Ειδικές Συνεχείς Κατανομές

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \forall x$$

f: πυκνότητα (πιθανότητας)

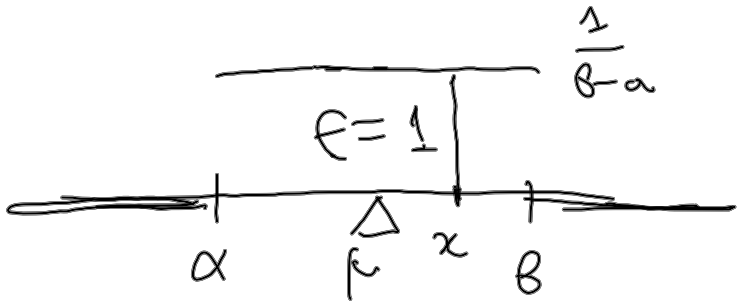
$$F' = f, \qquad \int_{-\infty}^{\infty} f = 1 \quad (f \geq 0)$$

Νοε 9-3:10 μμ

Ομοιόμορφη στο (α,β) $\quad -\infty<\alpha<\beta<\infty$

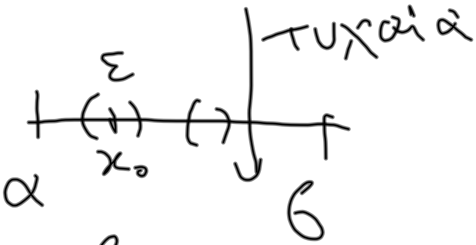
$X \sim U(\alpha,\beta)$

$$\overset{\text{ορ}}{\iff} \quad f_X(x) = \begin{cases} \dfrac{1}{\beta-\alpha}, & \alpha<x<\beta \\ 0, & \text{αλλού} \end{cases}$$



$$F(x)=\begin{cases}0, & x\le a\\ \dfrac{x-a}{\beta-a}, & \alpha\le x\le \beta\\ 1, & x\ge \beta\end{cases}$$

$X \sim U(a,\beta)$



$$\mu=\int_{-\infty}^{\infty} x f(x)\,dx=\int_{\alpha}^{\beta} x\cdot\frac{1}{\beta-a}\,dx$$

$$= \frac{1}{\beta-\alpha} \left.\frac{x^2}{2}\right|_{\alpha}^{\beta} = \frac{\beta^2-a^2}{2(\beta-a)} = \frac{a+\beta}{2}$$

$$\boxed{\mu = \frac{\alpha+\beta}{2}.}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,dx$$

$$= \frac{1}{\beta-\alpha} \cdot \int_a^\beta x^n\,dx = \frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-a)}.$$

$$E(X^2) = \frac{\beta^3-\alpha^3}{3(\beta-a)} = \frac{a^2 + a\beta + \beta^2}{3}.$$

Νοε 9-3:24 μμ

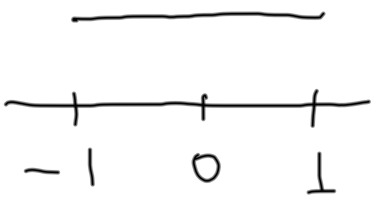
$$\sigma^2 = Var(X) = E(X^2) - \mu^2$$

$$= \frac{\alpha^2 + \alpha\beta + \beta^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2$$

$$= \cdots = \boxed{\frac{(\beta-\alpha)^2}{12} = \sigma^2} \quad \boxed{\mu = \frac{\alpha+\beta}{2}}$$

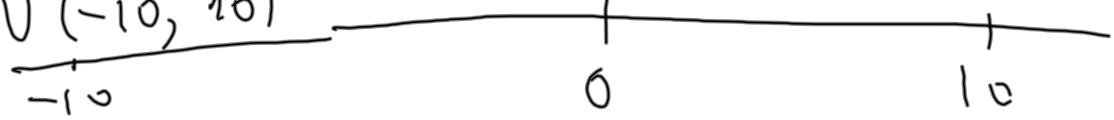
$$\sigma = \frac{\beta-\alpha}{\sqrt{12}}$$

$U(-1,1)$



$\mu = 0, \quad \sigma^2 = \frac{1}{3}$

$U(-10, 10)$



$$\sigma^2 = \frac{(10-(-10))^2}{12} = \frac{2^2 \cdot 10^2}{12} = \frac{10^2}{3} = 100 \cdot \frac{1}{3}$$

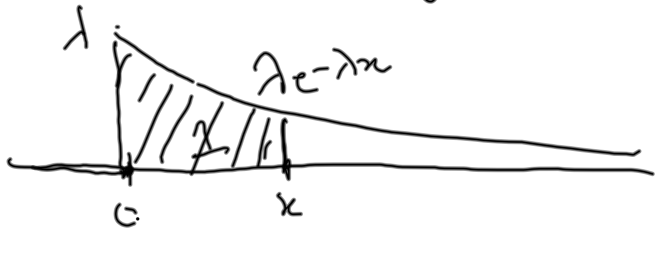
Εκθετική με παράμετρο $\lambda > 0$

$X \sim Exp(\lambda)$ όταν η πυκνότητα

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{διαφ.} \end{cases}$$

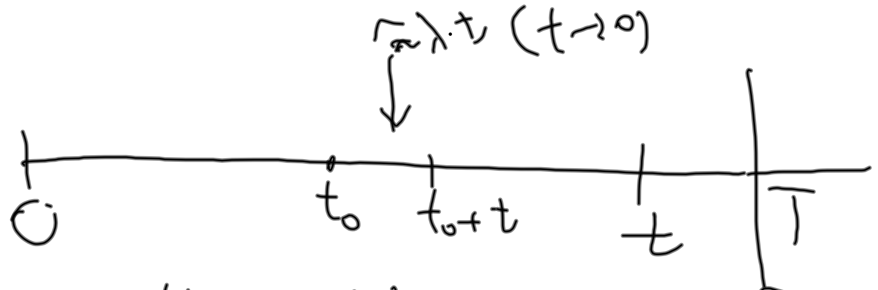
$f \geq 0$ $\quad \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$

$$= \int_0^{\infty} \left(-e^{-\lambda x}\right)' dx = -e^{-\lambda x}\Big|_0^{\infty} = -0 + e^0 = 1$$



$$F(x) = \int_0^x \left(-e^{-\lambda t}\right)' dt = -e^{-\lambda x} + 1$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$



$X(t) = \#$ συμβάντων στο $[0, t]$

$$X(t) \sim \text{Poisson}(\lambda t)$$

$$P(X(t)=x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}, \quad x = 0, 1, \ldots$$

$$P(X(t)=0) = e^{-\lambda t} = 1-(1-e^{-\lambda t}) = 1-F_X(t)$$

$$\{X(t)=0\} = \{T > t\}$$

$T =$ χρονική στιγμή του συμβάντος.

$$P(T>t) = P(X(t)=0) = e^{-\lambda t}$$

$$P(T \le x) = 1 - e^{-\lambda x} = F_T(x)$$

Δηλ. $T \sim \mathcal{E}xp(\lambda)$

$x, y > 0$

$$P(X > x+y \mid X > x) = P(A \mid B) \cdot$$

$$A = \{w : X(w) > x+y\}, \subseteq B = \{w : X(w) > x\}$$

$$\cdot = \frac{P(A \cap B)}{P(B)} = \frac{P(X > x+y)}{P(X > x)} \overset{Exp(\lambda)}{=} \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}}$$

$$X \sim Exp(\lambda) \quad P(X > t) = e^{-\lambda t}$$

$$= e^{-\lambda y}$$

$$= P(X > y)$$

Νοε 9-3:54 μμ

$$P(X > x+y \mid X > x) = \underline{P(X>y)}$$

$$\forall\, x, y > 0.$$

$$\boxed{\mu = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}}$$

Μέση τιμή / διασπορά Εκθετικής.

$$\mu = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx = \int_0^{\infty} x \left(-e^{-\lambda x}\right)' dx$$

$$= x\left(-e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty} (x)' \left(-e^{-\lambda x}\right) dx$$

$$= \int_0^{\infty} e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx = \int_0^{\infty} x^2 \left(-e^{-\lambda x}\right)' dx$$

$$= \ldots = \frac{2}{\lambda^2}. \qquad \sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

Νοε 9-4:12 μμ

$$E(X^n) = \frac{n!}{\lambda^n} \quad n = 1, 2, \ldots$$

$\Gamma(\alpha, \lambda)$

Συνάρτηση $\Gamma(\alpha)$ του Euler

$$\forall \alpha > 0 \quad \Gamma(\alpha) \overset{\text{ορ.}}{=} \int_0^\infty w^{\alpha-1} e^{-w}\, dw.$$

Νοε 9-4:20 μμ

$$\Gamma(1) = \int_0^\infty w^{1-1} e^{-w}\, dw = \int_0^\infty e^{-w}\, dw = 1$$

$$w^{1-1} \to 1$$

$$\Gamma(\alpha+1) = \int_0^\infty w^{\alpha} \left(-e^{-w}\right)' dw$$

$$= \cancel{-e^{-w} \cdot w^{\alpha} \Big|_0^\infty} + \int_0^\infty \alpha w^{\alpha-1} e^{-w}\, dw$$

$$\boxed{\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)} \quad \boxed{\Gamma(1) = 1}$$

$$\alpha > 0$$

Νοε 9-4:23 μμ

$$\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2 = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3\cdot 2 = 6 = 3!$$

$$\Gamma(5) = 4\Gamma(4) = 4\cdot 6 = 24 = 4!$$

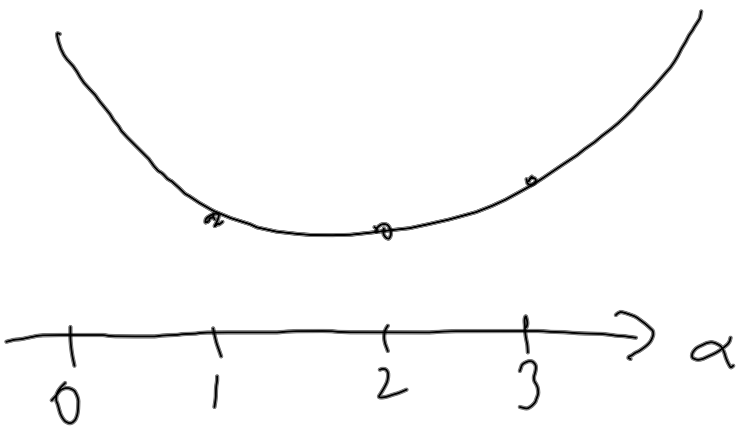
$$\Gamma(6) = 5\,\Gamma(5) = 5\cdot 24 = 120 = 5!$$

$$\Gamma(n) = (n-1)! \qquad n = 1, 2, \ldots$$

$$\Gamma(1/2) = \sqrt{\pi} \qquad \text{(Euler)}$$

$$\Gamma(3/2) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\frac{\sqrt{\pi}}{2} = \frac{3}{4}\sqrt{\pi}$$



$X \sim \Gamma(\alpha, \lambda)$ όταν

$$f_X(x) = \frac{\lambda^\alpha}{\underbrace{\Gamma(\alpha)}} x^{\alpha-1} e^{-\lambda x}, \quad x > 0$$

$\lambda x = w \quad x = \frac{w}{\lambda} \quad dx = \frac{1}{\lambda} dw$

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \int_0^\infty \left(\frac{w}{\lambda}\right)^{\alpha-1} e^{-w} \frac{1}{\lambda} dw$$

$$= \frac{1}{\lambda^\alpha} \underbrace{\int_0^\infty w^{\alpha-1} e^{-w} dw} = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

Ειδικά όταν $\alpha = n \in \{1, 2, \dots\}$

$$f_X(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \quad x > 0$$

$$\text{Erlang}(n, \lambda) \equiv \Gamma(n, \lambda)$$

$n = 1$, $\quad f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$$\text{Exp}(\lambda) \equiv \Gamma(1, \lambda)$$

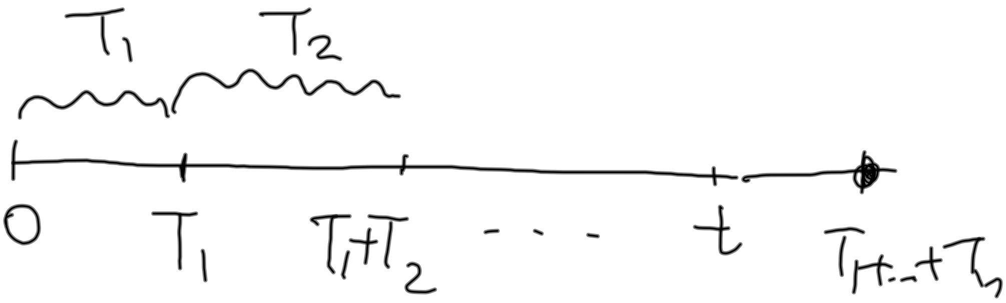
Νοε 9-4:35 μμ

$$\boxed{\mu = \frac{\alpha}{\lambda}, \quad \sigma^2 = \frac{\alpha}{\lambda^2}} \qquad \Gamma(\alpha, \lambda)$$

$$\boxed{\mu = \frac{n}{\lambda}, \quad \sigma^2 = \frac{n}{\lambda^2}} \qquad \mathrm{Erl}(n, \lambda)$$

Νοε 9-4:37 μμ

$$\{T_1 + \cdots T_n > t\} \Longleftrightarrow \{X(t) \leq n-1\}$$



$$T_1, \ldots, T_n \sim \mathcal{E}xp(\lambda) \Rightarrow T_1 + \cdots + T_n \sim \mathcal{E}rl\,(n, \lambda)$$

$$P(X(t) \leq n-1) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda \cdot t)^k}{k!}$$

$$= P(T_1 + \dots + T_n > t) = 1 - P(\dots \leq t)$$

$$= 1 - \int_0^t \underbrace{\frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}} dx$$

$$F_X(x) = 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}$$

όταν $X \sim \text{Erlang}(n, \lambda)$.

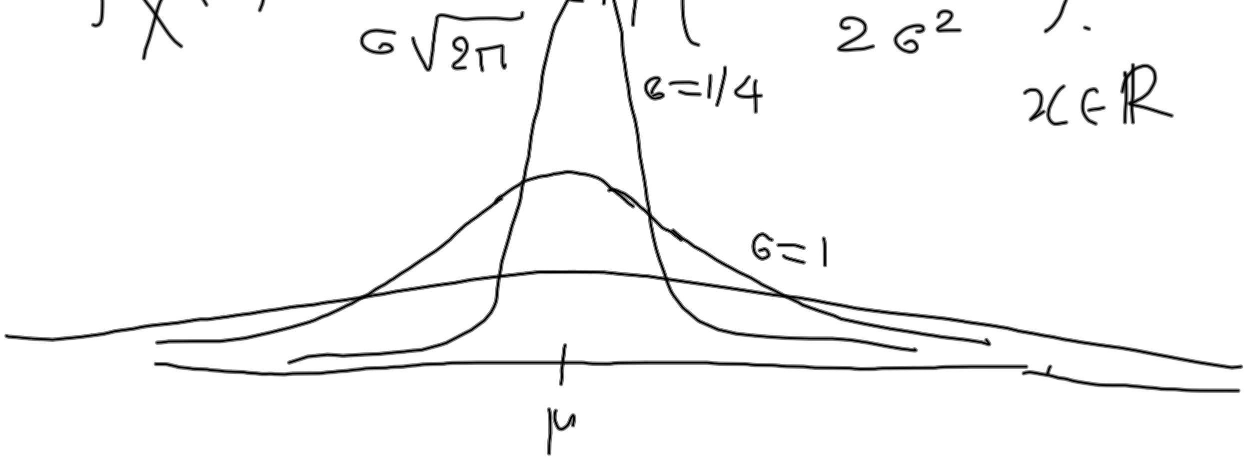
Νοε 9-4:47 μμ

$N(\mu, \sigma^2)$ κανονική μ, σ^2.

$\mu \in \mathbb{R}, \ \sigma^2 > 0$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

$x \in \mathbb{R}$



$$I = \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2\right) dx$$

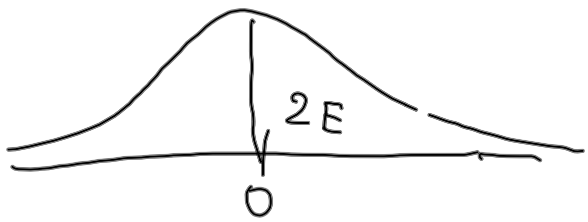
$$y = \frac{x-\mu}{\sigma\sqrt{2}} \Rightarrow x = \mu + \sigma\sqrt{2}\cdot y$$

$$dx = \sigma\sqrt{2}\cdot dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2}\sigma\sqrt{2}\, dy = \sigma\sqrt{2}\cdot \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{\sqrt{\pi}}$$

Νοε 9-4:52 μμ

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = 2\int_0^{\infty} \underline{e^{-y^2}}\,dy \quad \text{συμμετρ.}$$



$$y^2 = w$$

$$y = \sqrt{w}$$

$$dy = \frac{1}{2\sqrt{w}}\,dw$$

$$= 2\int_0^{\infty} e^{-w}\frac{1}{2\sqrt{w}}\,dw = \int_0^{\infty} w^{\frac{1}{2}-1} e^{-w}\,dw = \Gamma\left(\frac{1}{2}\right)$$

$N(0,1)$ $\mu = 0$, $\sigma^2 = 1$

τυποποιημένη κανονική

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}$$

$$= \varphi(x)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$