\alpha)$$

$$\Rightarrow \frac{c-\mu}{\sigma} = z_{2/3} = -z_{1/3} \Rightarrow c - \mu = -\sigma z_{1/3} \qquad c = \mu - \sigma z_{1/3}$$

$$z_{1/3} = 0.43.$$

4.5) Χρόνος ζωής λυχνίας είναι μία συνεχής τ.μ. X με Εκθετική κατανομή και μέσο $E(X)=1000$ ώρες.

Να βρεθεί η τιμή του α ώστε αν δοθεί εγγύηση α ωρών, το 95% των λυχνιών θα επιζούν του χρόνου εγγύησης.

$P(X>\alpha) \geq 0.95$ (*)

$X \sim Exp(\lambda) \Rightarrow E(X) = \frac{1}{\lambda} = 1000$ ώρες

$\lambda = \frac{1}{1000}$.

$f(x) = \lambda e^{-\lambda x} = \frac{1}{1000} e^{-x/1000} \quad x>0$

$F(x) = 1-e^{-\lambda x}, \quad P(X>\alpha) = 1-F(\alpha) = 1-(1-e^{-\lambda \alpha}) = e^{-\lambda \alpha}$.

(*) $\Leftrightarrow e^{-\lambda \alpha} \geq 0.95 \quad -\lambda\alpha \geq \log(0.95) \quad \alpha \leq \frac{-1}{\lambda}\log(0.95)$

$$\alpha \le \frac{-1}{\lambda} \log\left(\frac{19}{20}\right) = \frac{1}{\lambda} \log\left(\frac{20}{19}\right) = 1000 \cdot \log\left(\frac{20}{19}\right)$$

$$\alpha \le 51.29 \text{ ώρες.}$$

$$\alpha^{+} = 51.29 \text{ ώρες.}$$

9.7) $X \sim Exp(\lambda)$, $Y = 1 - e^{-\lambda X}$. Τι κατανομή ακολουθεί η Y;

$X \in (0,\infty)$, $\lambda > 0$, $\Rightarrow e^{-\lambda X} \in (0,1) \Rightarrow 1 - e^{-\lambda X} \in (0,1)$.

$\Rightarrow Y \in (0,1)$. $\quad F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y \ge 1 \end{cases}$ $= ?$ αν $y \in [0,1)$

Για $y \in [0,1)$ έχουμε:

$$F_Y(y) = P(Y \le y) = P(1 - e^{-\lambda X} \le y) = P(e^{-\lambda X} \ge 1-y)$$
$$= P(-\lambda X \ge \log(1-y))$$
$$= P\left(X \le -\frac{1}{\lambda}\log(1-y)\right) = F_X\left(-\frac{1}{\lambda}\log(1-y)\right)$$

(1ο s): $F_X(x) = 1 - e^{-\lambda x}$
$$\Rightarrow F_Y(y) = F_X\left(-\frac{1}{\lambda}\log(1-y)\right) = 1 - e^{-\lambda\left(-\frac{1}{\lambda}\log(1-y)\right)}$$
$$= 1 - e^{\log(1-y)} = 1 - (1-y) = y$$

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ y, & y \in [0,1] \\ 1, & y \ge 1 \end{cases} \Rightarrow Y \sim U(0,1)$$

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$\underline{\underline{2o\varsigma}}$ $F_Y(y) = F_X\left(-\frac{1}{\lambda}\log(1-y)\right)$ $(y \in (0,1))$

$$\Rightarrow f_Y(y) = f_X\left(-\frac{1}{\lambda}\log(1-y)\right)\cdot\left(-\frac{1}{\lambda}\log(1-y)\right)'$$

$$= \lambda e^{-\lambda\left(-\frac{1}{\lambda}\log(1-y)\right)} \cdot \frac{1}{\lambda}\cdot\frac{1}{1-y}$$

$$= e^{\log(1-y)}\cdot\frac{1}{1-y} = \frac{1-y}{1-y} = 1$$

$f_Y(y) = 1, \quad y \in (0,1) \qquad$ (0 αλλού).

Άρα $Y \sim U(0,1)$.

4.11 $X \sim U(a,b)$. $\mu = 1$, $\sigma^2 = 3$

Βρείτε: a, b, την πυκνότητα της $Y = |X|$

και τις ροπές EY^n $(n = 1, 2, \ldots)$ της Y,

καθώς και τη μέση τιμή και διασπορά της Y.

Στην $U(a,b)$ γνωρίζω ότι $a < b$, $\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$

$$\frac{a+b}{2} = 1 \qquad \frac{(b-a)^2}{12} = 3$$

$$\begin{matrix} (1) \\ (2) \end{matrix} \begin{pmatrix} a+b=2 \\ b-a=6 \end{pmatrix} \quad (b-a)^2 = 36 \overset{a<b}{\Longleftrightarrow} b-a=6$$

$(1)+(2) \Rightarrow 2b = 8 \Rightarrow b = 4$

$(1)-(2) \Rightarrow 2a = -4 \Rightarrow a = -2$

$X \sim U(-2, 4)$

Βρίσκω τις ροπές $E(Y^n)$ $n=1,2,\dots$ της $Y=|X|$.

$$E(Y^n)=E(|X|^n)=\int_{-\infty}^{\infty}|x|^n f(x)\,dx=\int_{-2}^{4}|x|^n\frac{1}{6}\,dx$$

$$=\frac{1}{6}\left(\underbrace{\int_{-2}^{0}|x|^n dx}+\underbrace{\int_{0}^{2}|x|^n dx}+\int_{2}^{4}|x|^n dx\right)$$

$$=\frac{1}{6}\left(2\int_{0}^{2}x^n dx+\int_{2}^{4}x^n dx\right)=\frac{1}{6}\left(2\,\frac{2^{n+1}}{n+1}+\frac{4^{n+1}-2^{n+1}}{n+1}\right)$$

$$=\boxed{\frac{1}{6(n+1)}\left(2^{n+1}+4^{n+1}\right)=E(Y^n)}$$

$$E(Y)\overset{n=1}{=}\frac{1}{6\cdot 2}\cdot\left(2^2+4^2\right)=\frac{20}{12}=\frac{5}{3}$$

$$E(Y^2)\overset{n=2}{=}\frac{1}{6\cdot 3}\left(2^3+4^3\right)=E(|X|^2)=E(X^2)=\mu^2+\sigma^2=1+3=4$$

$$Var(Y)=4-\left(\frac{5}{3}\right)^2=\frac{11}{9}.$$

Πυκνότητα της $Y = |X|$:

$$F_Y(y) = P(Y \le y) = P(|X| \le y) \overset{y>0}{=} P(-y \le X \le y)$$

$$= F_X(y) - F_X(-y) \quad \text{γιατί } F_X \text{ συνεχής}$$

$$f_Y(y) = F_Y'(y) = [F_X(y) - F_X(-y)]' = f_X(y) - f_X(-y)(-y)'$$

$$\boxed{f_Y(y) = f_X(y) + f_X(-y) \quad y>0}$$

$$f_X(y) = \frac{1}{6}, \quad y \in (-2,4)$$

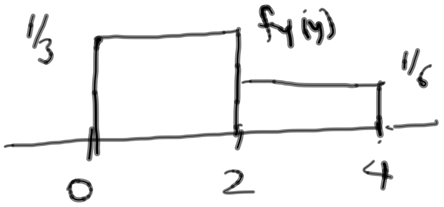
(1η) $0<y<2 \Rightarrow -y \in (-2,0) \Rightarrow -y \in (-2,4)$

$$\Rightarrow f_Y(y) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

(2η) $2<y<4 \Rightarrow -y \in (-4,-2) \Rightarrow -y < -2 \Rightarrow f_X(-y) = 0$

Άρα $f_Y(y) = \frac{1}{6} + 0 = \frac{1}{6}$, $y \in (2,4)$.

$$f_Y(y) = \begin{cases} 1/3, & 0 < y < 2 \\ 1/6, & 2 < y < 4 \end{cases} \qquad (0 \text{ αλλού}).$$



$$E(Y^n) = \int_{-\infty}^{\infty} y^n f_Y(y)\,dy = \int_0^2 y^n \cdot \frac{1}{3}\,dy + \int_2^4 y^n \frac{1}{6}\,dy$$

$$= \frac{1}{3} \cdot \frac{2^{n+1}}{n+1} + \frac{1}{6} \cdot \frac{4^{n+1} - 2^{n+1}}{n+1} = \frac{1}{6(n+1)} \left(2 \cdot 2^{n+1} + 4^{n+1} - 2^{n+1}\right)$$

$$= \frac{1}{6(n+1)} \left(2^{n+1} + 4^{n+1}\right).$$

Πολυδιάστατες τυχαίες μεταβλητές

Παρατηρούμε 2 χαρακτηριστικά ενός ατόμου, π.χ.

X = ύψος του ατόμου
Y = βάρος του ατόμου

(X, Y). Βασικά X και Y είναι τ.μ. στον ίδιο χώρο πιθανότητας.

$$F_{X,Y}(x,y) \overset{\text{op}}{=} P(X \leq x, Y \leq y) = P\left(\{\omega : X(\omega) \leq x\} \cap \{\omega : Y(\omega) \leq y\}\right)$$

καλείται συνάρτηση κατανομής του ζεύγους (X,Y)
ή από κοινού κατανομή των X, Y.

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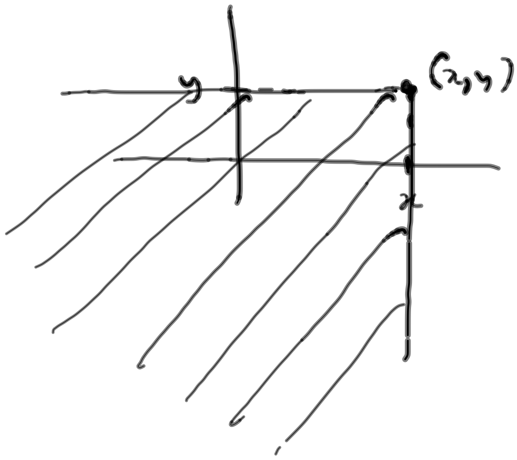


$$\lim_{y\to\infty} F(x,y) = P(X\le x) = F_X(x)$$

$$\lim_{x\to\infty} F(x,y) = P(Y\le y) = F_Y(y)$$

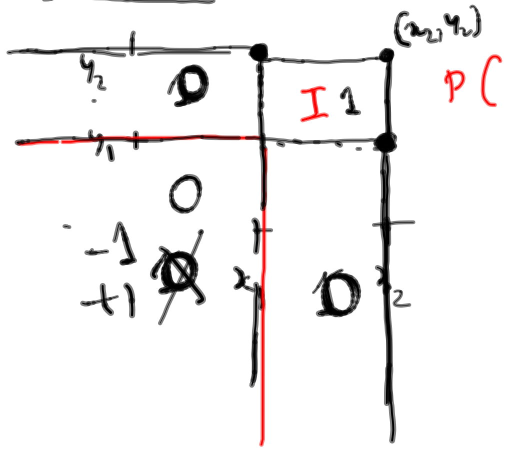
$$F(x,\infty) = F_X(x), \quad F(\infty,y) = F_Y(y)$$

Νοε 22-6:40 μμ

Μονοτονία: $\forall\ x_1 < x_2$ και $y_1 < y_2$ ισχύει:

$$F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1) \geq 0$$

Απόδειξη: $P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) = ?$



$$P((X,Y)\in I) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1) \geq 0$$

Νοε 22-6:43 μμ