$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$-\infty < x < +\infty$$

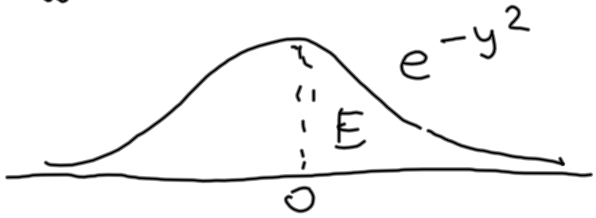
$\mu \in R, \quad \sigma^2 > 0$

$\sigma = \sqrt{\sigma^2}$. Υπολογισμός:

$$\int_{-\infty}^{\infty} \exp\left(-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2\right) dx \qquad \frac{x-\mu}{\sigma\sqrt{2}} = y$$

$$= \underbrace{\sigma\sqrt{2} \int_{-\infty}^{\infty} e^{-y^2} dy} = \sigma\sqrt{2\pi} \qquad \begin{array}{l} x = \mu + y\,\sigma\sqrt{2} \\ dx = \sigma\sqrt{2}\, dy \end{array}$$

$$I = \int_{-\infty}^{\infty} e^{-y^2}\, dy = 2E = 2\int_0^{\infty} e^{-y^2}\, dy$$



$y^2 = t \qquad y = \sqrt{t} \qquad dy = \frac{1}{2\sqrt{t}}\, dt \qquad t \in (0, \infty)$

$$I = 2 \cdot \int_0^{\infty} \frac{1}{2\sqrt{t}} e^{-t}\, dt = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\, dt$$

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt \qquad = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Πρόταση: Αν $X \sim N(\mu, \sigma^2)$ τότε

η $Z \overset{\text{ορ}}{=} \dfrac{X-\mu}{\sigma} \sim N(0,1)$.

Αντίστροφα, αν η $Z \sim N(0,1)$ τότε η

$$X \overset{\text{ορ}}{=} \mu + \sigma Z \sim N(\mu, \sigma^2)$$

Γενικότερα: Αν $X \sim N(\mu, \sigma^2)$, και $\alpha \in \mathbb{R}^*$,

$\beta \in \mathbb{R} \Rightarrow \quad Y = \alpha X + \beta \sim N(\alpha\mu + \beta,\ \alpha^2 \sigma^2)$.

Νοε 11-4:20 μμ

$X \sim N(\mu, \sigma^2)$. Θέτω $Z = \frac{X-\mu}{\sigma}$

$$P(Z \le t) = F_Z(t) = P\left(\frac{X-\mu}{\sigma} \le t\right)$$

$$\overset{\sigma>0}{=} P(X \le \mu+\sigma t) = F_X(\mu+\sigma t)$$

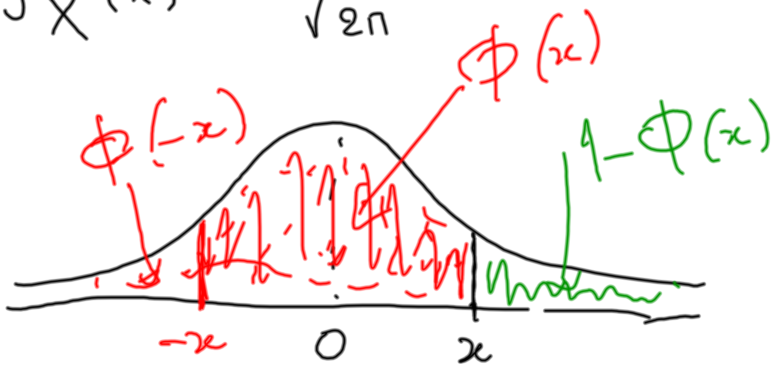
$$f_Z(t) = F_Z'(t) = F_X'(\mu+\sigma t)\cdot(\mu+\sigma t)'$$

$$= \sigma f_X(\mu+\sigma t) = \frac{\sigma}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\cancel{\mu}+\sigma t-\cancel{\mu})^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right) \qquad Z\sim N(0,1).$$

Τυποπ. Κανονική $\equiv N(0,1)$

$$\varphi(x) = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \in \mathbb{R}$$



$$\Phi(x) \overset{op}{=} \int_{-\infty}^{x} \varphi(t)\,dt \qquad \varphi = \Phi'$$

$$\Phi(0) = \frac{1}{2}$$

$$\Phi(-x) = 1 - \Phi(x) \quad \forall x \in \mathbb{R}$$

για $x=0$ παίρνω $\Phi(0) = 1/2$

Μέση τιμή: $Z \sim N(0,1)$ τότε

(i) $E(Z) = 0$ (ii) $Var(Z) = 1$.

$X = \mu + \sigma Z$ (όταν $X \sim N(\mu, \sigma^2)$)

$E(X) = \mu + \sigma \cancelto{0}{E(Z)} = \mu$, $Var(X) = Var(\mu + \sigma Z)$

$= \sigma^2 Var(Z) = \sigma^2$.

$Z \sim N(0,1)$

$$E(Z) = \int_{-\infty}^{\infty} \underbrace{x\,\varphi(x)}_{\text{περιττή}}\,dx = 0$$

$$E(Z^2) = \int_{-\infty}^{\infty} x^2 \varphi(x)\,dx = 2\int_0^{\infty} x^2 \varphi(x)\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-x^2/2}\,dx$$

$x^2/2 = y$

$x = \sqrt{2y}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2y\, e^{-y} \frac{1}{\sqrt{2y}}\,dy$$

$dx = \frac{1}{2\sqrt{2y}} \cdot 2\,dy$

Νοε 11-4:39 μμ

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{y}\, e^{-y}\, dy = \frac{2}{\sqrt{\pi}} \cdot \underbrace{\int_0^\infty y^{\frac{3}{2}-1} e^{-y}\, dy}_{\Gamma(3/2)}$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right) = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} = 1.$$

$$Var(Z) = E(Z^2) - (E(Z))^2 = 1 - 0^2 = 1.$$

π.χ.

$$\Phi(1) = 0.8413 \quad \Phi(1.2) = 0.8849$$

$$\Phi(3) = 0.9997$$

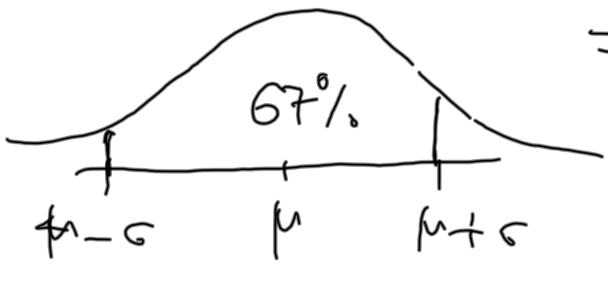
Νοε 11-4:43 μμ

$$P(\mu-\sigma < X < \mu+\sigma) = ? \qquad \underline{X \sim N(\mu,\sigma^2)}$$

$$= P\Big(-1 < \underbrace{\frac{X-\mu}{\sigma}}_{=\,Z \sim N(0,1)} < 1\Big) = \Phi(1^-) - \Phi(-1)$$

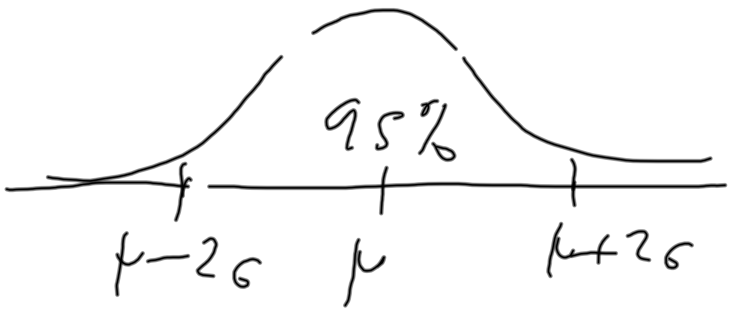
$$\overset{X_{\sigma\omega}}{=} \Phi(1) - \Phi(-1) = \Phi(1) - (1-\Phi(1))$$

$$= 2\Phi(1) - 1 \simeq 67\%$$



$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 2\Phi(2) - 1$$

$$= 2 \cdot 0.9773 - 1 \simeq 95\%$$



Ασκήσεις Κεφ 4

4.1, 4.2, 4.5, 4.7, 4.8, 4.11, 4.12,

4.13, 4.14, 4.15, 4.17, 4.18

4.7 $X \sim Exp(\theta)$ $(\theta > 0)$. Βρείτε την πυκνότητα της $Y = 1 - e^{-\theta X}$.

Νοε 11-4:53 μμ

$$P(Y \leq y) = P(1 - e^{-\theta X} \leq y) \qquad y \in (0,1)$$

$$= P(e^{-\theta X} \geq 1-y)$$

$$= P(-\theta X \geq \log(1-y))$$

$$= P\left(X \leq \frac{-1}{\theta} \log(1-y)\right) = F_X\left(-\frac{1}{\theta} \log(1-y)\right)$$

$$f_Y(y) = F_Y'(y) = f_X\left(-\frac{1}{\theta} \log(1-y)\right) \cdot \left(\frac{-1}{\theta} \log(1-y)\right)'$$

$$= \theta e^{-\theta \cdot \left(-\frac{1}{\theta} \log(1-y)\right)} \cdot \frac{-1}{\theta} \cdot \frac{1}{1-y} \cdot (1-y)' = \frac{1-y}{1-y} = 1.$$

$f_Y(y) = 1$, για $y \in (0,1)$
(0 αλλού). $Y \sim U(0,1)$

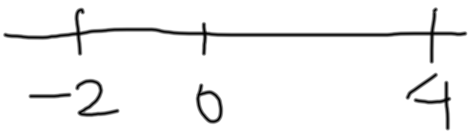
4.1) $\underbrace{X \sim U(a,b)}$, $E(X)=1$, $Var(X)=3$

$a, b = ?$

$$\left.\begin{aligned} 1 = \mu &= \frac{a+b}{2} \\ 3 = \sigma^2 &= \frac{(b-a)^2}{12} \end{aligned}\right\} \quad \left.\begin{aligned} a+b &= 2 \\ (b-a)^2 &= 36 \\ a &< b \end{aligned}\right\} \quad \begin{aligned} a+b &= 2 \\ b-a &= 6 \\ b &= 4 \\ a &= -2 \end{aligned}$$

$X \sim U(-2,4)$

$Y = |X|$. Βρείτε την συνκ. της Y.



$Y = |X| \in (0,4)$. $\quad y \in (0,4)$

$$F_Y(y) = P(|X| \le y) = P(-y \le X \le y)$$

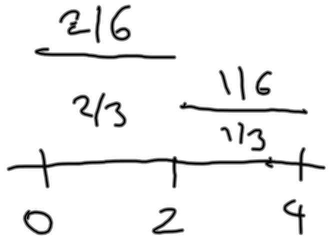
$$= F_X(y) - F_X(-y-) \overset{\text{συν.}}{=} F_X(y) - F_X(-y)$$

$$f_Y(y) = \left(F_X(y) - F_X(-y)\right)'$$

$$= f_X(y) + f_X(-y)$$

$$f_X(x) = \begin{cases} 1/6, & x \in (-2,4) \\ 0, & \text{διαφορ.} \end{cases}$$

$$= \begin{cases} 1/6 + 1/6, & y \in (0,2) \\ 1/6 + 0, & y \in (2,4) \end{cases}$$



Βρείτε $E(Y)$, $Var(Y)$ $\quad Y=|X|$

$$E(Y)=\int_0^4 y\, f_Y(y)\, dy=\int_0^2 \frac{2}{6}\, y+\int_2^4 \frac{1}{6}\cdot y$$

$$=\ldots=5/3$$

$$E(Y^2)=\int_0^2 y^2\,\frac{2}{6}+\int_2^4 y^2\,\frac{1}{6}=4=EX^2$$

$$Var(Y)=E(Y^2)-(E(Y))^2=\frac{11}{9}.$$