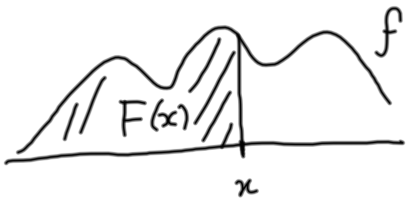
11ο 8/11/2017

Βασικές συνεχείς κατανομές.

$f(x)$ πυκνότητα (πιθανότητας)

$\iff$ (1) $f \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

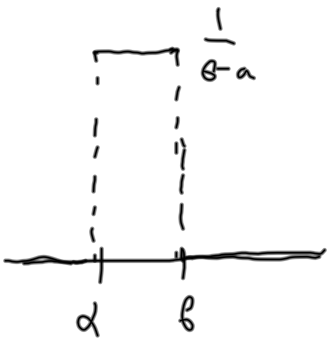


$$f(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = F'(x_0)$$

$P(x_0 < X \leq x_0 + h) \simeq h \cdot f(x_0)$

Ομοιόμορφη $U(\alpha,\beta)$ $(\alpha<\beta,\ -\infty<\alpha<\beta<\infty)$.

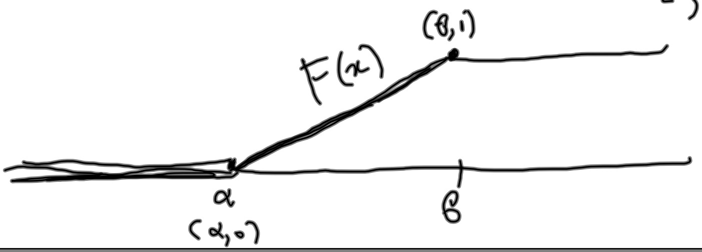
Ορισμός: $X \sim U(\alpha,\beta)$ όταν η πυκνότητα της X δίνεται από τον τύπο



$$f(x)=\frac{1}{\beta-\alpha},\quad \alpha<x<\beta$$

(0 αλλού)

$$F(x)=\begin{cases} 0, & x\le \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \alpha\le x\le\beta, \\ 1, & x\ge\beta \end{cases}$$



Πρόταση $\quad X \sim U(\alpha,\beta)$ τότε $\mu = \dfrac{\alpha+\beta}{2}, \quad \sigma^2 = \dfrac{(\beta-\alpha)^2}{12}.$

Απόδ.:

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\alpha}^{\beta} x f(x)\,dx = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x\,dx$$

$$= \frac{1}{\beta-\alpha}\left.\frac{x^2}{2}\right|_{\alpha}^{\beta} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{\alpha+\beta}{2}.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x^2\,dx = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\alpha^2+\alpha\beta+\beta^2}{3}$$

$$\sigma^2 = E(X^2)-\mu^2 = \frac{\alpha^2+\alpha\beta+\beta^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{1}{12}\left[4\alpha^2+4\alpha\beta+4\beta^2-3(\alpha+\beta)^2\right]$$

$$= \frac{\alpha^2+\beta^2-2\alpha\beta}{12} = \frac{(\beta-\alpha)^2}{12}$$

π.χ.
ροπές: $E(X^n) =$ ροπή n-οστής τάξης

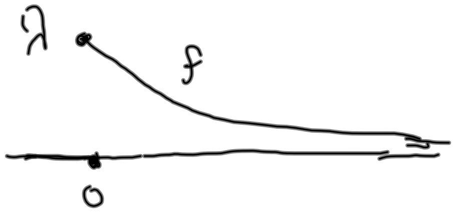
$$= \int_{-\infty}^{\infty} x^n f(x)\,dx = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x^n\,dx$$

$$= \frac{\beta^{n+1}-\alpha^{n+1}}{(\beta-\alpha)(n+1)}.$$

Εκθετική κατανομή $(\mathrm{Exp}(\lambda))$.

Η X καλείται εκθετική με παράμετρο λ όταν έχει πυκνότητα

$$f(x) = \lambda e^{-\lambda x}, \quad x>0 \qquad (\lambda>0).$$

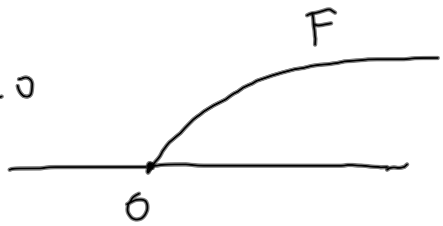


$$S_X = (0,\infty)$$

$$F(x) \overset{x>0}{=} \int_{-\infty}^{x} f(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt = \int_0^x \left(-e^{-\lambda t}\right)'\,dt$$

$$= \left(-e^{-\lambda t}\right)\Big|_0^x = -e^{-\lambda x} + 1, \quad x>0$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1-e^{-\lambda x}, & x \geq 0 \end{cases}$$



Ιδιότητα αμνήμονος: $\forall \; t_1, t_2 > 0$

$$P(X > t_1 + t_2 \mid X > t_1) = P(X > t_2) \qquad (*)$$

Διότι $P(X>t) = 1 - F(t) = e^{-\lambda t} \quad (t>0)$

$$P(X > t_1 + t_2 \mid X > t_1) = \frac{P(\{X > t_1 + t_2\} \cap \{X > t_1\})}{P(X > t_1)} =$$

$$\left[\left(P(A|B) = \frac{P(A \cap B)}{P(B)} \right) \qquad \{X > t_1 + t_2\} \subseteq \{X > t_1\} \right]$$

$$= \frac{P(X > t_1 + t_2)}{P(X > t_1)} = \frac{e^{-\lambda(t_1 + t_2)}}{e^{-\lambda t_1}} = \frac{\cancel{e^{-\lambda t_1}}\, e^{-\lambda t_2}}{\cancel{e^{-\lambda t_1}}}$$

$$= e^{-\lambda t_2} = P(X > t_2)$$

Πρόταση: $E(X) = \mu = \frac{1}{\lambda}$, $\sigma^2 = Var(X) = \frac{1}{\lambda^2}$

$$\mu = \int_0^\infty x \lambda e^{-\lambda x} dx = \ldots\ldots = \frac{1}{\lambda}$$

$$\int_0^\infty x e^{-\lambda x} dx = \frac{1}{\lambda^2} \qquad \lambda x = y$$

$$\int_0^\infty y e^{-y} dy = 1 \qquad e^{-y} = (-e^{-y})' \quad \text{κ.ο.κ.}$$

$$E(X^2) = \frac{2}{\lambda^2} \qquad \left(\Rightarrow \sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \right).$$

$$\lambda \int_0^\infty x^2 e^{-\lambda x} dx \rightsquigarrow \int y^2 e^{-y} dy \qquad e^{-y} = (-e^{-y})' \quad \text{κ.ο.κ.}$$

Συνάρτηση Γάμμα - Κατανομή Γάμμα.

$$\forall\ \alpha>0, \quad \Gamma(\alpha) \overset{\text{ορ}}{=} \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left(-e^{-x}\right)\Big|_0^\infty = -e^{-\infty} + e^{-0} = -0+1=1$$

$$\Gamma(a+1) = \int_0^\infty x^{a} e^{-x}\, dx = \int_0^\infty x^{\alpha} \left(-e^{-x}\right)'\, dx$$

$$= \cancelto{0}{-x^{\alpha} e^{-x}\Big|_0^\infty} + a \int_0^\infty x^{\alpha-1} e^{-x}\, dx = \alpha\, \Gamma(\alpha)$$

$\Gamma(1)=1, \quad \Gamma(2)=1\cdot\Gamma(1)=1, \quad \Gamma(3)=2\Gamma(2)=2 \quad \Gamma(4)=3\Gamma(3)=6$

$\Gamma(n)=(n-1)! \qquad n=1,2,3,\dots$ (επαγωγή στο n)

Άρα η $\Gamma(\alpha)$ υπολογίζεται για $\alpha = 1, 2, \ldots$

Ερώτημα: Υπάρχει $\alpha > 0$, όχι ακέραιος, ώστε το $\Gamma(\alpha)$ να υπολογίζεται;

Euler: $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$ $\qquad \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = \sqrt{\pi}$

$$\Gamma(3.5) = \Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\,\Gamma\left(\tfrac{3}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$
$$= \frac{5\cdot 3}{2^3}\cdot\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}.$$

Κατανομή Γάμμα: $X \sim \Gamma(\alpha, \lambda)$ $\quad (\alpha, \lambda > 0)$

όταν η πυκνότητα $\boxed{f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0}$

$\alpha = n \in \{1, 2, \dots\}$ η $\Gamma(\alpha, \lambda)$ καλείται Erlang(n, λ)

δηλ. $\Gamma(n, \lambda) \equiv \mathcal{E}(n, \lambda)$ $\quad f(x) = \dfrac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \; x > 0$

$$\Gamma(1, \lambda) \equiv \mathcal{E}(1, \lambda) \equiv \mathcal{E}xp(\lambda)$$

<u>Πρόταση:</u> Οι ροπές της $\Gamma(\alpha, \lambda)$ δίνονται από τον τύπο

$$\boxed{E(X^n) = \frac{[\alpha]_n}{\lambda^n}}, \quad [\alpha]_n = \underbrace{\alpha(\alpha+1)\cdots(\alpha+n-1)}_{n \text{ παρ.}}$$

<u>Παρατήρηση:</u> Η $f(x) = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \; x > 0$ είναι πράγματι μία πυκνότητα, διότι είναι ≥ 0 (προφανές)

και $\displaystyle\int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx \overset{\lambda x = y \quad x=y/\lambda \quad dx=\frac{1}{\lambda}dy}{=} \int_0^\infty \left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-y}\,\frac{1}{\lambda}\,dy$

$$= \frac{1}{\lambda^\alpha} \underbrace{\int_0^\infty y^{\alpha-1} e^{-y}\,dy}_{\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\lambda^\alpha}.$$

Απόδειξη για ροπές:

$$E(X^n) = \int_{-\infty}^\infty x^n f(x)\,dx = \frac{\lambda^a}{\Gamma(\alpha)} \int_0^\infty x^{n+\alpha-1} e^{-\lambda x}\,dx$$

$\lambda x = y \quad x = \frac{y}{\lambda},\ dx = \frac{1}{\lambda}dy$

$$= \frac{\lambda^a}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{\lambda}\right)^{n+\alpha-1} e^{-y}\,\frac{1}{\lambda}\,dy = \frac{\lambda^a}{\Gamma(\alpha)\,\lambda^{n+\alpha}} \overbrace{\int_0^\infty y^{n+a-1} e^{-y}\,dy}^{\Gamma(n+a)}$$

$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\cdot\lambda^n}$$

Όμως $\dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)} = \dfrac{(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\cdot\alpha\cdot\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = [\alpha]_n$

δηλ. $E(X^n) = \dfrac{[\alpha]_n}{\lambda^n}$.

$\alpha=1 \quad [\alpha]_n = [1]_n = n! \quad \Rightarrow \quad E(X^n) = \dfrac{n!}{\lambda^n}$.

Για την $\Gamma(\alpha,\lambda)$ ισχύει

$$\mu = \frac{\alpha}{\lambda}, \qquad \sigma^2 = \frac{\alpha}{\lambda^2}$$

διότι $E(X) = \dfrac{[\alpha]_1}{\lambda^1} = \dfrac{\alpha}{\lambda}$, $\quad E(X^2) = \dfrac{[\alpha]_2}{\lambda^2} = \dfrac{\alpha(\alpha+1)}{\lambda^2}$

$\Rightarrow \sigma^2 = E(X^2) - \mu^2 = \dfrac{\alpha(\alpha+1)}{\lambda^2} - \left(\dfrac{\alpha}{\lambda}\right)^2 = \dfrac{\alpha}{\lambda^2}$

Ισχύει το εξής: Αν $X_1, X_2, \dots, X_n$ είναι ανεξ. εκθ. (λ)

τότε η $X_1 + \dots + X_n \sim \mathcal{E}(n,\lambda) \equiv \Gamma(n,\lambda)$.

Κατανομή χ^2_ν με ν βαθμούς ελευθερίας

$$\chi^2_\nu \equiv \Gamma\left(\frac{\nu}{2}, \frac{1}{2}\right) \qquad \alpha = \frac{\nu}{2}, \quad \lambda = 1/2$$

πυκνότητα $\boxed{f(x) = \dfrac{(1/2)^{\nu/2}}{\Gamma(\frac{\nu}{2})} x^{\nu/2-1} e^{-x/2}, \quad x > 0}$

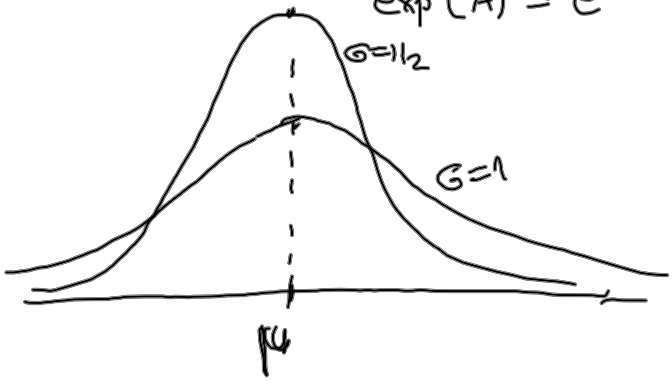
$$\mu = \frac{\alpha}{\lambda} = \frac{\nu/2}{1/2} = \nu \qquad \sigma^2 = \frac{\alpha}{\lambda^2} = \frac{\nu/2}{(1/2)^2} = 2\nu$$

Η πιο σπουδαία συνεχής κατανομή.

$\mu \in \mathbb{R}$, $\sigma^2 > 0$, $X \sim N(\mu, \sigma^2)$ όταν η πυκνότητα

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}$$

$$\exp(A) = e^A$$



Η κατανομή $N(0,1)$ $(\mu=0,\ \sigma=1)$ καλείται τυποποιημένη κανονική κατανομή (standard Normal)

$$f(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 2\int_0^{\infty} e^{-x^2/2}\,dx \qquad x^2/2 = y$$

$$x = \sqrt{2y}$$

$$= 2\int_0^{\infty} e^{-y}\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{y}}\,dy = \sqrt{2}\int_0^{\infty} y^{\frac{1}{2}-1} e^{-y}\,dy \qquad dx = \frac{1}{2\sqrt{2y}}\cdot 2\,dy = \frac{1}{\sqrt{2y}}\,dy$$

$$= \sqrt{2}\,\Gamma(1/2) = \sqrt{2}\cdot\sqrt{\pi} = \sqrt{2\pi}$$

Νοε 8-6:50 μμ

$$E(Z) = \int_{-\infty}^{\infty} z\,\varphi(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z\,e^{-z^2/2}}_{\text{περιττή}}\,dz = 0$$

$Z \sim N(0,1)$

$$E(Z^2) = \int_{-\infty}^{\infty} z^2\,\varphi(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^2 e^{-z^2/2}}_{\text{άρτια}}\,dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2}\,dz \qquad z^2/2 = y \quad z = \sqrt{2y} \quad dz = \frac{1}{\sqrt{2y}}\,dy$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} 2y\,e^{-y}\,\frac{1}{\sqrt{2y}}\,dy = \frac{2\cdot\cancel{2}}{\cancel{2}\cdot\sqrt{\pi}} \underbrace{\int_{0}^{\infty} y^{1/2} e^{-y}\,dy}_{\Gamma(3/2)}$$

$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1 \quad \Rightarrow \quad \sigma^2 = E(Z^2) - \mu^2 = 1 - 0^2 = 1$$

Αν $X \sim N(0,1)$ τότε $\mu = 0$, $\sigma^2 = 1$.