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INVISCID LIMIT OF LINEARLY DAMPED AND FORCED NONLINEAR SCHRÖDINGER EQUATIONS

NIKOLAOS GIALELIS

Abstract. We approximate a solution of the nonlinear Schrödinger Cauchy problem by solutions of the linearly damped and driven nonlinear Schrödinger Cauchy problems in any open subset of $\mathbb{R}^n$ and, for the case $n = 1$, we provide an estimate of the convergence rate. In doing so, we extract a sufficient relation between the external force and the constant of damping.

1. Introduction

In this work we are interested in the n-dimensional linearly damped, driven nonlinear Schrödinger equation (LDDNLS), with the common case of pure power nonlinearity, i.e.

$$iu_t + \Delta u + \lambda |u|^{\alpha} u + i\gamma u = f, \quad \forall (t,x) \in [0,T] \times U, \tag{1.1}$$

where $\lambda \in \mathbb{R}^*$ and $\alpha > 0$, $\gamma > 0$ and $u = u(t,x;\gamma)$, $f = f(t,x;\gamma)$ are complex-valued functions for $t \in [0,T]$ with $T > 0$ and $x \in \overline{U}$ with $U \subseteq \mathbb{R}^n$ being an arbitrary open set. γ is the constant of zero order dissipation and f an external excitation. The goal is to show, under certain conditions, that (1.1) can be considered as a perturbation of the associated nonlinear Schrödinger equation (NLS), i.e.

$$iv_t + \Delta v + \lambda |v|^{\alpha} v = 0, \quad \forall (t,x) \in [0,T] \times U. \tag{1.2}$$

NLS models with gain and loss effects have found applications to many physical fields such as nonlinear optics and fluid mechanics (see [3] and the references therein). The use of damping and forcing effects for (1.2) is not a novelty for physicists (see e.g. [6] and [20]). On the other hand, some cases of (1.1) have already been studied, concerning the solvability and the long time behavior of solutions and their attractors of Cauchy problems (see [2, 13, 14, 15, 16, 17, 18, 24]). Comparisons between the two equations have also been made (see [12] about some blowup issues). Even though these two equations seem quite similar, they share important differences. In particular, many of the symmetries of (1.2) do not hold for (1.1), such as the known scaling symmetry, the Galilean invariance and the time reversal symmetry (see [22]). To the author's best knowledge, some questions of "inviscid limit" type for these equations still remain unasked. In [5], (1.1) arises from a perturbation study of the sine-Gordon equation and in [26] it is shown that (1.2)

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is the inviscid limit of complex Ginzburg-Landau equation. However, it is natural for us to expect that (1.1) could be a perturbation of (1.2) and this viewpoint is the scope of this study.

Here, we extract a sufficient relation between f and γ of the form $\|f\| = O(\gamma)$, as $\gamma \searrow 0$ (see (6.1)), to obtain two approximation results in Section 6. First (see Proposition 6.1 and Corollary 6.2), we approximate a solution (or the solution in case of uniqueness) v of the NLS initial-boundary value problem

$$\begin{aligned} iv_t + \Delta v + \lambda |v|^\alpha v = 0, \quad & \forall (t,x) \in (0,T] \times U \\ v = v_0, \quad & \text{on } \{t=0\} \times \overline{U} \\ v = 0, \quad & \text{on } [0,T] \times \partial U, \end{aligned} \tag{1.3}$$

by a sequence $\{u_m\}_{m=1}^\infty$ of solutions of the LDDNLS initial-boundary value problems of the form

$$\begin{aligned} iu_t + \Delta u + \lambda |u|^\alpha u + i\gamma u = f, \quad & \forall (t,x) \in (0,T] \times U \\ u = u_0, \quad & \text{on } \{t=0\} \times \overline{U} \\ u = 0, \quad & \text{on } [0,T] \times \partial U, \end{aligned} \tag{1.4}$$

as $\gamma_m \searrow 0$, $f_m \to 0$ and $u_{0m} \to v_0$. Second (see Proposition 6.3), we estimate the rate of this approximation for certain cases. We note that the convergences above will be rigorously interpreted.

In proving the above results, we first show, in Sections 4 and 5, the existence of a bounded solution of (1.4), which satisfies a certain estimate (see Theorems 4.1, 4.2 and 5.1). The aforementioned sufficient condition $\|f\| = O(\gamma)$, as $\gamma \searrow 0$, comes naturally from that estimate. We emphasize that the technique we use differs from the classic one of "regularized nonlinearities" presented in [9] and this is also a third goal that we reach with the present work.

We note that, since our main interest lies in inviscid limit results, we deal with the defocusing and the subcritical focusing case, as well as the critical focusing case with sufficiently small initial datum (see (4.1)), where the analysis for the extraction of energy estimates is not that extended in comparison with the supercritical focusing case for sufficiently small initial datum. Hence, we exclude this case, not bacause of inefficiency of our approach, but to keep the work as compact as possible and stay focused on our main result.

2. Notation

We denote by $* \vee \star := \max\{*, \star\}$ and by $B_\varrho(x) \subset \mathbb{R}^n$ the open ball of radius $\varrho > 0$ centered at x. If $p, r \in [1,\infty]$ and $k, m \in \mathbb{N}_0$, then we write

$$\begin{gathered} |\cdot|_{m,r,U} := \|\cdot\|_{W^{m,r}(U)}, \quad |\cdot|_{-m,U} := \|\cdot\|_{H^{-m}(U)} \\ |\cdot|_{k,p,T;m,r,U} := \|\cdot\|_{W^{k,p}(0,T;W^{m,r}(U))}, \quad |\cdot|_{k,p,T;-m,U} := \|\cdot\|_{W^{k,p}(0,T;H^{-m}(U))}. \end{gathered}$$

We omit $p = \infty$, $T = \infty$ and $U = \mathbb{R}^n$ from the notation.

For $m \in \mathbb{N}_0$ and U, we consider that the space $H^m(U) \equiv W^{m,2}(U)$ is equipped with the inner product $(*, \star)_{H^m(U)} \to \mathbb{C}$ defined as

$$(u,v)_{H^m(U)} := \sum_{0 \le |\alpha| \le m} \int_U (D_w^\alpha u)(D_w^\alpha \overline{v}) dx, \quad \forall u, v \in H^m(U).$$

When $m = 0$, we simply write $(*, \star) := (*, \star)_{H^0(U)} \equiv (*, \star)_{L^2(U)}$.

Let $\mathcal{F}(U_1;\mathbb{C})$ be a function space over $U_1 \subset U_2 \subseteq \mathbb{R}^n$ and $f \in \mathcal{F}(U_1)$. We denote by $\mathcal{E}_{U_2} f$ its extension by zero in $U_2 \setminus U_1$ and $\mathcal{E}_{U_2}\mathcal{F}(U_1) := \{\mathcal{E}_{U_2} f \big| f \in \mathcal{F}(U_1)\}$. We omit $U_2 = \mathbb{R}^n$ from these notations. Moreover, if $g \in \mathcal{F}(U_2)$, we denote by $\mathcal{R}_{U_1} g$ and $\mathcal{R}_{U_1}\mathcal{F}(U_2)$ the restriction of g in U_1 and the set of these restricted functions, respectively.

We write C and c for any non-negative constant factor and exponent, respectively. These constants may be explicitly calculated in terms of known quantities and may change from line to line and also within a certain line in a given computation. We also employ the letter K for any increasing function $K : [0,\infty)^m \to [0,\infty)$, as well as $\widetilde{K} : [0,\infty)^2 \times (0,\infty) \to [0,\infty)$, such that

(1) $\widetilde{K}(\cdot,\cdot,z_0)$ is increasing, for fixed $z_0 > 0$ and also
(2) there exists K such that $\widetilde{K}(x, O(z), z) \to K(x_0)$, as $(x,z) \to (x_0, 0)$.

When U appears as subscript in an element, it denotes that this depends on it, while its absence designates independence. If $u : [0,T]\times U \to \mathbb{C}$, with $u(t,\cdot) \in \mathcal{F}(U)$ for each $t \in [0,T]$, then, following the notation of, e.g., [11] and [23], we associate with u the mapping $\mathbf{u} : [0,T] \to \mathcal{F}(U;\mathbb{C})$, defined by $[\mathbf{u}(t)](x) := u(t,x)$, for every $x \in U$ and $t \in [0,T]$.

3. Preliminaries

Lemma 3.1. *Let $u, v \in L^{\alpha+2}(U)$. Then*

$$\int_U |u|^{\alpha+1}|v|dx \le |u|^{\alpha+1}_{0,\alpha+2,U}|v|_{0,\alpha+2,U}, \tag{3.1}$$

$$||u|^\alpha u - |v|^\alpha v|_{0,\frac{\alpha+2}{\alpha+1},U} \le C(|u|^c_{0,\alpha+2,U} + |v|^c_{0,\alpha+2,U})|u-v|_{0,\alpha+2,U}. \tag{3.2}$$

Proof. The first inequality follows from (7.4) for $p = \frac{\alpha+2}{\alpha+1}$ and $q = \alpha+2$. As for the second one, we apply (7.2), (7.4) for $p = \alpha+1$ and $q = \frac{\alpha+1}{\alpha}$ and (7.1). □

Next, we set

$$\alpha \in \begin{cases} (0,\infty), & \text{if n} = 1,2 \\ (0, \frac{4}{\text{n}-2}], & \text{otherwise.} \end{cases} \tag{3.3}$$

In view of (3.1) and the scaling invariant embedding $H^1_0(U) \hookrightarrow L^{\alpha+2}(U)$ (notice that U is assumed to be just an open set and then see Remark 7.5, we define $g : H^1_0(U) \to L^{\frac{\alpha+2}{\alpha+1}}(U) \hookrightarrow H^{-1}(U)$ to be the nonlinear and bounded operator such that

$$\langle g(u;\alpha), v\rangle := \lambda \int_U |u|^\alpha \overline{u}\, v dx, \quad \text{for } v \in H^1_0(U).$$

Next, we recall the following well establish result.

Lemma 3.2. *For every $f \in H^{-1}(U)$ there exists $\{f_j\}_{j=0}^n \subset L^2(U)$ such that*

$$\langle f, v\rangle = \int_U v\overline{f_0} + \sum_{j=1}^n (\partial^j v)\overline{f_j} dx, \ \forall v \in H^1_0(U)$$

and, in particular, we have

$$(v, f) = \langle f, v\rangle, \quad \forall v \in H^1_0(U), \ \forall f \in L^2(U).$$

Proof. The first result follows from a direct application the complex version of Riesz-Fréchet representation theorem (see [8, Proposition 11.27]). The second is a direct consequence of the first one. □

Now, for the above operator we have the following estimate.

Proposition 3.3. *Let $u, v \in H_0^1(U)$. Then*

$$|g(u) - g(v)|_{0,\frac{\alpha+2}{\alpha+1},U} \le K\big(|u|_{1,2,U}, |v|_{1,2,U}\big)|u - v|_{0,\alpha+2,U}. \tag{3.4}$$

The proof of the above proposition is a direct application of (3.2) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$.

We further define $\mathcal{N}[\cdot,\cdot], \mathcal{N}_\gamma[\cdot,\cdot] : \big(H_0^1(U)\big)^2 \to \mathbb{C}$ to be the forms which are associated with the operators $\Delta+g$ and $\Delta+g+i\gamma I$, respectively, such that $\mathcal{N}[u,v] := \langle \Delta u, v\rangle + \langle g(u), v\rangle$ and $\mathcal{N}_\gamma[u,v] := \langle \Delta u, v\rangle + \langle g(u), v\rangle + i\gamma\langle u, v\rangle$, for every $u, v \in H_0^1(U)$.

We then restate problems (1.3) and (1.4) as Cauchy ones: for $\mathbf{f} : [0,T] \to L^2(U)$, we seek solutions $\mathbf{v}, \mathbf{u} \in L^\infty(0,T; H_0^1(U)) \cap W^{1,\infty}(0,T; H^{-1}(U))$ of

$$\begin{gathered} \langle i\mathbf{v}', u\rangle + \mathcal{N}[\mathbf{v}, u] = 0, \quad \forall u \in H_0^1(U), \text{ a.e. in } [0,T] \\ \mathbf{v}(0) = v_0. \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} \langle i\mathbf{u}', v\rangle + \mathcal{N}_\gamma[\mathbf{u}, v] = \langle \mathbf{f}, v\rangle, \quad \forall v \in H_0^1(U), \text{ a.e. in } [0,T] \\ \mathbf{u}(0) = u_0. \end{gathered} \tag{3.6}$$

Also, we provide an estimate for the forms $\mathcal{N}$ and $\mathcal{N}_\gamma$.

Proposition 3.4. *Let $u, v \in H_0^1(U)$. Then*

$$|\mathcal{N}[u,v]| + |\mathcal{N}_\gamma[u,v]| \le K\big(|u|_{1,2,U}, |v|_{1,2,U}\big). \tag{3.7}$$

The proof of the above proposition is and application of (7.4) ($p = p = 2$), (3.1) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$. Some useful results also follow.

Lemma 3.5. *Let α be as in* (3.3) *and $u \in H_0^1(U)$. Then*

$$|u|_{0,\alpha+2,U}^{\alpha+2} \le C|Du|_{0,2,U}^{\frac{n\alpha}{2}}|u|_{0,2,U}^{\frac{4-n\alpha}{2}+\alpha}. \tag{3.8}$$

If, in addition, $n = 2$ and $\tau \in (1,\infty)$, then

$$|u|_{0,2\tau,U}^{2\tau} \le C|Du|_{0,2,U}^{2(\tau-1)}|u|_{0,2,U}^{2}. \tag{3.9}$$

Proof. The first inequality is direct from Theorem 7.4 (and Remark 7.5) for $p = \alpha + 2$, $r = q = 2$, $j = 0$, $m = 1$ and $\theta = \frac{n\alpha}{2(\alpha+2)}$. As for the second one we set $\alpha = 2(\tau - 1)$ in (3.8). □

Remark 3.6. If

$$\alpha \in \begin{cases} (0,\infty), & \text{if } n = 1, 2 \\ (0, \frac{4}{n-2}), & \text{otherwise,} \end{cases} \tag{3.10}$$

then the exponent of the term $|u|_{0,2,U}$ in (3.8) is strictly positive and hence that term does not vanish. Moreover, an estimate of the constant in (3.9) is

$$C \le (4\pi)^{(1-\tau)}\tau^\tau, \tag{3.11}$$

for an elegant proof of which we refer to [21] and the references therein.

Lemma 3.7. *Let $\alpha \in (0,4/n)$, $\epsilon > 0$ and $u \in H_0^1(U)$. Then*

$$|u|_{0,\alpha+2,U}^{\alpha+2} \le \epsilon |Du|_{0,2,U}^2 + C|u|_{0,2,U}^c. \tag{3.12}$$

The above lemma is an application of (7.3) for $p = \frac{4}{n\alpha}$ and $q = \frac{4}{4-n\alpha}$ into (3.8).

Proposition 3.8. *(i) Let $\mathcal{H}$ be a Hilbert space, as well as $\{\mathbf{u}_k\}_{k=1}^\infty \subset L^\infty(0,T;\mathcal{H})$ and $\mathbf{u} : [0,T] \to \mathcal{H}$ with $\mathbf{u}_k(t) \rightharpoonup \mathbf{u}(t)$ in $\mathcal{H}$, for a.e. $t \in [0,T]$. If $\|\mathbf{u}_k\|_{L^\infty(0,T;\mathcal{H})} \le C$ uniformly for all $k \in \mathbb{N}^*$, then $\mathbf{u} \in L^\infty(0,T;\mathcal{H})$ with $\|\mathbf{u}\|_{L^\infty(0,T;\mathcal{H})} \le C$, where C is the same in both inequalities.*

(ii) Let $\mathcal{F}$ be a Banach space with the Radon-Nikodym property with respect to the Lebesgue measure in $(0,T,\mathscr{B}([0,T]))$ and $\{\mathbf{u}_k\}_{k=1}^\infty \cup \{\mathbf{u}\} \subset L^\infty(0,T;\mathcal{F}^*)$ with $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;\mathcal{F}^*)$ (That is, $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ in $\sigma(L^\infty(0,T;\mathcal{F}^*), L^1(0,T;\mathcal{F}))$. Note that $L^\infty(0,T;\mathcal{F}^*) \cong (L^1(0,T;\mathcal{F}))^*$ (see, e.g., [10, Theorem 1, §IV.1]*.) If $\|\mathbf{u}_k\|_{L^\infty(0,T;\mathcal{F}^*)} \le C$ uniformly for all $k \in \mathbb{N}^*$, then $\|\mathbf{u}\|_{L^\infty(0,T;\mathcal{F}^*)} \le C$, where C is the same in both inequalities.*

Proof. (i) We derive that $\|\mathbf{u}(t)\|_{\mathcal{H}} \le C$, for a.e. $t \in [0,T]$, from the (sequentially) weak lower semi-continuity of the norm. The result follows directly.

(ii) Let $v \in \mathcal{F}$ be such that $\|v\|_{\mathcal{F}} \le 1$ and set $\mathbf{v} : [0,T] \to \mathcal{F}$ the constant function with $\mathbf{v}(t) := v$, for all $t \in [0,T]$. We have

$$\int_s^{s+h} \langle \mathbf{u}_k, \mathbf{v} \rangle dt \le Ch,$$

] for every $s \in (0,T)$ and every sufficiently small $h > 0$. Letting $k \to \infty$, dividing both parts by h and then letting $h \to 0$, we obtain $\langle \mathbf{u}(s), v \rangle \le C$, for every $s \in (0,T)$. Since v arbitrary, the proof is complete. □

Proposition 3.9. *Let $U_1 \subset U_2 \subseteq \mathbb{R}^n$, $m \in \mathbb{N}_0$ and $\{u_k\}_{k=1}^\infty \cup \{u\} \subset H^m(U_2)$ such that $u_k \rightharpoonup u$ in $H^m(U_2)$. Then $\mathcal{R}_{U_1} u_k \rightharpoonup \mathcal{R}_{U_1} u$ in $H^m(U_1)$. The analogous result for L^p, with $p \in (1,\infty)$, instead of H^m also holds.*

Proof. We show the first result and in analogous fashion we obtain the second one. Let $v \in C_c^\infty(U_1)$, then we have

$$\begin{aligned}
(\mathcal{R}_{U_1} u_k - \mathcal{R}_{U_1} u, v)_{H^m(U_1)} &= \sum_{|\beta|=0}^m \int_{U_1} D^\beta (\mathcal{R}_{U_1} u_k - \mathcal{R}_{U_1} u) D^\beta \overline{v} dx \\
&= \sum_{|\beta|=0}^m \int_{U_2} D^\beta (u_k - u) D^\beta \mathcal{E}_{U_2} \overline{v} dx \\
&= (u_k - u, \mathcal{E}_{U_2} v)_{H^m(U_2)} \to 0,
\end{aligned}$$

hence, the result follows from a denseness argument. □

Proposition 3.10. *Let $\{u_m\}_{m=1}^\infty \cup \{u\} \subset H^1(U)$ such that $u_m \rightharpoonup u$ in $H^1(U)$ and $u_m \rightharpoonup u$ in $L^2(U)$. Then $Du_m \rightharpoonup Du$ in $L^2(U)$.*

Proof. Let $v \in C_c^\infty(U)$. Then

$$(Du_m - Du, v) = (u_m - u, v)_{H^1(U)} - (u_m - u, v) \to 0,$$

hence, the result follows from a denseness argument. □

4. LDDNLS Cauchy problem in bounded open sets

In this section we assume $U \subset \mathbb{R}^n$ is bounded.

Theorem 4.1. *Let α be as in (3.10), $\mathbf{f} \in W^{1,\infty}(0,T;L^2(U))$ and $u_0 \in H_0^1(U)$. If*

$$\lambda < 0, \quad \text{or}$$
$$\lambda > 0 \text{ and } \alpha \in (0, \frac{4}{n}), \quad \text{or} \tag{4.1}$$
$$\lambda > 0, \quad \alpha = \frac{4}{n} \text{ and } |u_0|_{0,2,U} \vee \frac{1}{\gamma}|\mathbf{f}|_{0,T;0,2,U} < \lambda^{-1/\alpha}|R|_{0,2},$$

where R as in Theorem 7.6, then there exist a solution $\mathbf{u} \in L^\infty(0,T;H_0^1(U)) \cap W^{1,\infty}(0,T;H^{-1}(U))$ of (3.6), such that

$$|\mathbf{u}|_{0,T;1,2,U} + |\mathbf{u}'|_{0,T;-1,U} \le \widetilde{\mathcal{K}} := \widetilde{K}(|u_0|_{1,2,U}, |\mathbf{f}|_{1,T;0,2,U}, \gamma). \tag{4.2}$$

Proof. **Step 1.** We use the standard Faedo-Galerkin method. It holds true that $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U)$ (see Remark 7.5), hence there exists a countable subset of $H_0^1(U) \cap C^\infty(U)$, which is an orthogonal basis of $L^2(U)$, e.g., the complete set of eigenfunctions for the operator $-\Delta$ in $H_0^1(U)$ (This specific subset is an orthogonal basis of both $H_0^1(U)$ and $L^2(U)$). Let $\{w_k\}_{k=1}^\infty \subset H_0^1(U) \cap C^\infty(U)$ be that basis, appropriately normalized so that $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2(U)$. Fixing any $m \in \mathbb{N}^*$, we define $\mathbf{d}_m : J_m \to \mathbb{C}^m$, with $\mathbf{d}_m(t) := [d_m^1(t), \dots, d_m^m(t)]^{\mathrm{T}}$, to be the unique, absolutely continuous, maximal solution (i.e. J_m with $0 \in J_m$ is the maximal interval on which the solution is defined) of the initial-value problem

$$\mathbf{d}_m{}'(t) = F_m(t, \mathbf{d}_m(t)), \quad \forall t \in J_m^*$$
$$\mathbf{d}_m(0) = [(u_0, w_1), \dots, (u_0, w_m)]^{\mathrm{T}},$$

where $F_m \in C([0,T]^{2m+1}; \mathbb{C}^m)$ with

$$F_m^k(t, d_m(t)) := i\mathcal{N}_\gamma[\sum_{l=1}^m d_m^l(t) w_l, w_k] - i(w_k, \mathbf{f}(t)), \quad \forall k = 1, \dots, m.$$

Now, we define $\mathbf{u}_m : J_m \to H_0^1(U) \cap C^\infty(U)$, with

$$\mathbf{u}_m(t) := \sum_{k=1}^m \overline{d_m^k}(t) w_k.$$

It is then trivial to verify that

$$\langle i\mathbf{u}_m', w_k \rangle + \mathcal{N}_\gamma[\mathbf{u}_m, w_k] = \langle \mathbf{f}, w_k \rangle, \tag{4.3}$$

everywhere in J_m and for all $k \in \{1, \dots, m\}$. Note that $u_{0m} := u_m(0, \cdot) = \mathbf{u}_m(0) \to u_0$ in $L^2(U)$ and $|u_{0m}|_{0,2,U} \le |u_0|_{0,2,U}$. Furthermore, $|u_{0m}|_{1,2,U} \le |u_0|_{1,2,U}$. Indeed, we can argue as in Step 3. of the proof of [11, Theorem 2, Section 6.5] to deduce $|Du_{0m}|_{0,2,U} \le |Du_0|_{0,2,U}$. Moreover, we set $f_0 := \mathbf{f}(0)$, since $\mathbf{f} \in C([0,T]; L^2(U))$.

Step 2. We multiply the variational equation (4.3) by $d_m^k(t)$, sum for $k = 1, \dots, m$ and take imaginary parts of both sides to find

$$\frac{d}{dt}|\mathbf{u}_m|_{0,2,U}^2 + 2\gamma|\mathbf{u}_m|_{0,2,U}^2 \le 2|(\mathbf{f}, \mathbf{u}_m)|,$$

hence, from (7.3) for $\epsilon = \gamma/2$ ($p = q = 2$),

$$\frac{d}{dt}|\mathbf{u}_m|^2_{0,2,U} + \gamma|\mathbf{u}_m|^2_{0,2,U} \le \frac{1}{\gamma}|\mathbf{f}|^2_{0,2,U} \le \frac{1}{\gamma}|\mathbf{f}|^2_{0,T;0,2,U},$$

which implies the estimate

$$|\mathbf{u}_m|_{0,2,U} \le |u_0|_{0,2,U} \vee \frac{1}{\gamma}|\mathbf{f}|_{0,T;0,2,U}, \quad \forall t \in [0,T], \tag{4.4}$$

therefore, since $m \in \mathbb{N}^*$ is arbitrary, $J_m \equiv [0,T]$, for all $m \in \mathbb{N}^*$ and

$$|\mathbf{u}_m|_{0,2,U} \le \widetilde{\mathcal{K}}, \quad \forall t \in [0,T],\ \forall m \in \mathbb{N}^*. \tag{4.5}$$

Step 3α. We multiply the variational equation (4.3) by ${d_m^k}'(t) + \gamma d_m^k(t)$, sum for $k = 1, \dots, m$ and take real parts of both sides to find

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m, \mathbf{f}] + \gamma\mathcal{J}[\mathbf{u}_m, \mathbf{f}] + \frac{\gamma}{2}|Du_m|^2_{0,2,U} - \frac{\gamma\lambda(\alpha+1)}{\alpha+2}|\mathbf{u}_m|^{\alpha+2}_{0,\alpha+2,U} = \mathrm{Re}(\mathbf{f}', \mathbf{u}_m), \tag{4.6}$$

where

$$\mathcal{J}[v,g] := \frac{1}{2}|Dv|^2_{0,2,U} - \frac{\lambda}{\alpha+2}|v|^{\alpha+2}_{0,\alpha+2,U} + \mathrm{Re}(g,v), \quad \forall v \in H^1_0(U), g \in L^2(U).$$

Note that $\mathcal{J}[u_{0m}, f_0] \le K(|u_0|_{1,2,U}, |\mathbf{f}|_{0,T;0,2,U})$. To show that

$$|Du_m|_{0,2,U} \le \widetilde{\mathcal{K}}, \quad \forall m \in \mathbb{N}^*, \tag{4.7}$$

we consider the following cases.
(i) Since $\frac{\gamma}{2}|D\mathbf{u}_m|^2_{0,2,U} - \frac{\gamma\lambda(\alpha+1)}{\alpha+2}|\mathbf{u}_m|^{\alpha+2}_{0,\alpha+2,U} \ge 0$, from (7.4) ($p = q = 2$) and (4.5) we obtain

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m, \mathbf{f}] + \gamma\mathcal{J}[\mathbf{u}_m, \mathbf{f}] \le |\mathbf{u}_m|_{0,2,\Omega}|\mathbf{f}'|_{0,2,U} \le \widetilde{\mathcal{K}}|\mathbf{f}'|_{0,T;0,2,U},$$

which implies

$$\mathcal{J}[\mathbf{u}_m, \mathbf{f}] \le \mathcal{J}[u_{0m}, f_0] \vee \frac{1}{\gamma}\widetilde{\mathcal{K}}|\mathbf{f}'|_{0,T;0,2,U}.$$

Hence

$$\frac{1}{2}|Du_m|^2_{0,2,U} \le \widetilde{\mathcal{K}}|\mathbf{f}|_{0,T;0,2,U} + \mathcal{J}[u_{0m}, f_0] \vee \frac{1}{\gamma}\widetilde{\mathcal{K}}|\mathbf{f}'|_{0,T;0,2,U},$$

therefore we obtain (4.7).
(ii) Using (3.12) for $\epsilon = \frac{\alpha+2}{2\lambda(\alpha+1)}$ to estimate the last term on the left-hand side of (4.6), we have

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m, \mathbf{f}] + \gamma\mathcal{J}[\mathbf{u}_m, \mathbf{f}] \le \widetilde{\mathcal{K}}(\gamma + |\mathbf{f}'|_{0,T;0,2,U}),$$

which implies

$$\mathcal{J}[\mathbf{u}_m, \mathbf{f}] \le \mathcal{J}[u_{0m}, f_0] \vee \widetilde{\mathcal{K}}(1 + \frac{1}{\gamma}|\mathbf{f}'|_{0,T;0,2,U}).$$

Therefore, applying again (3.12) for $\epsilon = \frac{\widetilde{\epsilon}(\alpha+2)}{\lambda}$ and some $\widetilde{\epsilon} \in (0, 1/2)$, we obtain

$$\frac{1}{2}|Du_m|^2_{0,2,U} \le \widetilde{\mathcal{K}}(1 + |\mathbf{f}|_{0,T;0,2,U}) + \mathcal{J}[u_{0m}, f_0] \vee \widetilde{\mathcal{K}}(1 + \frac{1}{\gamma}\widetilde{\mathcal{K}}|\mathbf{f}'|_{0,T;0,2,U}),$$

hence (4.7) follows.

Using (7.6) for C_{cr} to estimate the last term on the left-hand side of (4.6), as well as (4.4), we have

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\gamma\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\leq\widetilde{\mathcal{K}}(\gamma+|\mathbf{f}'|_{0,T;0,2,U}),$$

since $\frac{1}{2}-\frac{\lambda}{\alpha+2}C_{cr}\big(|u_0|_{0,2,U}\vee\frac{1}{\gamma}|\mathbf{f}|_{0,T;0,2,U}\big)^{\alpha}>0$. (4.7) then follows.

Step 3β. From (4.5) and (4.7) we conclude that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is uniformly bounded in $L^{\infty}(0,T;H_0^1(U))$, with

$$|\mathbf{u}_m|_{0,T;1,2,U}\leq\widetilde{\mathcal{K}},\quad\forall m\in\mathbb{N}^*. \tag{4.8}$$

Notice that we avoid to use the Poincaré inequality along with (4.7) for the above bound.

Step 4. We fix an arbitrary $v\in H_0^1(U)$ with $|v|_{1,2,U}\leq 1$ and write $v=\mathcal{P}v\oplus(I-\mathcal{P})v$, where $\mathcal{P}$ is the projection in $\operatorname{span}\{w_k\}_{k=1}^m$. Since $\mathbf{u}_m'\in\operatorname{span}\{w_k\}_{k=1}^m$ and $\mathcal{N}[h,g]$ linear for g, from the variational equation (4.3) we obtain that

$$\langle i\mathbf{u}_m',v\rangle=-\mathcal{N}_{\gamma}[\mathbf{u}_m,\mathcal{P}v]+\langle\mathbf{f},\mathcal{P}v\rangle.$$

Applying (3.7) we derive $|\langle i\mathbf{u}_m',v\rangle|\leq\widetilde{\mathcal{K}}+|\mathbf{f}|_{0,T;0,2,U}$. Hence $\{\mathbf{u}_m'\}_{m=1}^{\infty}$ is uniformly bounded in $L^{\infty}(0,T;H^{-1}(U))$, with

$$|\mathbf{u}_m'|_{0,T;-1,U}\leq\widetilde{\mathcal{K}},\quad\forall m\in\mathbb{N}^*. \tag{4.9}$$

Step 5α. From (4.8), (4.9), [9, Theorem 1.3.14 i)] and Proposition 3.8 (i), there exist a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}\subseteq\{\mathbf{u}_m\}_{m=1}^{\infty}$ and a function $\mathbf{u}\in L^{\infty}(0,T;H_0^1(U))\cap W^{1,\infty}(0,T;H^{-1}(U))$, such that

$$\mathbf{u}_{m_l}(t)\rightharpoonup\mathbf{u}(t)\text{ in }H_0^1(U), \tag{4.10}$$

for every $t\in[0,T]$ and $|\mathbf{u}|_{0,T;1,2,U}\leq\widetilde{\mathcal{K}}$.

Step 5β. $H^{-1}(U)$ is separable since $H_0^1(U)$ is separable, hence by the Dunford-Pettis theorem (see [10, Theorem 1, §III.3]) we have

$$L^{\infty}(0,T;H^{-1}(U))\cong\big(L^1(0,T;H_0^1(U))\big)^*.$$

From the the above, (4.9), the Banach-Alaoglu-Bourbaki theorem (see [8, Theorem 3.16]) and Proposition 3.8 (ii), there exist a subsequence of $\{\mathbf{u}_{\mathrm{m_l}}\}_{\mathrm{l}=1}^{\infty}$, which we still denote as such and a function $\mathbf{h}\in L^{\infty}(0,T;H^{-1}(U))$, such that

$$\mathbf{u}_{m_l}'\overset{*}{\rightharpoonup}\mathbf{h}\text{ in }L^{\infty}(0,T;H^{-1}(U))\text{ and }|\mathbf{h}|_{0,T;-1,U}\leq\widetilde{\mathcal{K}}. \tag{4.11}$$

From the convergence in (4.10), [23, Lemma 1.1, Chapter 3], along with the Leibniz rule, we can derive that

$$\int_0^T\langle\mathbf{u}_{m_l}',\psi v\rangle dt\to\int_0^T\langle\mathbf{u}',\psi v\rangle dt,\qquad forall\psi\in C_c^1([0,T]),\ v\in H_0^1(U),$$

hence $\mathbf{h}\equiv\mathbf{u}'$.

Step 6α. Since U is bounded, $H_0^1(U)\hookrightarrow\hookrightarrow L^2(U)\hookrightarrow H^{-1}(U)$. Hence, from (4.8), (4.9) and the Aubin-Lions-Simon lemma (see [7, Theorem II.5.16]), there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$, which we still denote as such and a function $\mathbf{y}\in C([0,T];L^2(U))$, such that

$$\mathbf{u}_{m_l}\to\mathbf{y}\quad\text{in }C([0,T];L^2(U)). \tag{4.12}$$

From the convergence in (4.10), we deduce that $\mathbf{y} \equiv \mathbf{u}$.

Step 6β. From (4.8), (4.12), (3.8) and Remark 3.6 we have

$$\mathbf{u}_{m_l} \to \mathbf{u} \quad \text{in } C([0,T]; L^{\alpha+2}(U)). \tag{4.13}$$

Step 6γ. From (3.4), (4.8), the bound in (4.10), (4.12) and (4.13) we obtain

$$g(\mathbf{u}_{m_l}) \to g(\mathbf{u}) \quad \text{in } C([0,T]; L^{\frac{\alpha+2}{\alpha+1}}(U)). \tag{4.14}$$

Step 7α. Let now $\psi \in C_c^\infty([0,T])$ and fix $N \in \mathbb{N}^*$. We choose m_l such that $N \le m_l$ and $v \in \operatorname{span}\{w_k\}_{k=1}^N$, hence, by the linearity of the inner product, we obtain from (4.3) that

$$\int_0^T \langle i\mathbf{u}'_{m_l}, \psi v\rangle + \mathcal{N}_\gamma[\mathbf{u}_{m_l}, \psi v] dt = \int_0^T \langle \mathbf{f}, \psi v\rangle dt.$$

In view of Proposition 3.10, we then pass to the weak, $*$-weak and strong limits (since $\psi v \in L^1(0,T; H_0^1(U))$), to obtain

$$\int_0^T \langle i\mathbf{u}', \psi v\rangle + \mathcal{N}_\gamma[\mathbf{u}, \psi v] dt = \int_0^T \langle \mathbf{f}, \psi v\rangle dt.$$

Since ψ is arbitrary, $\mathbf{u}$ satisfies the variational equation in (3.6) for every $v \in \operatorname{span}\{w_k\}_{k=1}^N$. By the linear and continuous dependence on v, we obtain the desired result, after letting $N \to \infty$.

Step 7β. Finally, $\mathbf{u}$ satisfies the initial condition, i.e. $\mathbf{u}(0) \equiv u_0$, which follows from (4.12) for $t = 0$ combined with $\mathbf{u}_m(0) \to u_0$ in $L^2(U)$ from Step 1. □

We can also get the following well-known result, by slightly modifying, in an evident way, the above proof.

Theorem 4.2. *Let α be as in* (3.10) *and $v_0 \in H_0^1(U)$. If*

$$\begin{gathered} \lambda < 0, \quad \text{or} \\ \lambda > 0 \text{ and } \alpha \in (0, \frac{4}{n}), \quad \text{or} \\ \lambda > 0, \quad \alpha = \frac{4}{n} \text{ and } |v_0|_{0,2,U} < \lambda^{-1/\alpha} |R|_{0,2}, \end{gathered} \tag{4.15}$$

where R as in Theorem 7.6, then there exist a solution $\mathbf{v} \in L^\infty(0,T; H_0^1(U)) \cap W^{1,\infty}(0,T; H^{-1}(U))$ of (3.5)*, such that*

$$|\mathbf{v}|_{0,T;1,2,U} + |\mathbf{v}'|_{0,T;-1,U} \le K(|v_0|_{1,2,U}). \tag{4.16}$$

5. LDDNLS Cauchy problem in unbounded sets

In this section, we assume that $U \subseteq \mathbb{R}^n$ is unbounded. The concept behind the proof of the following result is that of [4, Theorem 1.3].

Theorem 5.1. *Let $U \subseteq \mathbb{R}^n$ be unbounded, α be as in* (3.10)*, $\mathbf{f} \in W^{1,\infty}(0,T; L^2(U))$ and $u_0 \in H^1(U)$. Then the conclusions of Theorem 4.1 and Theorem 4.2 still hold.*

Proof. We deal with the extension of Theorem 4.1 for unbounded sets. The second result follows similarly.

Step 1. Since U open, we fix an arbitrary $B_\varrho(x_0) \subset U$. Let $u_{0k} := \mathcal{R}_U \eta_k u_0$, for all $k \in \mathbb{N}^*$, where $\{\eta_k\}_{k=1}^\infty$ as in Appendix 8. Hence, for all $k \in \mathbb{N}^*$, we have

$$|u_{0k}|_{0,2,U} \le |u_0|_{0,2,U} \quad \text{and} \quad |u_{0k}|_{1,2,U} \le C|u_0|_{1,2,U}. \tag{5.1}$$

From the first inequality in (5.1), the required bound of $|u_0|_{0,2,U}$ for the critical focusing case *iii*) in (4.1) remains the same, as in the corresponding case of bounded open sets. We also notice that

$$u_{0k} = 0, \text{ in } B_{a_k}(x_0)^{\mathrm{T}} \cap U,$$

hence, by fixing a $\delta = \delta(\varrho, a_1)$ such that $\delta < a_1 - \varrho$ and by setting $B_k := B_{a_k+\delta}(x_0) \cap U$, for every $k \in \mathbb{N}^*$, we obtain that $\{\mathcal{R}_{B_k} u_{0k}\}_{k=1}^\infty \subset H_0^1(B_k)$ (see also [8, Lemma 9.5]). Moreover,

$$u_{0k} \to u_0 \text{ in } L^2(U). \tag{5.2}$$

Indeed,

$$|u_{0k} - u_0|_{0,2,U} = |(\eta_k - 1)u_0|_{0,2,U} \le |u_0|_{0,2,B_{a_{k-1}}(x_0)^{\mathrm{T}} \cap U} \to 0.$$

Step 2α. Fixing any $k \in \mathbb{N}^*$, we consider (3.6) in $U = B_k$, where we take $\mathcal{R}_{B_k} u_{0k}$ as our initial datum. and we set $\mathbf{u}^k \in L^\infty(0,T; H_0^1(B_k)) \cap W^{1,\infty}(0,T; H^{-1}(B_k))$ to be a solution that Theorem 4.1 provides. From its proof, it follows that there exist a sequence $\{\mathbf{u}_m^k\}_{m=1}^\infty$ of absolutely continuous functions from $[0,T]$ to $H_0^1(B_k) \cap C^\infty(B_k)$, such that

$$|\mathbf{u}_m^k|_{0,T;1,2,B_k} + |\mathbf{u}_m^{k\,\prime}|_{0,T;-1,B_k} \le \widetilde{K}(|u_{0k}|_{1,2,B_k}, |\mathbf{f}|_{1,T;0,2,B_k}, \gamma), \quad \forall m \in \mathbb{N}^*. \tag{5.3}$$

and

$$\begin{gathered} \mathbf{u}_m^k(t) \rightharpoonup \mathbf{u}^k(t) \quad \text{in } H_0^1(B_k), \text{ for every } t \in [0,T], \\ \mathbf{u}_m^{k\,\prime} \overset{*}{\rightharpoonup} \mathbf{u}^{k\prime} \quad \text{in } L^\infty(0,T; H^{-1}(B_k)). \end{gathered} \tag{5.4}$$

From (5.1) and (5.3) we deduce that

$$|\mathbf{u}_m^k|_{0,T;1,2,B_k} + |\mathbf{u}_m^{k\,\prime}|_{0,T;-1,B_k} \le \widetilde{\mathcal{K}}, \quad \forall m \in \mathbb{N}^*. \tag{5.5}$$

Step 2β. From the fact that the local regularity of the eigenfunctions at the boundary depends on the local smoothness of the boundary and also that $\partial B_k \backslash \partial U \in C^\infty$, we obtain that $\mathbf{u}_m^k(t)$ and $\mathbf{u}_m^{k\,\prime}(t)$ are smooth on $\partial B_k \setminus \partial U$ for every $t \in [0,T]$, with

$$\mathcal{R}_{\partial B_k \backslash \partial U} \mathbf{u}_m^k = \mathcal{R}_{\partial B_k \backslash \partial U} \mathbf{u}_m^{k\,\prime} = 0, \quad \forall m \in \mathbb{N}^*.$$

Therefore, the extensions by zero $\mathbf{v}_m^k := \mathcal{E}_U \mathbf{u}_m^k$, for all $m \in \mathbb{N}^*$, are continuous in $\partial B_k \setminus \partial U$ and thus $\{\mathbf{v}_m^k\}_{m=1}^\infty$ and $\{\mathbf{v}_m^{k\,\prime}\}_{m=1}^\infty$ are sequences of functions mapping to $H_0^1(U)$. Evidently,

$$|\mathbf{v}_m^k|_{0,T;1,2,U} = |\mathbf{u}_m^k|_{0,T;1,2,B_k} \quad \text{and} \quad |\mathbf{v}_m^{k\,\prime}|_{0,T;-1,U} = |\mathbf{u}_m^{k\,\prime}|_{0,T;-1,B_k},$$

hence, from (5.5), we obtain

$$|\mathbf{v}_m^k|_{0,T;1,2,U} + |\mathbf{v}_m^{k\,\prime}|_{0,T;-1,U} \le \widetilde{\mathcal{K}}, \quad \forall m \in \mathbb{N}^*.$$

Step 2γ. Dealing as in Step 4 of the proof of Theorem 4.1, there exist a subsequence $\{\mathbf{v}^k_{m_l}\}_{l=1}^\infty \subseteq \{\mathbf{v}^k_m\}_{m=1}^\infty$ and a function $\mathbf{v}^k \in L^\infty(0,T;H^1_0(U))\cap W^{1,\infty}(0,T;H^{-1}(U))$, such that

$$\begin{gathered} \mathbf{v}^k_{m_l}(t) \rightharpoonup \mathbf{v}^k(t) \quad \text{in } H^1_0(U), \text{ for every } t\in[0,T], \\ {\mathbf{v}^k_{m_l}}' \overset{*}{\rightharpoonup} \mathbf{v}^{k\prime} \quad \text{in } L^\infty(0,T;H^{-1}(U)), \\ |\mathbf{v}^k|_{0,T;1,2,U} + |\mathbf{v}^{k\prime}|_{0,T;-1,U} \le \widetilde{\mathcal{K}}. \end{gathered} \tag{5.6}$$

Since $k\in\mathbb{N}^*$ is arbitrary, $\{\mathbf{v}^k\}_{k=1}^\infty \subset L^\infty(0,T;H^1_0(U))\cap W^{1,\infty}(0,T;H^{-1}(U))$ and the above estimate is satisfied for each $k\in\mathbb{N}^*$.

Step 3α. Dealing again as before, there exist a subsequence $\{\mathbf{v}^{k_l}\}_{l=1}^\infty \subseteq \{\mathbf{v}^k\}_{k=1}^\infty$ and a function $\mathbf{u}\in L^\infty(0,T;H^1_0(U))\cap W^{1,\infty}(0,T;H^{-1}(U))$, such that

$$\begin{gathered} \mathbf{v}^{k_l}(t) \rightharpoonup \mathbf{u}(t) \quad \text{in } H^1_0(U), \text{ for every } t\in[0,T], \\ {\mathbf{v}^{k_l}}' \overset{*}{\rightharpoonup} \mathbf{u}' \quad \text{in } L^\infty(0,T;H^{-1}(U)), \\ |\mathbf{u}|_{0,T;1,2,U} + |\mathbf{u}'|_{0,T;-1,U} \le \widetilde{\mathcal{K}}. \end{gathered} \tag{5.7}$$

Step 3β. From (3.4), (3.8), Remark 3.6, the estimate in (5.6) and [9, Lemma 3.3.6] we deduce that $\{g(\mathbf{v}^{k_l})\}_{l=1}^\infty$ is bounded in $C^{0,\frac12}([0,T];L^{\frac{\alpha+2}{\alpha+1}}(U))$. Hence, from Proposition 1.1.2 in the same book, there exist a subsequence of $\{\mathbf{v}^{k_l}\}_{l=1}^\infty$, which we still denote as such, and a function $\mathbf{y}\in C([0,T];L^{\frac{\alpha+2}{\alpha+1}}(U))$, such that

$$g(\mathbf{v}^{k_l}(t)) \rightharpoonup \mathbf{y}(t) \quad \text{in } L^{\frac{\alpha+2}{\alpha+1}}(U), \text{ for every } t\in[0,T]. \tag{5.8}$$

Step 4α. Let Ω be any bounded $\subset U$, such that $H^1(\Omega)\hookrightarrow\hookrightarrow L^2(\Omega)$, e.gȧ ball. For $k\in\mathbb{N}^*$ big enough so that $\Omega\subseteq B_k$, we have

$$\langle \mathbf{v}^k, \mathcal{E}_U v\rangle = (\mathbf{u}^k, \mathcal{E}_{B_k} v), \quad \langle g(\mathbf{v}^k), \mathcal{E}_U v\rangle = \langle g(\mathbf{u}^k), \mathcal{E}_{B_k} v\rangle, \quad \langle \mathbf{v}^{k\prime}, \mathcal{E}_U v\rangle = \langle \mathbf{u}^{k\prime}, \mathcal{E}_{B_k} v\rangle, \tag{5.9}$$

for every $v\in C_c^\infty(\Omega)$. Indeed, for the first equality, from (5.6) we obtain

$$\int_U \overline{\mathbf{v}^k_{m_l}} \mathcal{E}_U v dx \to \int_U \overline{\mathbf{v}^k} \mathcal{E}_U v dx\,,$$

and from (5.4) we obtain

$$\int_U \overline{\mathbf{v}^k_{m_l}} \mathcal{E}_U v dx = \int_{B_k} \mathcal{R}_{B_k} \overline{\mathbf{v}^k_{m_l}} \mathcal{E}_{B_k} v dx \to \int_{B_k} \overline{\mathbf{u}^k} \mathcal{E}_{B_k} v dx.$$

The second equality follows similarly. The third equality follows from the first one and Lem1.1, Ch3, in [23]. Now, since $\mathbf{u}^k$ is a solution of (3.6) in B_k,

$$\langle i\mathbf{u}^{k\prime}, \mathcal{E}_{B_k} v\rangle + \mathcal{N}_\gamma[\mathbf{u}^k, \mathcal{E}_{B_k} v] = \langle \mathbf{f}, \mathcal{E}_{B_k} v\rangle, \quad \forall v\in C_c^\infty(\Omega), \text{ a.e. in } [0,T],$$

hence, from (5.9),

$$\langle i\mathbf{v}^{k\prime}, \mathcal{E}_U v\rangle + \mathcal{N}_\gamma[\mathbf{v}^k, \mathcal{E}_U v] = \langle \mathbf{f}, \mathcal{E}_U v\rangle, \quad \forall v\in C_c^\infty(\Omega), \text{ a.e. in } [0,T]. \tag{5.10}$$

Step 4β. From the first convergence in (5.7), the weak lower semi-continuity of the H^1-norm and the aforementioned compact embedding, we obtain that there exist a subsequence of $\{\mathbf{v}^{k_l}\}_{l=1}^\infty$, which we still denote as such, for which we have

$$\mathbf{v}^{k_l}(t) \to \mathbf{u}(t) \text{ in } L^2(\Omega), \text{ for every } t\in[0,T]. \tag{5.11}$$

We set $k = k_l$ in (5.10) and we pass to the limit $l \to \infty$. From (5.7), (5.8), (5.11) and Proposition 3.10, we deduce that

$$\int_0^T (\langle i\mathbf{u}', \mathcal{E}_U v\rangle + \langle \Delta \mathbf{u}, \mathcal{E}_U v\rangle + \langle \mathbf{y}, \mathcal{E}_U v\rangle + i\gamma\langle \mathbf{u}, \mathcal{E}_U v\rangle)\overline{\psi} dt = \int_0^T \langle \mathbf{f}, \mathcal{E}_U v\rangle \overline{\psi} dt,$$

for every $v \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty([0,T])$, hence

$$\langle i\mathbf{u}', \mathcal{E}_U v\rangle + \langle \Delta \mathbf{u}, \mathcal{E}_U v\rangle + \langle \mathbf{y}, \mathcal{E}_U v\rangle + i\gamma\langle \mathbf{u}, \mathcal{E}_U v\rangle = \langle \mathbf{f}, \mathcal{E}_U v\rangle, \tag{5.12}$$

for all $v \in C_c^\infty(\Omega)$, a.e. in $[0,T]$.

Step 4γ. From (5.8) and [restr]Proposition 3.9 we have

$$g(\mathcal{R}_\Omega \mathbf{v}^{k_l}(t)) = \mathcal{R}_\Omega g(\mathbf{v}^{k_l}(t)) \rightharpoonup \mathcal{R}_\Omega \mathbf{y}(t) \quad \text{in } L^{\frac{\alpha+2}{\alpha+1}}(U), \text{ for every } t \in [0,T]. \tag{5.13}$$

On the other hand, from (5.11) and Proposition 3.9,

$$\mathcal{R}_\Omega \mathbf{v}^{k_l}(t) \to \mathcal{R}_\Omega \mathbf{u}(t) \quad \text{in } L^2(\Omega), \text{ for every } t \in [0,T].$$

From (3.4), (3.8), Remark 3.6 and the latter convergence we obtain

$$g(\mathcal{R}_\Omega \mathbf{v}^{k_l}(t)) \to g(\mathcal{R}_\Omega \mathbf{u}(t)) = \mathcal{R}_\Omega g(\mathbf{u}(t)) \quad \text{in } L^{\frac{\alpha+2}{\alpha+1}}(U), \text{ for every } t \in [0,T]. \tag{5.14}$$

From (5.13) and (5.14) we derive $\mathcal{R}_\Omega g(\mathbf{u}) \equiv \mathcal{R}_\Omega \mathbf{y}$ and so (5.12) gets the form

$$i\langle \mathbf{u}', \mathcal{E}_U v\rangle + \mathcal{N}_\gamma[\mathbf{u}, \mathcal{E}_U v] = \langle \mathbf{f}, \mathcal{E}_U v\rangle, \quad \forall v \in C_c^\infty(\Omega), \text{ a.e. in } [0,T].$$

Since Ω is arbitrary, $\mathbf{u}$ satisfies the variational equation in (3.6).

Step 5. As far as the initial condition is concerned, we fix an arbitrary $t_0 \in (0,T]$. Let $v \in H_0^1(\mathrm{U})$ be arbitrary and $\phi \in C^1([0,T])$ such that $\phi(0) \neq 0$ and $\phi(t_0) = 0$. We then have from [23, Lemma 1.1, Chapter 3], along with the Leibniz rule, that

$$\begin{gathered}\int_0^{t_0} \langle \mathbf{v}_m^{k\,\prime}, \phi v\rangle dt = -\int_0^{t_0} \langle \mathbf{v}_m^k, \phi' v\rangle dt - \langle \mathbf{v}_m^k(0), \phi(0) v\rangle, \\ \int_0^{t_0} \langle \mathbf{u}', \phi v\rangle dt = -\int_0^{t_0} \langle \mathbf{u}, \phi' v\rangle dt - \langle \mathbf{u}(0), \phi(0) v\rangle.\end{gathered} \tag{5.15}$$

Moreover, $\langle \mathbf{v}_m^k(0), \phi(0) v\rangle = \langle \mathbf{u}_m^k(0), \phi(0)\mathcal{R}_{B_k} v\rangle$, hence, by setting $m = m_l$ and letting $l \to 0$, we obtain

$$\int_0^{t_0} \langle \mathbf{v}^{k\prime}, \phi v\rangle dt = -\int_0^{t_0} \langle \mathbf{v}^k, \phi' v\rangle dt - \langle \mathcal{R}_{B_k} u_{0k}, \phi(0)\mathcal{R}_{B_k} v\rangle.$$

Since $\langle \mathcal{R}_{B_k} u_{0k}, \phi(0)\mathcal{R}_{B_k} v\rangle = \langle u_{0k}, \phi(0) v\rangle$, we set $k = k_l$ and we pass to the limit $l \to \infty$, applying (5.2), to obtain

$$\int_0^{t_0} \langle \mathbf{u}', \phi v\rangle dt = -\int_0^{t_0} \langle \mathbf{u}, \phi' v\rangle dt - \langle u_0, \phi(0) v\rangle. \tag{5.16}$$

From the second equation in (5.15) and (5.16), we conclude that $\mathbf{u}(0) = u_0$. □

6. NLS as limit case $\gamma \to 0$ of LDDNLS

Here we consider $\{u_{0m}\}_{m=1}^\infty \cup \{v_0\} \subset H_0^1(U)$, $\{\mathbf{f}_m\}_{m=1}^\infty \subset W^{1,\infty}(0,T;L^2(U))$ and $\{\gamma_m\}_{m=1}^\infty \subset (0,\infty)$ with $\gamma_m \searrow 0$, such that

$$\begin{gathered}|f_m|_{1,T;0,2,U} = O(\gamma_m), \quad \text{as } m \to \infty, \\ u_{0m} \to v_0, \quad \text{in } H_0^1(U).\end{gathered} \tag{6.1}$$

Proposition 6.1. *For every v_0 and $\{(u_{0m}, \mathbf{f}_m, \gamma_m)\}_{m=1}^{\infty}$ as above, as well as every corresponding sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ of solutions of (3.6), which Theorem 4.1 or 5.1 provides, there exist a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty} \subseteq \{\mathbf{u}_m\}_{m=1}^{\infty}$ and a solution $\mathbf{v} \in L^\infty(0,T;H_0^1(U)) \cap W^{1,\infty}(0,T;H^{-1}(U))$ of (3.5), such that*

$$\mathbf{u}_{m_l}(t) \rightharpoonup \mathbf{v}(t) \text{ in } H_0^1(U), \quad \text{for every } t \in [0,T],$$
$$\mathbf{u}'_{m_l} \overset{*}{\rightharpoonup} \mathbf{v}' \quad \text{in } L^\infty(0,T;H^{-1}(U)),$$
$$|\mathbf{u}_{m_l}|_{0,T;1,2,U} + |\mathbf{u}_{m_l}{}'|_{0,T;-1,U} + |\mathbf{v}|_{0,T;1,2,U} + |\mathbf{v}'|_{0,T;-1,U} \leq K(|v_0|_{1,2,U}),$$

for all $m \in \mathbb{N}^*$.

Proof. In view of the From the above proofs, it is sufficient to show that $\{|\mathbf{u}_m|_{0,T;1,2,U} + |\mathbf{u}_m{}'|_{0,T;-1,U}\}_{m=1}^{\infty}$ is bounded. Indeed, it is direct from the limit property of $\widetilde{\mathcal{K}}$ that

$$|\mathbf{u}_m|_{0,T;1,2,U} + |\mathbf{u}_m{}'|_{0,T;-1,U} \leq K(|v_0|_{1,2,U}), \quad \forall m \in \mathbb{N}^*. \qquad \square$$

Before we proceed to the next result, we make a short, needed note about the uniqueness of solutions of the problems (3.5) and (3.6). It is easy to see that uniqueness results for (3.6) follow exactly as for (3.5). In particular (see [9]), for the case $n = 1$ as well as for $n = 2$, $\alpha \in (0,2]$, we obtain uniqueness in every open $U \subseteq \mathbb{R}^n$, from the embedding $H_0^1(U) \hookrightarrow L^\infty(U)$ and Trudinger's inequality respectively. One can also utilize (3.9) and (3.11) instead of Trudinger's inequality (see also the proof of point (ii) in Proposition 6.3 below. As for the case $U = \mathbb{R}^n$, uniqueness follows for all $n \in \mathbb{N}^*$ from the dispersive properties (see also the Strichartz estimates) of every solution.

Corollary 6.2. *If the solutions of (3.5) and (3.6) are unique, then, for every v_0 and $\{(u_{0m}, \mathbf{f}_m, \gamma_m)\}_{m=1}^{\infty}$ as above, the corresponding sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ of solutions of (3.6) converges to the corresponding solution $\mathbf{v} \in L^\infty(0,T;H_0^1(U)) \cap W^{1,\infty}(0,T;H^{-1}(U))$ of (3.5), in the sense that*

$$\mathbf{u}_m(t) \rightharpoonup \mathbf{v}(t) \text{ in } H_0^1(U), \quad \text{for every } t \in [0,T],$$
$$\mathbf{u}'_m \overset{*}{\rightharpoonup} \mathbf{v}' \quad \text{in } L^\infty(0,T;H^{-1}(U))$$
$$|\mathbf{u}_m|_{0,T;1,2,U} + |\mathbf{u}_m{}'|_{0,T;-1,U} \leq K(|v_0|_{1,2,U}), \quad \forall m \in \mathbb{N}^*.$$

Proof. From Proposition 6.1 and uniqueness, we have that, for every such v_0 and $\{(u_{0m}, \mathbf{f}_m, \gamma_m)\}_{m=1}^{\infty}$, there exists a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty} \subseteq \{\mathbf{u}_m\}_{m=1}^{\infty}$ such that

$$\begin{gathered} \mathbf{u}_{m_l}(t) \rightharpoonup \mathbf{v}(t) \quad \text{in } H_0^1(U), \text{ for every } t \in [0,T], \\ \mathbf{u}'_{m_l} \overset{*}{\rightharpoonup} \mathbf{v}' \quad \text{in } L^\infty(0,T;H^{-1}(U)). \end{gathered} \tag{6.2}$$

Seeking a contradiction, we assume that a sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ does not converge to $\mathbf{v}$ in the above sense, e.g. there exists $t_0 \in [0,T]$ such that

$$\mathbf{u}_m(t_0) \not\rightharpoonup \mathbf{v}(t_0) \quad \text{in } H_0^1(U).$$

The second case follows similarly. Then there exist $\epsilon > 0$, $v_0 \in H_0^1(U)$ and a subsequence of $\{\mathbf{u}_m\}_{m=1}^{\infty}$, that we still denote as such, for which we have

$$|(\mathbf{u}_m(t_0), v_0)_{H_0^1(U)} - (\mathbf{v}(t_0), v_0)_{H_0^1(U)}| \geq \epsilon, \quad \forall m \in \mathbb{N}^*,$$

which is a contradiction to (6.2). The estimate follows from the limit property of $\widetilde{\mathcal{K}}$. $\square$

Next, we extract some estimates for the rate of the above convergence. We note that they involve the uniqueness cases, even though we do not make use of this property in the process.

Proposition 6.3. *For every convergent sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ of solutions of (3.6) to a solution $\mathbf{v}$ of (3.5), as in Proposition 6.1 or Corollary 6.2, we set $\mathbf{w}_m := \mathbf{u}_m - \mathbf{v}$, for all $m \in \mathbb{N}^*$. If $n = 1$, then there exist $C_{11} = C_{11}(|v_0|_{1,2,U})$, $C_{12} = C_{12}(|v_0|_{1,2,U}, |\mathbf{f}_m|_{1,T;0,2,U}, \gamma_m)$ with $C_{12} = O(\gamma_m^2)$, as $m \to \infty$, such that*

$$|\mathbf{w}_m|_{0,2,U}^2 \le |u_{0m} - v_0|_{0,2,U}^2 e^{C_{11}t} + C_{12}(1 - e^{C_{11}t}), \quad \forall t \in [0,T], \tag{6.3}$$

for every $m \in \mathbb{N}^*$. In particular, if $|u_{0m} - v_0|_{0,2,U} = O(\gamma_m)$, as $m \to \infty$, then

$$|\mathbf{w}_m|_{0,T;0,2,U} = O(\gamma_m), \text{ as } m \to \infty.$$

Proof. Let $m \in \mathbb{N}^*$. Then

$$i\mathbf{w}_m' + \Delta \mathbf{w}_m + g(\mathbf{u}_m) - g(\mathbf{v}) + i\gamma_m \mathbf{u}_m \overset{H^{-1}(U)}{=} \mathbf{f}_m, \text{ a.e. in } [0,T]. \tag{6.4}$$

Applying (7.2) and dealing as usual we obtain

$$\begin{aligned} \frac{d}{dt}|\mathbf{w}_m|_{0,2,U}^2 \le C \int_U |\mathbf{w}_m|^2 (|\mathbf{u}_m|^\alpha + |\mathbf{v}|^\alpha) dx + |\mathbf{w}_m|_{0,2,U}^2 \\ + C\gamma_m^2 |\mathbf{u}_m|_{0,2,U}^2 + C|\mathbf{f}_m|_{1,T;0,2,U}^2, \end{aligned}$$

a.e. in $[0,T]$. From the embedding $H_0^1(U) \hookrightarrow L^\infty(U)$ we obtain (6.3) with

$$C_{11} = 1 + K_1(|v_0|_{1,2,U}) \text{ and } C_{12} = \frac{C}{C_{11}}(K_2(|v_0|_{1,2,U})\gamma_m^2 + |\mathbf{f}_m|_{1,T;0,2,U}^2),$$

for increasing, non-negative K_1 and K_2. □

7. Useful inequalities

We first mention two elementary inequalities.

Theorem 7.1. *Let* $p > 0$, $\alpha \ge 0$ *and* $z_1, z_2 \in \mathbb{C}$. *Then*

$$|z_1 + z_2|^p \le C(|z_1|^p + |z_2|^p), \tag{7.1}$$

$$||z_1|^\alpha z_1 - |z_2|^\alpha z_2| \le C|z_1 - z_2|(|z_1|^\alpha + |z_2|^\alpha). \tag{7.2}$$

We also mention the Young inequality with constant ϵ and the Hölder inequality.

Theorem 7.2. *Let $a, b \in [0,\infty)$ and $p, q \in (1,\infty)$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \le \epsilon a^p + Cb^q, \quad \forall \epsilon > 0, \quad \text{where } C = \frac{1}{(\epsilon p)^{\frac{q}{p}} q}. \tag{7.3}$$

Theorem 7.3. *Let $p, q \in [1,\infty]$, such that $\frac{1}{p} + \frac{1}{q} = 1$, $u \in L^p(U)$ and $v \in L^q(U)$. Then*

$$\int_U |uv| dx \le |u|_{0,p,U} |v|_{0,q,U}. \tag{7.4}$$

The following result is a version of the Gagliardo-Nirenberg interpolation inequality (see [9]).

Theorem 7.4. *Let $q, r \in [1, \infty]$ and $j, m \in \mathbb{N}_0$ such that $j < m$. Then*

$$\sum_{|\beta|=j} |D^\beta u|_{0,p} \le C\Big(\sum_{|\beta|=m} |D^\beta u|_{0,r}\Big)^\theta |u|_{0,q}^{1-\theta}, \quad \forall u \in C_c^m(\mathbb{R}^n), \tag{7.5}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta\big(\frac{1}{r} - \frac{m}{n}\big) + (1-\theta)\frac{1}{q}, \quad \forall \theta \in [\frac{j}{m}, 1],$$

where C is a constant depending only on n, m, j, q, r and θ.

There is an exception: If $r > 1$ and $m - j - \frac{n}{r} \in \mathbb{N}_0$, then (7.5) *holds only for all $\theta \in [\frac{j}{m}, 1)$.*

Remark 7.5. The following Sobolev embeddings are true (see [8])

$$\begin{gathered} W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \text{ with } mp < n, \\ W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad \text{where } q \in [p, \infty) \text{ with } mp = n, \\ W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n), \quad \text{with } mp > n. \end{gathered}$$

It is then easy to see that the following embeddings

$$\begin{gathered} W_0^{m,p}(U) \hookrightarrow L^q(U), \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \text{ with } mp < n, \\ W_0^{m,p}(U) \hookrightarrow L^q(U), \quad \text{where } q \in [p, \infty) \text{ with } mp = n, \\ W_0^{m,p}(U) \hookrightarrow L^\infty(U), \quad \text{with } mp > n \end{gathered}$$

are also true for every $U \subseteq \mathbb{R}^n$. These embeddings are, additionally, scaling invariant, since, for every inequality of the corresponding embedding, we have $C_U = C_{\mathbb{R}^n} = C$ for every $U \subseteq \mathbb{R}^n$. Indeed, we only have to notice that

$$\mathcal{E}C_c^m(U) \subset C_c^m(\mathbb{R}^n) \text{ and } |D^\beta u|_{0,p,U} = |D^\beta \mathcal{E}u|_{0,p},$$

for every $u \in C_c^m(U)$, every multi-index β such that $0 \le |\beta| \le m$, and every $p \in [1, \infty]$ (see also [1]). Using the above arguments, we see that Theorem 7.4 is also true for every $u \in W_0^{m,p}(U)$ and also (7.5) is scaling invariant in the aforementioned space.

We note that the embeddings

$$\begin{gathered} W^{m,p}(U) \hookrightarrow L^q(U), \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \text{ with } mp < n, \\ W^{m,p}(U) \hookrightarrow L^q(U), \quad \text{where } q \in [p, \infty) \text{ with } mp = n, \\ W^{m,p}(U) \hookrightarrow L^\infty(U), \quad \text{with } mp > n, \end{gathered}$$

are true for appropriate choices of $U \subseteq \mathbb{R}^n$. Possible such choices are: (i) $\mathbb{R}^n_+$, (ii) any U that satisfies the cone condition, (iii) any bounded U with a locally Lipschitz boundary, (iv) any Lipschitz domain, etc. (see [8, 1, 19] for definitions and more examples/counterexamples). Evidently, these embeddings and the corresponding inequalities depend on the choice of U. Moreover, for the above special cases of $U \subseteq \mathbb{R}^n$, the (compact) Rellich-Kondrachov embeddings

$$\begin{gathered} W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U), \quad \text{where } q \in [1, p^*) \text{ and } \frac{1}{q^*} = \frac{1}{p} - \frac{1}{n} \text{ with } p < n, \\ W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U), \quad \text{where } q \in [p, \infty) \text{ with } p = n, \\ W^{1,p}(U) \hookrightarrow\hookrightarrow C(\overline{U}), \quad \text{with } p > n, \end{gathered}$$

are true if, in addition, U is bounded. On the contrary, if we replace $W^{1,p}(U)$ with $W_0^{1,p}(U)$, there is no restriction on the choice of U, except for being bounded. The latter follows from the fact that we only need the aforementioned continuous embeddings and the boundedness of U, in order to prove the compact ones.

In particular, applying the above remark, we modify a well-known result from [25] (see also [9, 22]).

Theorem 7.6. *Let $\alpha = \frac{4}{n}$ and $R \in H^1(\mathbb{R}^n)$ be the spherically symmetric, positive ground state of the elliptic equation $-\Delta R + R = |R|^\alpha R$, in $H^{-1}(\mathbb{R}^n)$. Then, the best constant C in*

$$|u|_{0,\alpha+2,U}^{\alpha+2} \le C|Du|_{0,2,U}^2 |u|_{0,2,U}^\alpha, \quad \forall u \in H_0^1(U), \text{ for any open } U \subseteq \mathbb{R}^n \tag{7.6}$$

is $C = C_{cr} := \frac{\alpha+2}{2|R|_{0,2}^\alpha}$.

8. Cut-off functions

If $\delta > 0$, we set $U^\delta \supset \overline{U}$ for

$$U^\delta := U \cup \cup_{x \in \partial U} B(x,\delta).$$

Proposition 8.1. *Let U and $\delta > 0$. Then there exists $\phi \in C_c^\infty(\mathbb{R}^n;[0,1])$ such that*

1. *$\operatorname{supp}(\phi) \subseteq \overline{U^\delta}$,*
2. *$\phi \equiv 1$ in $\overline{U}$, and*
3. *$\|\nabla^k \phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{C_k}{\delta^k}$, for every $k \in \mathbb{N}_0$ ($C_0 = 1$).*

Proof. We consider $\phi = \varphi_\delta * \chi_U$, i.e.

$$\phi(x) = \int_{\mathbb{R}^n} \varphi_\delta(x-y)\chi_U(y)dy = \int_{B(x,\delta)} \varphi_\delta(x-y)\chi_U(y)dy, \quad \forall x \in \mathbb{R}^n,$$

where φ_δ stands for the standard mollifier with $\operatorname{supp}(\varphi) \subseteq \overline{B(0,\delta)}$ and also χ_U for the characteristic function of U. It is well known that $\phi \in C^\infty(\mathbb{R}^n)$ with $D^\alpha \phi = D^\alpha \varphi_\delta * \chi_U$, for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$. If $x \in \overline{U}$, then $B(x,\delta) \subset U$, thus

$$\phi(x) = \int_{B(x,\delta)} \varphi_\delta(x-y)dy = 1, \quad \forall x \in \overline{U}.$$

Similarly we can obtain $\phi(x) \in [0,1]$ for every $x \in \mathbb{R}^n$, since the same is true for χ_U. If $x \in \overline{U^\delta}^{\mathsf{c}}$, then $B(x,\delta) \cap U = \varnothing$, thus $\phi(x) = 0$ for every such x and so $\operatorname{supp}(\phi) \subseteq \overline{U^\delta}$. Lastly, from the Faá di Bruno formula, we have

$$|D^\alpha \phi(x)| \le \int_{\mathbb{R}^n} |D^\alpha \varphi_\delta(x-y)||\chi_U(y)|dy \le \|\nabla^{|\alpha|}\varphi_\delta\|_{L^1(\mathbb{R}^n)} \le \frac{C_{|\alpha|}}{\delta^{|\alpha|}}, \quad \forall \alpha \in \mathbb{N}_0^n. \ \square$$

If $B_\varrho(x_0) \subset \mathbb{R}^n$ fixed and $\{a_k\}_{k=1}^\infty \subset \mathbb{R}_+$ increasing, such that $a_k > \varrho$ for all $k \in \mathbb{N}^*$ and $a_k \nearrow \infty$, we can obtain $\{\eta_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\eta_k(x) = \begin{cases} 1, & x \in \overline{B_{a_{k-1}}(x_0)} \\ 0, & x \in B_{a_k}(x_0)^{\mathrm{T}}, \end{cases} \quad \forall k \in \mathbb{N}^* \setminus \{1\} \quad \text{and} \quad \eta_1(x) = \begin{cases} 1, & x \in \overline{B_\varrho(x_0)} \\ 0, & x \in B_{a_1}(x_0)^{\mathrm{T}}. \end{cases}$$

In view of the above result, if, in addition, $a_{k+1} - a_k = a_1 - \varrho = C$ uniformly for all $k \in \mathbb{N}^*$ (i.e. C is independent of k), then $|D^\beta \eta_k|_{0,\infty} \le C_m$, for some

$\{C_m\}_{m=0}^{\infty} \subset \mathbb{R}_+$, uniformly for all $k \in \mathbb{N}^*$ and every multi-index β such that $|\beta| = m$. In particular, $C_0 = 1$. In fact, if $f \in C^\infty(\mathbb{R})$ with

$$f(t) := \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0, \end{cases}$$

then we can directly construct such a sequence as follows

$$\eta_k(x; x_0, a_{k-1}, a_k) := \frac{f(a_k - |x - x_0|)}{f(|x - x_0| - a_{k-1}) + f(a_k - |x - x_0|)},$$

for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{N}^* \setminus \{1\}$, and

$$\eta_1(x; B_\varrho(x_0), a_1) := \frac{f(a_1 - |x - x_0|)}{f(|x - x_0| - \varrho) + f(a_1 - |x - x_0|)}, \quad \forall x \in \mathbb{R}^n.$$

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Nikolaos Gialelis
Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-157 84, Athens, Greece
Email address: ngialelis@math.uoa.gr