On the 1-dim Defocusing NLS Equation with Non-vanishing Initial Data at Infinity

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Abstract We show global well-posedness of certain type of strong-in-time and weak-in-space solutions for the Cauchy problem of the 1-dimensional nonlinear Schrödinger equation, in various cases of open sets, bounded and unbounded. These solutions do not vanish at the boundary or at infinity.

Keywords NLS equation · Cauchy problem · Strong-in-time solutions · Non-vanishing solutions · Global well-posedness · Zhidkov space

1 Introduction

The 1-dimensional nonlinear Schrödinger equation (NLS) emerges as a first order model in a variety of fields—from high intensity laser beam propagation, to Bose-Einstein condensation, to water waves theory, etc. The NLS is completely integrable, hence solvable, in one dimension on the infinite line, or with periodic boundary conditions.

In this work we consider the one-dimensional defocusing NLS equation

$$i v_t + v_{xx} - |v|^{\alpha} v = 0, \ \forall\, (t, x) \in J^* \times U, \tag{1.1}$$

where $v : J \times \overline{U} \to \mathbb{C}$, with J an interval $\subseteq \mathbb{R}$ such that $0 \in J$, U an open set $\subseteq \mathbb{R}$ and $\alpha > 0$.

We are interested in solutions with a prescribed initial condition on $\{t = 0\} \times \overline{U}$. Moreover, v is either not necessarily equal to zero on $J \times \partial U$, or not necessarily such

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that $\lim_{|x|\to\infty} v = 0$ on $J \times \overline{U}$. In the case that U is unbounded we assume that v has a constant amplitude at infinity.

Since we are interested in all possible cases of open sets, U could be bounded (e.g., a ball) or unbounded—with or without empty boundary (e.g., $\mathbb{R}_+$ or $\mathbb{R}$, respectively). Let us recall that when $U = \mathbb{R}$, the existence of many such solutions is well-known, e.g. the black soliton $v(t,x) = e^{-it}\tanh\left(2^{-\frac{1}{2}}x\right)$, for $\alpha = 2$. In the present work we look for solutions of the form $v(t,x) = e^{irt}\,(u(t,x) + \zeta(x))$, for $r \in \mathbb{R}$ and u, ζ complex-valued functions over $J \times \overline{U}$ and $\overline{U}$, respectively, such that u vanishes at the boundary and at infinity, but ζ, in contrast, survives. The arising problem then becomes

$$\begin{cases} iu_t + (u+\zeta)_{xx} - (|u+\zeta|^\alpha + r)\,(u+\zeta) = 0, \ \forall\,(t,x) \in J^* \times U \\ u = u_0, \ \text{on } \{t=0\} \times \overline{U} \\ u = 0, \ \text{on } J \times \partial U \text{ and } u \xrightarrow{|x|\to\infty} 0, \ \text{on } J \times \overline{U}, \end{cases} \tag{1.2}$$

for given r, ζ and also $u_0 : \overline{U} \to \mathbb{C}$ which vanishes at the boundary and at infinity.

The problem (1.2) for $U = \mathbb{R}$ and

$$\alpha = 2\tau \text{ for } \tau \in \mathbb{N} := \{1, 2, \dots\} \text{ and } r = -\rho^\tau \text{ for } \rho > 0, \tag{1.3}$$

has been studied in [8]. There it is stated that if $\zeta \in C_b^2(\mathbb{R})$, $D\zeta \in H^2(\mathbb{R})$, and additionally $\left(|\zeta|^2 - \rho\right) \in L^2(\mathbb{R})$, then (1.2) is globally well-posed in $H^1(\mathbb{R})$ and the energy of the solution is conserved.

Recently, in [10], the above result is extended not only by weakening the assumptions on ζ but also by considering more general cases of $U \subseteq \mathbb{R}$, other than the Euclidean space itself. Namely, it is shown that the problem (1.2) is globally and uniquely solvable in $H_0^1(U)$ for any open $U \subseteq \mathbb{R}$, if $\zeta \in X^1(U)$, the Zhidkov space over U (see the notations below) and additionally if α, r are as in (1.3); for unbounded U it is further assumed that $\left(|\zeta|^2 - \rho\right) \in L^2(U)$.

In this work we introduce sufficient conditions on ζ that establish the continuous dependence on the initial data, as well as the conservation of energy. We show that in bounded sets there is no need for extra assumptions on ζ. On the other hand, in unbounded sets the assumptions on ζ are stronger than the ones for the bounded case, yet still, they remain weaker than the ones in [8] for the case $U = \mathbb{R}$ and they ascertain the rigorous proof of the well-posedness of the problem.

The present paper is organized as follows: the problem is formulated for two different sets of assumptions in Sect. 2, where the necessary notation is also introduced. The well-posedness of the problem for bounded U—with "minimal" assumptions on ζ—is treated in Sect. 3. We note that both the strong H_0^1-regularity and the continuous dependence of the solution on the initial data require the conservation of energy; the latter is established without any additional assumption in the case of bounded U. However, as shown in Sect. 4, for unbounded U the energy is proved to be conserved under stronger assumptions on ζ. The underlying reason for this, is that in the case of unbounded U the backward-in-time existence is not guaranteed by the "minimal" assumptions of Sect. 3; to surpass this obstacle we employ an approximation by regular solutions for which the energy is actually conserved.

2 Preliminaries

We start with some notation used throughout the paper:

1. J denotes any bounded interval such that $0 \in J$, $J_\pm := J \cap \mathbb{R}_\pm$ and U for any open $\subseteq \mathbb{R}$.
2. If $p, q, r \in [1, \infty]$ and $k, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, we write

$$|\cdot|_{m,r,U} := \|\cdot\|_{W^{m,r}(U)}, \ |\cdot|_{-m,U} := \|\cdot\|_{H^{-m}(U)}$$
$$|\cdot|_{k,p,J;m,r,U} := \|\cdot\|_{W^{k,p}(J;W^{m,r}(U))}, \ |\cdot|_{K,p,J;-m,U} := \|\cdot\|_{W^{k,p}(J;H^{-m}(U))}.$$

   We omit $p = \infty$, $J = \mathbb{R}$ and $U = \mathbb{R}$ from the notation.
3. If $m \in \mathbb{N}$, then $X^m(U)$ stands for the Zhidkov space over U, defined as

$$X^m(U) := \left\{ u \in L^\infty(U) \;\middle|\; D^k u \in L^2(U), \ \forall k \in \{1, \ldots, m\} \right\}$$

   and equipped with its natural norm $\|\cdot\|_{X^m(U)} := |\cdot|_{0,\infty,U} + \sum_{k=1}^m \left|D^k \cdot\right|_{0,2,U}$. The first version of such spaces over $\mathbb{R}$ is introduced in [13] and a generalization for higher dimensions (along with certain modifications) is done in [7–9, 14]. In this work, however, we consider X^m over any open set.
4. Let $\mathcal{F}(U_1; \mathbb{C})$ be a function space over $U_1 \subset U_2 \subseteq \mathbb{R}$ and $f \in \mathcal{F}(U_1)$. We denote by $\mathcal{E}_{U_2} f$ its extension by zero in $U_2 \backslash U_1$ and $\mathcal{E}_{U_2}\mathcal{F}(U_1) := \left\{ \mathcal{E}_{U_2} f \;\middle|\; f \in \mathcal{F}(U_1) \right\}$. We omit $U_2 = \mathbb{R}$ from these notations. Moreover, if $g \in \mathcal{F}(U_2)$, we denote by $\mathcal{R}_{U_1} g$ and $\mathcal{R}_{U_1}\mathcal{F}(U_2)$ the restriction of g in U_1 and the set of these restricted functions, respectively.
5. We write C and c for any non-negative constant factor and exponent, respectively. These constants may be explicitly calculated in terms of known quantities and may change from line to line and also within a certain line in a given computation. We also employ the letter K for any increasing function $K : [0, \infty)^n \to [0, \infty)$. When J and U appear as subscripts in an element, they denote that this depends on them, while their absence designates independence.
6. If $u : J \times U \to \mathbb{C}$, with $u(t, \cdot) \in \mathcal{F}(U)$ for each $t \in J$, then, following the notation of, e.g., [6, 11], we associate with u the mapping $\mathbf{u} : J \to \mathcal{F}(U; \mathbb{C})$, defined by $[\mathbf{u}(t)](x) := u(t, x)$, for every $x \in U$ and $t \in J$.

Next, recall Hölder's inequality: let $U \subseteq \mathbb{R}^n$, $m \in \mathbb{N} \backslash \{1\}$, $\{p_k\}_{k=1}^m \subset [1, \infty]$, such that $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$ and $u_k \in L^{p_k}(U)$ for $k = 1, \ldots, m$. Then

$$\int_U |u_1 \cdots u_m| \, dx \leq \prod_{k=1}^m |u_k|_{0,p_k,U}. \tag{2.1}$$

From (2.1) for $p_1 = \frac{\alpha+2}{\alpha+1}$ and $p_2 = \alpha+2$, $\alpha \geq 0$, it clearly holds that

$$\int_U |u|^{\alpha+1} \, |v| \, dx \leq |u|^c_{0,\alpha+2,U} |v|_{0,\alpha+2,U}, \tag{2.2}$$

for $u,\, v \in L^{\alpha+2}(U)$.

In the sequel, we assume that $\alpha > 0$, $\zeta \in L^{\alpha+2}(U)$ and $r \in \mathbb{R}$.
From (2.2) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$, we define $g : H_0^1(U) \to Y_\alpha := L^{\frac{\alpha+2}{\alpha+1}}(U) + L^2(U) \hookrightarrow H^{-1}(U)$ to be the nonlinear and bounded operator such that

$$\langle g(u;\, \alpha, \zeta, r)\, ,\, v \rangle := \int_U (|u+\zeta|^\alpha + r)\, (u+\zeta)\, \bar{v} dx, \text{ for } v \in H_0^1(U)\, .$$

For the above operator we have the following estimate.

Proposition 1 *Let $u,\, v \in H_0^1(U)$. Then*

$$\begin{aligned} \|g(u) - g(v)\|_{Y_\alpha} \leq K \left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{0,\alpha+2,U} \right) \times \\ \times \left(|u-v|_{0,\alpha+2,U} + |u-v|_{0,2,U} \right). \end{aligned} \tag{2.3}$$

Proof For $u,\, v \in L^{\alpha+2}(U)$, we have

$$\left| |u|^\alpha u - |v|^\alpha v \right|_{0, \frac{\alpha+2}{\alpha+1}, U} \leq C \left(|u|^c_{0,\alpha+2,U} + |v|^c_{0,\alpha+2,U} \right) |u-v|_{0,\alpha+2,U}. \tag{2.4}$$

This inequality follows by direct application of

$$\left| |z_1|^\alpha z_1 - |z_2|^\alpha z_2 \right| \leq C \, |z_1 - z_2| \, (|z_1|^\alpha + |z_2|^\alpha)\, , \ z_1, z_2 \in \mathbb{C}, \tag{2.5}$$

(2.1) for $p_1 = \alpha+1$ and $p_2 = \frac{\alpha+1}{\alpha}$, as well as $|z_1+z_2|^p \leq C\, (|z_1|^p + |z_2|^p)$, $p > 0$. From (2.4) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$ we get

$$\begin{aligned} \|g(u) - g(v)\|_{Y_\alpha} \leq C \left(|u|^c_{1,2,U} + |v|^c_{1,2,U} + |\zeta|^c_{0,\alpha+2,U} \right) |u-v|_{0,\alpha+2,U} + \\ + C|u-v|_{0,2,U} \end{aligned}$$

and the result follows.

Now, we further assume that $\zeta \in H^1(U)$ and we define $\mathcal{N}[\cdot, \cdot] : \left(H_0^1(U)\right)^2 \to \mathbb{C}$ to be the form which is associated with the operator $D^2(\cdot + \zeta) - g$, such that $\mathcal{N}[u, v] := \left\langle D^2(u+\zeta)\, ,\, v \right\rangle - \langle g(u)\, ,\, v \rangle$, for every $u,\, v \in H_0^1(U)$.

We then restate the problem (1.2): we seek a solution $\mathbf{u}_J \in L^\infty\left(J;\, H_0^1(U)\right) \cap W^{1,\infty}\left(J;\, H^{-1}(U)\right)$ of

$$\begin{cases} i \left\langle \mathbf{u}_J', v \right\rangle + \mathcal{N}[\mathbf{u}_J, v] = 0, \ \forall v \in H_0^1(U)\, , \text{ a.e. in } J \\ \mathbf{u}_J(0) = u_0. \end{cases} \tag{2.6}$$

We also provide an estimate for the form $\mathcal{N}$.

Proposition 2 *Let* $u, v \in H_0^1(U)$. *Then*

$$|\mathcal{N}[u, v]| \le K\left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha+2,U}\right). \quad (2.7)$$

Proof From (2.1) ($p_1 = p_2 = 2$), (2.2) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$, we get $|\mathcal{N}[u, v]| \le C|D(u+\zeta)|_{0,2,U}|v|_{1,2,U} + C|u+\zeta|^c_{0,\alpha+2,U}|v|^c_{1,2,U}$, hence the result follows.

We further define the energy functional $E : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$E(\cdot; \alpha, \zeta, r) := \frac{1}{2}|D(\cdot+\zeta)|^2_{0,2,U} + G(\cdot; \alpha, \zeta, r),$$

where $G : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$, with

$$G(\cdot; \alpha, \zeta, r) := \int_U V(|\cdot+\zeta|; \alpha, r)\, dx,$$

where $V : [0, \infty) \to [0, \infty)$ is defined as

$$V(x; \alpha, r) := \frac{1}{\alpha+2}x^{\alpha+2} + \frac{1}{2}rx^2 + \frac{\alpha}{2(\alpha+2)}|r|^{\frac{\alpha+2}{\alpha}}. \quad (2.8)$$

It easily follows that for every constant $C_\alpha > \alpha+2$ there exists a constant $A_\alpha > 0$, such that

$$x^{\alpha+2} \le C_\alpha V(x), \ \forall x \ge A_\alpha. \quad (2.9)$$

For the functional G we have the following estimates.

Proposition 3 *Let* $u, v \in H_0^1(U)$. *If* $(G(u) - G(v)) \in \mathbb{R}$, *then*

$$\begin{aligned} |G(u) - G(v)| \le K\left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha+2,U}\right) \times \\ \times \left(|u-v|_{0,\alpha+2,U} + |u-v|_{0,2,U}\right) \end{aligned} \quad (2.10)$$

and

$$G(u) \le K\left(|u|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha+2,U}, |U|\right). \quad (2.11)$$

Proof From

$$\begin{aligned} G(u) - G(v) &= \int_0^1 \frac{d}{ds} G(su + (1-s)\, v)\, ds = \\ &= \int_0^1 \mathrm{Re}\langle g(su + (1-s)\, v), u - v\rangle\, ds, \end{aligned} \quad (2.12)$$

(2.2) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$ we get

$$|G(u)-G(v)| \leq C\left(|u|_{1,2,U}^c+|v|_{1,2,U}^c+|\zeta|_{0,\alpha+2,U}^c+|\zeta|_{1,2,U}^c\right)\times \\ \times\left(|u-v|_{0,\alpha+2,U}+|u-v|_{0,2,U}\right).$$

As for (2.11), we notice that

$$G(0)=\int_U V(|\zeta|)\,dx=\frac{1}{\alpha+2}|\zeta|_{0,\alpha+2,U}^{\alpha+2}+\frac{1}{2}r|\zeta|_{0,2,U}^2+\frac{\alpha}{2(\alpha+2)}|r|^{\frac{\alpha+2}{\alpha}}|U| \leq \\ \leq K\left(|\zeta|_{1,2,U},|\zeta|_{0,\alpha+2,U},|U|\right).$$

Then the result follows from (2.10) and the triangle inequality.

Let us now assume that $\zeta \in L^{\alpha+2}(U)\cap L^{\infty}(U)$. Two fine properties concerning the operator g follow.

Proposition 4 *Let $u,\, v \in H_0^1(U)$. Then $(g(u)-g(v)) \in L^2(U)$ with*

$$|g(u)-g(v)|_{0,2,U} \leq K\left(|u|_{1,2,U},|v|_{1,2,U},|\zeta|_{0,\infty,U}\right)|u-v|_{0,2,U}. \tag{2.13}$$

Proof By simple application of (2.5), we get

$$\int_U |g(u)-g(v)|^2 dx \leq \int_U \left(|u|^{2\alpha}+|v|^{2\alpha}\right)|u-v|^2 dx+ \\ +\left(|\zeta|_{0,\infty,U}^c+C\right)|u-v|_{0,2,U}^2.$$

We then employ the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\infty}(U)$.

Proposition 5 *Let $u,\, v \in H_0^1(U)$. If either $|U|<\infty$, or $|U|=\infty$, α and r be as in (1.3), as well as $\left(|\zeta|^2-\rho\right) \in L^2(U)$, then g maps to $L^2(U)$ and*

$$|g(u)|_{0,2,U} \leq \begin{cases} K_U\left(|u|_{1,2,U},|\zeta|_{0,\infty,U}\right), & \text{if } |U|<\infty \\ K\left(|u|_{1,2,U},|\zeta|_{0,\infty,U},\left||\zeta|^2-\rho\right|_{0,2,U}\right), & \text{otherwise.} \end{cases} \tag{2.14}$$

Proof We notice that $g(0)=(|\zeta|^{\alpha}+r)\zeta$, which belongs to $L^2(U)$. Indeed, for $|U|<\infty$ this is straightforward. For $|U|=\infty$, by expanding via

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\right), \tag{2.15}$$

we get $|g(0)|_{0,2,U} \leq K\left(|\zeta|_{0,\infty,U},\left||\zeta|^2-\rho\right|_{0,2,U}\right)$. The result then follows from (2.13) and the triangle inequality.

Let us now notice that ζ being in $L^{\alpha+2}(U)$ plays no essential role at any of the above results. Hence, for

$$\alpha, r \text{ as in (1.3) and } \zeta \in L^{\infty}(U) \text{ with } \big(|\zeta|^2-\rho\big) \in L^2(U)\,,$$

we define $g_s : H_0^1(U) \to L^2(U)$ by

$$\langle g_s(u;\tau,\zeta,\rho)\,,v\rangle := \int_U \big(|u+\zeta|^{2\tau}-\rho^{\tau}\big)\,(u+\zeta)\,\bar{v}dx, \text{ for } v\in H_0^1(U)\,,$$

which satisfies the above estimates.

Now, we further assume that $\zeta \in X^1(U)$ and we define $\mathcal{N}_s[\cdot,\cdot] : \big(H_0^1(U)\big)^2 \to \mathbb{C}$ to be the form which is associated with the operator $D^2(\cdot+\zeta)-g_s$, such that $\mathcal{N}_s[u,v] := \big\langle D^2(u+\zeta)\,,v\big\rangle - \langle g_s(u)\,,v\rangle$, for every $u, v \in H_0^1(U)$. We note that apart from belonging to $\mathcal{L}\big(H^1(U)\,;H^{-1}(U)\big)$, $D^2 \in \mathcal{L}\big(X^1(U)\,;H^{-1}(U)\big)$ also, with its usual definition. Now, the problem (1.2) becomes: find a solution $\mathbf{u}_J \in L^{\infty}\big(J;H_0^1(U)\big) \cap W^{1,\infty}\big(J;H^{-1}(U)\big)$ of

$$\begin{cases} i\,\big\langle \mathbf{u}_J', v\big\rangle + \mathcal{N}_s[\mathbf{u}_J, v] = 0, \ \forall v \in H_0^1(U)\,, \text{ a.e. in } J \\ \mathbf{u}_J(0) = u_0. \end{cases} \tag{2.16}$$

From (2.14) and (2.1) (for $p_1 = p_2 = 2$), we derive the following estimate

$$|\mathcal{N}_s[u,v]| \le K\Big(|u|_{1,2,U}, |v|_{1,2,U}, \|\zeta\|_{X^1(U)}, \big||\zeta|^2-\rho\big|_{0,2,U}\Big) \tag{2.17}$$

for every $u, v \in H_0^1(U)$.

We also define the respective energy functional $E_s : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$E_s(\cdot;\tau,\zeta,\rho) := \frac{1}{2}|D(\cdot+\zeta)|_{0,2,U}^2 + G_s(\cdot;\tau,\zeta,\rho)\,,$$

where $G_s : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$, with

$$G_s(\cdot;\tau,\zeta,\rho) := \int_U V(|\cdot+\zeta|\,;\tau,\rho)\,dx,$$

for which we have

$$\begin{aligned} |G_s(u)-G_s(v)| \le K\Big(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{0,\infty,U}, \big||\zeta|^2-\rho\big|_{0,2,U}\Big) \times \\ \times |u-v|_{0,2,U}, \end{aligned} \tag{2.18}$$

from (2.12) and (2.14). Moreover, $G_s(0) < K\left(|\zeta|_{0,\infty,U},\, \left||\zeta|^2-\rho\right|_{0,2,U}\right)$, which is obtained easily from

$$a^{n+1} - a\,(n+1)\,b^n + nb^{n+1} = (a-b)^2\left(a^{n-1} + 2a^{n-2}b + \ldots + (n-1)ab^{n-2} + nb^{n-1}\right).$$

Hence, from (2.18) and the triangle inequality we get

$$G_s(u) \le K\left(|u|_{1,2,U},\, |\zeta|_{0,\infty,U},\, \left||\zeta|^2-\rho\right|_{0,2,U}\right), \tag{2.19}$$

for every $u \in H_0^1(U)$ and so $E_s, G_s : H_0^1(U) \to \mathbb{R}_+$.

We also need the following results.

Proposition 6 *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F}$ be a Banach space with the Radon-Nikodym property with respect to the Lebesgue measure in $(J, \mathscr{B}(J))$.*

1. *Let $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset L^{\infty}(J;\mathcal{H})$ and $\mathbf{u} : J \to \mathcal{H}$ with $\mathbf{u}_k(t) \rightharpoonup \mathbf{u}(t)$ in $\mathcal{H}$, for a.e. $t \in J$. If $\|\mathbf{u}_k\|_{L^{\infty}(J;\mathcal{H})} \le C$ uniformly for all $k \in \mathbb{N}$, then $\mathbf{u} \in L^{\infty}(J;\mathcal{H})$ with $\|\mathbf{u}\|_{L^{\infty}(J;\mathcal{H})} \le C$, where C is the same in both inequalities.*
2. *Let $\{\mathbf{u}_k\}_{k=1}^{\infty} \cup \{\mathbf{u}\} \subset L^{\infty}(J;\mathcal{F}^*)$ with $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^{\infty}(J;\mathcal{F}^*)$.[1] If $\|\mathbf{u}_k\|_{L^{\infty}(J;\mathcal{F}^*)} \le C$ uniformly for all $k \in \mathbb{N}$, then $\|\mathbf{u}\|_{L^{\infty}(J;\mathcal{F}^*)} \le C$, where C is the same in both inequalities.*
3. *Let $p \in [1,\infty)$ and $\{\mathbf{u}_k\}_{k=1}^{\infty} \cup \{\mathbf{u}\} \subset L^p(J;\mathcal{H})$ with $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $L^p(J;\mathcal{H})$. If $\|\mathbf{u}_k\|_{L^{\infty}(J;\mathcal{H})} \le C$ uniformly for all $k \in \mathbb{N}$, then $\|\mathbf{u}\|_{L^{\infty}(J;\mathcal{H})} \le C$, where C is the same in both inequalities.*

Proof 1. We derive that $\|\mathbf{u}(t)\|_{\mathcal{H}} \le C$, for a.e. $t \in J$, from the (sequentially) weak lower semi-continuity of the norm. Then, the result follows directly.

2. Let $v \in \mathcal{F}$ be such that $\|v\|_{\mathcal{F}} \le 1$ and set $\mathbf{v} : J \to \mathcal{F}$ the constant function with $\mathbf{v}(t) := v$, for all $t \in J$. We have

$$\int_s^{s+h} \langle \mathbf{u}_k, \mathbf{v}\rangle\, dt \le Ch, \text{ for every } s \in J^{\circ} \text{ and every sufficiently small } h > 0.$$

Letting $k \to \infty$, dividing both parts by h and then letting $h \to 0$, we get $\langle \mathbf{u}(s), v\rangle \le C$, for every $s \in J^{\circ}$. Since v arbitrary, the proof is complete.

3. We deal as in 2.

Lemma 1 *Let $\mathbf{u} : \overline{J} \to H^m(U)$ be such that $\mathbf{u} \in C\left(\overline{J}; L^2(U)\right)$. Then $\mathbf{u}$ is weakly continuous (as a function to $H^m(U)$).*

Proof Let $t_0 \in \overline{J}$ and $\{t_n\}_{n=1}^{\infty} \subset \overline{J}$ be such that $t_n \to t_0$. Let also $v \in C_c^{\infty}(U)$. Then

[1] That is, $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ in $\sigma\left(L^{\infty}(J;\mathcal{F}^*), L^1(J;\mathcal{F})\right)$. Note that $L^{\infty}(J;\mathcal{F}^*) \cong \left(L^1(J;\mathcal{F})\right)^*$ (see, e.g., [5] Theorem 1, Sect. IV.1).

$$\left|(\mathbf{u}(t_n)-\mathbf{u}(t_0)\,,\,v)_{H^m(U)}\right| = \left|\sum_{k=0}^{m}\int_U D^k(\mathbf{u}(t_n)-\mathbf{u}(t_0))\,D^k\bar{v}dx\right| =$$
$$= \left|\sum_{k=0}^{m}(-1)^k\int_U(\mathbf{u}(t_n)-\mathbf{u}(t_0))\,D^{2k}\bar{v}dx\right| \le$$
$$\overset{2.1}{\le} |\mathbf{u}(t_n)-\mathbf{u}(t_0)|_{0,2,U}\sum_{k=0}^{m}\left|D^{2k}v\right|_{0,2,U} \to 0,$$

hence, the result follows from the density argument and the fact that $t_0 \in \overline{J}$ is arbitrary.

Lemma 2 *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{u} : \overline{J} \to \mathcal{H}$ be weakly continuous. If $\|\mathbf{u}\|_{\mathcal{H}} \in C(\overline{J})$, then $\mathbf{u} \in C(\overline{J};\mathcal{H})$.*

Proof Let $t_0 \in \overline{J}$ and $\{t_n\}_{n=1}^{\infty} \subset \overline{J}$ be such that $t_n \to t_0$. Then

$$\|\mathbf{u}(t_n)-\mathbf{u}(t_0)\|_{\mathcal{H}}^2 = \|\mathbf{u}(t_n)\|_{\mathcal{H}}^2 - (\mathbf{u}(t_0)\,,\,\mathbf{u}(t_n))_{\mathcal{H}} -$$
$$- (\mathbf{u}(t_n)-\mathbf{u}(t_0)\,,\,\mathbf{u}(t_0))_{\mathcal{H}} \to 0,$$

hence, the result follows since $t_0 \in \overline{J}$ is arbitrary.

Lemma 3 *Let $U_1 \subset U_2 \subseteq \mathbb{R}$, $m \in \mathbb{N}_0$ and $\{u_k\}_{k=1}^{\infty} \cup \{u\} \subset H^m(U_2)$ such that $u_k \rightharpoonup u$ in $H^m(U_2)$. Then $\mathcal{R}_{U_1}u_k \rightharpoonup \mathcal{R}_{U_1}u$ in $H^m(U_1)$.*

Proof Let $v \in C_c^{\infty}(U_1)$. Then

$$\left(\mathcal{R}_{U_1}u_k - \mathcal{R}_{U_1}u,\, v\right)_{H^m(U_1)} = \sum_{k=0}^{m}\int_{U_1} D^k\left(\mathcal{R}_{U_1}u_k - \mathcal{R}_{U_1}u\right)\,D^k\bar{v}dx =$$
$$= \sum_{k=0}^{m}\int_{U_2} D^k(u_k-u)\,D^k\mathcal{E}_{U_2}\bar{v}dx = \left(u_k-u,\,\mathcal{E}_{U_2}v\right)_{H^m(U_2)} \to 0,$$

hence, the result follows from the density argument.

3 Solutions in Bounded Sets

In this section, we assume that $U \subset \mathbb{R}^n$ is bounded.

3.1 A General Result for $r \in \mathbb{R}$

Theorem 1 *Let $u_0 \in H_0^1(U)$. Then for every J, there exists a solution $\mathbf{u}_J \in L^{\infty}\left(J;\,H_0^1(U)\right) \cap W^{1,\infty}\left(J;\,H^{-1}(U)\right)$ of (2.6), such that*

$$|\mathbf{u}_J|_{0,J;1,2,U} + \left|\mathbf{u}_J'\right|_{0,J;-1,U} \leq \mathcal{K}, \tag{3.1}$$

where

$$\mathcal{K} := \begin{cases} K\left(|u_0|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha+2,U}\right), & \textit{if } r \geq 0 \\ K_U\left(|u_0|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha+2,U}\right), & \textit{if } r < 0 \end{cases}$$

and also

$$E(\mathbf{u}_J) \leq E(u_0), \text{ everywhere in } J. \tag{3.2}$$

Moreover, if u_0 and ζ are real-valued, then $\mathbf{u}_J(t) = \overline{\mathbf{u}_J(-t)}$, for all $t \in J$ with $|t| \leq \operatorname{dist}(0, \partial J)$.

Proof STEP 1: We make use of the standard Faedo-Galerkin method. It holds true that $H_0^1(U;\mathbb{C}) \hookrightarrow\hookrightarrow L^2(U;\mathbb{C})$, hence there exists a countable subset of $H_0^1(U;\mathbb{R}) \cap C^\infty\left(\overline{U};\mathbb{R}\right)$, which is an orthogonal basis of $L^2(U;\mathbb{C})$, e.g., the complete set of eigenfunctions for the operator $-D^2$ in $H_0^1(U;\mathbb{C})$.[2] Let $\{w_k\}_{k=1}^\infty \subset H_0^1(U;\mathbb{R}) \cap C^\infty\left(\overline{U};\mathbb{R}\right)$ be that basis, appropriately normalized so that $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2(U;\mathbb{C})$. Fixing any $m \in \mathbb{N}$, we define $\mathbf{d}_m \in C^\infty(J_m;\mathbb{C}^m)$, with $\mathbf{d}_m(t) := \left[d_m^1(t), \ldots, d_m^m(t)\right]^{\mathrm{T}}$, to be the unique maximal solution of the initial-value problem

$$\begin{cases} \mathbf{d}_m{}'(t) = F_m(\mathbf{d}_m(t)), \ \forall t \in J_m^* \\ \mathbf{d}_m(0) = [(u_0, w_1), \ldots, (u_0, w_m)]^{\mathrm{T}}, \end{cases}$$

where $F_m \in C^\infty\left(\mathbb{R}^{2m};\mathbb{C}^m\right)$ with

$$F_m^k(\mathbf{z}) := i\mathcal{N}[\sum_{l=1}^m z_l w_l, w_k], \text{ for all } \mathbf{z} \in \mathbb{C}^m, \text{ with } \mathbf{z} := [z_1, \ldots, z_m]^{\mathrm{T}},$$

for all $k \in \{1, \ldots, m\}$. Now, we define $\mathbf{u}_m \in C^\infty\left(J_m; H_0^1(U;\mathbb{C}) \cap C^\infty\left(\overline{U};\mathbb{C}\right)\right)$, with

$$\mathbf{u}_m(t) := \sum_{k=1}^m d_m^k(t)\, w_k.$$

It is then trivial to verify that

$$i\left(\mathbf{u}_m', w_k\right) + \mathcal{N}[\mathbf{u}_m, w_k] = 0, \text{ everywhere in } J_m, \tag{3.3}$$

[2]This specific subset is an orthogonal basis of both $H_0^1(U;\mathbb{C})$ and $L^2(U;\mathbb{C})$.

for all $k \in \{1, \dots, m\}$. Note that $u_{0m} := u_m(0, \cdot) = \mathbf{u}_m(0) \to u_0$ in $L^2(U)$ and $|u_{0m}|_{0,2,U} \leq |u_0|_{0,2,U}$. Furthermore, $|u_{0m}|_{1,2,U} \leq |u_0|_{1,2,U}$. Indeed, since $\sum_{k=1}^m a_k w_k \in \operatorname{span}\{w_k\}_{k=1}^m$ for some $\{a_k\}_{k=1}^m \subset \mathbb{C}$ we have that $\big(D^2 u_{0m}, u_{0m}\big) = \big(D^2 u_{0m}, u_0\big)$, hence we get

$$|Du_{0m}|_{0,2,U}^2 = -\big(D^2 u_{0m}, u_{0m}\big) = (Du_{0m}, Du_0) \leq \\ \leq \frac{1}{2}|Du_{0m}|_{0,2,U}^2 + \frac{1}{2}|Du_0|_{0,2,U}^2.$$

Therefore $|Du_{0m}|_{0,2,U} \leq |Du_0|_{0,2,U}$.

STEP 2: We multiply the variational equation in (3.3) by $-\overline{d_m^k}'(t)$, sum for $k = 1, \dots, m$, and take real parts of both sides, and thus obtain

$$\frac{d}{dt}E(\mathbf{u}_m) = 0, \text{ that is } E(\mathbf{u}_m) \leq E(u_0), \text{ everywhere in } J_m, \tag{3.4}$$

hence, if $r \geq 0$, we have that $|\mathbf{u}_m|_{1,2,U} \leq \mathcal{K}$ and thus $J_m \equiv \mathbb{R}$. Since $m \in \mathbb{N}$ is arbitrary, we get $|\mathbf{u}_m|_{1,2,U} \leq \mathcal{K}$, for all $m \in \mathbb{N}$. Hence, we conclude that $\{\mathbf{u}_m\}_{m=1}^\infty$ is uniformly bounded in $L^\infty\big(\mathbb{R}; H_0^1(U)\big)$, with

$$|\mathbf{u}_m|_{0;1,2,U} \leq \mathcal{K}, \ \forall m \in \mathbb{N}. \tag{3.5}$$

If $r < 0$, from (2.11) we have that $|D\mathbf{u}_m|_{0,2,U} \leq E(\mathbf{u}_m) \leq \mathcal{K}$ and thus $J_m \equiv \mathbb{R}$. Therefore, from the Poincaré inequality, we also get $|\mathbf{u}_m|_{0,2,U} \leq \mathcal{K}$ and thus (3.5) follows.

STEP 3: We fix an arbitrary $v \in H_0^1(U)$ with $|v|_{1,2,U} \leq 1$ and write $v = \mathcal{P}v \oplus (\mathcal{I} - \mathcal{P})v$, where $\mathcal{P}$ is the projection in $\operatorname{span}\{w_k\}_{k=1}^m$. Since $\mathbf{u}_m' \in \operatorname{span}\{w_k\}_{k=1}^m$ and $\mathcal{N}[h, g]$ is (conjugate) linear for g, from the variational equation in (3.3) we get that $\langle \mathbf{u}_m', v\rangle = \big(\mathbf{u}_m', v\big) = \big(\mathbf{u}_m', \mathcal{P}v\big) = i\mathcal{N}[\mathbf{u}_m, \mathcal{P}v]$. Applying (2.7) we derive $\big|\langle \mathbf{u}_m', v\rangle\big| \leq \mathcal{K}$. Hence $\big\{\mathbf{u}_m'\big\}_{m=1}^\infty$ is uniformly bounded in $L^\infty\big(\mathbb{R}; H^{-1}(U)\big)$, with

$$\big|\mathbf{u}_m'\big|_{0;-1,U} \leq \mathcal{K}, \ \forall m \in \mathbb{N}. \tag{3.6}$$

STEP 4α: We fix an arbitrary J. From (3.5), (3.6), Theorem 1.3.14 (i) in [4] and **Proposition** 6 1., there exist a subsequence $\big\{\mathbf{u}_{m_l}\big\}_{l=1}^\infty \subseteq \{\mathbf{u}_m\}_{m=1}^\infty$ and a function $\mathbf{u}_J \in L^\infty\big(J; H_0^1(U)\big) \cap W^{1,\infty}\big(J; H^{-1}(U)\big)$, such that

$$\mathbf{u}_{m_l}(t) \rightharpoonup \mathbf{u}_J(t) \text{ in } H_0^1(U), \text{ for every } t \in \overline{J} \text{ and also } |\mathbf{u}_J|_{0,J;1,2,U} \leq \mathcal{K}. \tag{3.7}$$

STEP 4β: $H^{-1}(U)$ is separable since $H_0^1(U)$ is separable, hence by the Dunford-Pettis theorem (see, e.g., [5], Theorem 1, Sect. III.3) we have $L^\infty\big(J; H^{-1}(U)\big) \cong \big(L^1\big(J; H_0^1(U)\big)\big)^*$. In virtue of the above, from (3.6), the Banach-Alaoglu-Bourbaki theorem (see, e.g., [3], Theorem 3.16) and **Proposition** 6 2., there exist a subsequence of $\big\{\mathbf{u}_{m_l}\big\}_{l=1}^\infty$, which we still denote as such and a function $\mathbf{h} \in L^\infty\big(J; H^{-1}(U)\big)$, such that

$$\mathbf{u}'_{m_l} \overset{*}{\rightharpoonup} \mathbf{h} \text{ in } L^\infty\big(J;\, H^{-1}(U)\big) \text{ and also } |\mathbf{h}|_{0,J;-1,U} \le \mathcal{K}. \tag{3.8}$$

From the convergence in (3.7), Lemma 1.1, Chap. 3 in [11], along with the Leibniz rule, we can derive that

$$\int_J \big(\mathbf{u}'_{m_l},\, \psi v\big)\, dt \to \int_J \big\langle \mathbf{u}'_J,\, \psi v\big\rangle\, dt,\ \forall\, \psi \in C^1_c(J^\circ)\,,\, v \in H^1_0(U)\,,$$

hence $\mathbf{h} \equiv \mathbf{u}'_J$.

STEP 5α: Since U is bounded, $H^1_0(U) \hookrightarrow\hookrightarrow L^2(U) \hookrightarrow H^{-1}(U)$. Hence, from (3.5), (3.6) and the Aubin–Lions–Simon lemma (see [2], Theorem II.5.16), there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$, which we still denote as such and a function $\mathbf{y} \in C\big(\bar{J};\, L^2(U)\big)$, such that

$$\mathbf{u}_{m_l} \to \mathbf{y} \text{ in } C\big(\bar{J};\, L^2(U)\big)\,. \tag{3.9}$$

From the convergence in (3.7), we deduce that $\mathbf{y} \equiv \mathbf{u}_J$.

STEP 5β: From (3.5), (3.9) and the Gagliardo–Nirenberg inequality (see, e.g., Theorem 1.3.7 in [4]) $|u|_{0,\alpha+2,U} \le C\,|Du|^c_{0,2,U}\,|u|^c_{0,2,U}$, we have

$$\mathbf{u}_{m_l} \to \mathbf{u}_J \text{ in } C\big(\bar{J};\, L^{\alpha+2}(U)\big)\,. \tag{3.10}$$

STEP 5γ: From (2.3), (3.5), the bound in (3.7), (3.9) and (3.10) we get

$$g\big(\mathbf{u}_{m_l}\big) \to g(\mathbf{u}_J) \text{ in } C\big(\bar{J};\, Y_\alpha\big)\,. \tag{3.11}$$

STEP 6α: Let now $\psi \in C^\infty_c(J^\circ)$ and fix $N \in \mathbb{N}$. We choose m_l such that $N \le m_l$ and $v \in \operatorname{span}\{w_k\}_{k=1}^N$, hence, by the linearity of the inner product, we get from (3.3) that

$$\int_J i\,\big(\mathbf{u}'_{m_l},\, \psi v\big) + \mathcal{N}\big[\mathbf{u}_{m_l},\, \psi v\big]\, dt = 0.$$

We then pass to the weak, $*$-weak and strong limits (since $\psi v \in L^1\big(J;\, H^1_0(U)\big)$) to get

$$\int_J i\,\langle \mathbf{u}'_J,\, \psi v\rangle + \mathcal{N}[\mathbf{u}_J,\, \psi v]\, dt = 0.$$

Since ψ is arbitrary, $\mathbf{u}_J$ satisfies the variational equation in (2.6) for every $v \in \operatorname{span}\{w_k\}_{k=1}^N$. By the linear and continuous dependence on v, we get the desired result, after letting $N \to \infty$.

STEP 6β: For the initial condition, we fix an arbitrary $t_0 \in J^*$. Let $v \in H^1_0(\mathrm{U})$ be arbitrary and $\phi \in C^1\big(\bar{J}\big)$ such that $\phi(0) \ne 0$ and $\phi(t_0) = 0$. We then have from [11], Lemma 1.1, Chap. 3, along with the Leibniz rule, that

$$\int_0^{t_0} \left(\mathbf{u}'_{m_l}, \phi v\right) dt = -\int_0^{t_0} \left(\mathbf{u}_{m_l}, \phi' v\right) dt - \left(u_{0m_l}, \phi(0)\, v\right),$$
$$\int_0^{t_0} \left\langle \mathbf{u}'_J, \phi v \right\rangle dt = -\int_0^{t_0} \left(\mathbf{u}_J, \phi' v\right) dt - \left(\mathbf{u}_J(0), \phi(0)\, v\right).$$

Passing to the $*$-weak limits in the first equality, using that $u_{0m} \to u_0$ in $L^2(U)$ and the fact that $v \in H_0^1(U)$ is arbitrary, we derive that $\mathbf{u}_J(0) = u_0$.

STEP 7α: Now, for (3.2), we first derive from (2.10), (3.9) and (3.10) that $G\left(\mathbf{u}_{m_l}\right) \to G(\mathbf{u}_J)$, everywhere in J. On the other hand, from the convergence in (3.7), (3.9), the fact that if $\mathbf{u}_{m_l} \rightharpoonup \mathbf{u}_J$ in $H^1(U)$ and $\mathbf{u}_{m_l} \rightharpoonup \mathbf{u}_J$ in $L^2(U)$ then $D\mathbf{u}_{m_l} \rightharpoonup \mathbf{u}_J$ in $L^2(U)$, as well as the weak lower semi-continuity of the L^2-norm we get $\left|D\mathbf{u}_{m_l}\right|_{0,2,U} \leq |D\mathbf{u}_J|_{0,2,U}$, everywhere in J. Combining these two results, we get (3.2) from (3.4).

STEP 7βFinally, if ζ is real-valued, then $\overline{F_m(\mathbf{z})} = F_m(\bar{\mathbf{z}})$, for all $\mathbf{z} \in \mathbb{C}^m$ and if u_0 is real-valued, then $\overline{\mathbf{d}_m(0)} \in \mathbb{R}^m$. Hence, under these two assumptions, it easily follows that $\mathbf{u}_m(t) = \overline{\mathbf{u}_m(-t)}$, for all $t \in \mathbb{R}$. Now, the symmetry $\mathbf{u}_J(t) = \overline{\mathbf{u}_J(-t)}$, for all $t \in J$ with $|t| \leq \operatorname{dist}(0, \partial J)$, follows directly from the respective symmetry $\mathbf{u}_m(t) = \overline{\mathbf{u}_m(-t)}$ for all $t \in \mathbb{R}$, $m \in \mathbb{N}$ and the convergence in (3.7).

3.2 Uniqueness and Globality

It is obvious that the uniqueness of the extracted local solutions implies the "globality" of those solutions.

Proposition 7 *The solution* $\mathbf{u}_J$ *of* ***Theorem*** *1 is unique everywhere in* J.

Proof Let $u_{0,1} = u_{0,2}$ and $\mathbf{u}_{J,1}$, $\mathbf{u}_{J,2}$ be the corresponding solutions. Setting $\mathbf{w} := \mathbf{u}_{J,1} - \mathbf{u}_{J,2}$, we have

$$i\mathbf{w}' + \Delta\mathbf{w} - \left(g\left(\mathbf{u}_{J,1}\right) - g\left(\mathbf{u}_{J,2}\right)\right) \overset{H^{-1}(U)}{=} 0, \text{ a.e. in } J. \tag{3.12}$$

We apply the functional of (3.12) on $\mathbf{w}(t)$, for arbitrary $t \in J$ and take the imaginary parts of both parts to get us

$$|\mathbf{w}|^2_{0,2,U} \leq C \left| \int_0^t \left| \left\langle g\left(\mathbf{u}_{J,1}\right) - g\left(\mathbf{u}_{J,2}\right), \mathbf{w} \right\rangle \right| ds \right|, \forall t \in J.$$

Since $H^1(U) \hookrightarrow L^\infty(U)$, from (2.13) we deduce that

$$|\mathbf{w}|^2_{0,2,U} \leq C \left| \int_0^t |\mathbf{w}|^2_{0,2,U} ds \right| \leq C|t| \|\mathbf{w}|^2_{0,[-t,t];0,2,U},$$

hence

$$|\mathbf{w}|^2_{0,[-t,t];0,2,U} \le C|t||\mathbf{w}|^2_{0,[-t,t];0,2,U}.$$

For $|t|$ sufficiently small we have $\mathbf{w}\equiv 0$. Now, we show that $\mathbf{w}\equiv 0$ in J_+ and in an analogous fashion we can have that $\mathbf{w}\equiv 0$ in J_-. Since $\mathbf{w}\in C\big(\overline{J},\, L^2(U)\big)$, we set

$$s := \sup\big\{t_0 \in J \;\big|\; \mathbf{w}=0 \text{ for all } t\in[0,\, t_0]\big\}.$$

If $s \neq \sup J$, then there exists $\delta>0$ such that $[s,\, s+\delta]\subset J$. Then the continuity of $\mathbf{w}$ implies $\mathbf{w}(s)=0$ and, by dealing as above, we deduce that $\mathbf{w}=0$ for "a little further" than s, which is a contradiction to the definition of s.

3.3 Conservation of Energy and Well-Posedness

Here, we utilize the existence backwards in time as well as the uniqueness of the solution, in order to complete the puzzle of the well-posedness of the problem. First, we show the following result.

Proposition 8 *The energy of the unique solution* $\mathbf{u}_J$ *of* ***Theorem*** *1 is conserved, that is*

$$E(\mathbf{u}_J)=E(u_0)\,,\ \textit{everywhere in } J. \tag{3.13}$$

Proof We show that the energy is conserved in J_+ and in an analogous fashion we can get conservation of the energy in J_-. Let $t_0\in J_+$. We set $\widehat{J}:=[-t_0,\,0]$ and we define as $\mathbf{v}_{\widehat{J}}\in L^\infty\big(\widehat{J};\, H_0^1(U)\big)\cap W^{1,\infty}\big(\widehat{J};\, H^{-1}(U)\big)$ a solution of

$$\begin{cases} i\left\langle \mathbf{v}'_{\widehat{J}},\, v\right\rangle + \mathcal{N}\big[\mathbf{v}_{\widehat{J}},\, v\big]=0,\ \ \forall v\in H_0^1(U)\,,\ \text{a.e. in } \widehat{J} \\ \mathbf{v}_{\widehat{J}}(0)=\mathbf{u}_J(t_0)\,, \end{cases}$$

which **Theorem** 1 provides us. From the uniqueness of the solution we have that $\mathbf{v}_{\widehat{J}}\big(\widehat{t}\big)=\mathbf{u}_J(t)\,,$ for all $\widehat{t}\in[-t_0,\,0]$ and all $t\in[0,\,t_0]$. Moreover, from (3.2) we have that

$$E(\mathbf{u}_J(t_0))\le E(u_0)\ \text{ and } E\big(\mathbf{v}_{\widehat{J}}(-t_0)\big)\le E(\mathbf{u}_J(t_0))$$

and applying $\mathbf{v}_{\widehat{J}}(t-t_0)=\mathbf{u}_J(t)\,,$ for all $t\in[0,\,t_0]$, i.e. an equivalent formulation of the above equality, we obtain

$$E(\mathbf{u}_J(t_0))\le E(u_0)\ \text{ and } E(u_0)\le E(\mathbf{u}_J(t_0))\,.$$

Since $t_0\in J_+$ is arbitrary, we deduce (3.13) with J_+ instead of J.

Corollary 1 *The unique solution* $\mathbf{u}_J$ *of* ***Theorem*** *1 is a strong* H_0^1*-solution in* J*, i.e.* $\mathbf{u}_J \in C\big(\overline{J};\, H_0^1(U)\big) \cap C^1\big(\overline{J};\, H^{-1}(U)\big)$*, and is also continuously dependent on the initial datum.*

Proof For the regularity, since $\mathbf{u}_J \in C\big(\overline{J};\, L^2(U)\big)$, we deduce that $\mathbf{u}_J$ is weakly continuous from **Lemma** 1 and also that $|\mathbf{u}_J|_{0,2,U} \in C\big(\overline{J}\big)$ by the triangle inequality. Moreover, from (2.10) we also deduce that $G(\mathbf{u}_J) \in C\big(\overline{J}\big)$. Therefore, from (3.13) we get that $|\mathbf{u}_J|_{1,2,U} \in C\big(\overline{J}\big)$ and thus, from **Lemma** 2, we obtain that $\mathbf{u}_J \in C\big(\overline{J};\, H_0^1(U)\big)$ and also, by the variational equation, that $\mathbf{u}_J' \in C\big(\overline{J};\, H^{-1}(U)\big)$.

As far as the continuous dependence is concerned, we fix an arbitrary $u_0 \in H_0^1(U)$. Let $\{u_{0,m}\}_{m=1}^{\infty} \subset H_0^1(U)$ be such that $u_{0,m} \to u_0$ in $H_0^1(U)$. We write as $\mathbf{u}_J$ and $\mathbf{u}_{J,m}$, for $m \in \mathbb{N}$, the unique corresponding solutions of the problem (2.6). We deduce that $\{\mathbf{u}_J\} \cup \{\mathbf{u}_{J,m}\}_{m=1}^{\infty} \subset C\big(\overline{J};\, H_0^1(U)\big)$ from above. We fix an arbitrary $m_0 \in \mathbb{N}$ and then there exists a constant C_{m_0} such that

$$\big|u_{0,m}\big|_{1,2,U} \leq |u_0|_{1,2,U} + C_{m_0}, \text{ for all } m \in \mathbb{N} \text{ such that } m \geq m_0.$$

From (3.1), the above estimate, as well as the increasing property of K we have

$$\big|\mathbf{u}_{J,m}\big|_{0,J;1,2,U} + \big|\mathbf{u}_{J,m}'\big|_{0,J;-1,U} \leq \mathcal{K},$$

for all m as above. Hence, by dealing as in the proof of **Theorem** 1 from *STEP* 4α to *STEP* 6β, there exist a subsequence $\{\mathbf{u}_{J,m_l}\}_{l=1}^{\infty} \subseteq \{\mathbf{u}_{J,m}\}_{m=m_0}^{\infty}$ and a function $\mathbf{y} \in L^{\infty}\big(J;\, H_0^1(U)\big) \cap W^{1,\infty}\big(J;\, \mathrm{H}^{-1}(\mathrm{U})\big)$, such that $\mathbf{y}$ solves the problem (2.6) and also $\mathbf{u}_{J,m_l} \to \mathbf{y}$ in $C\big(\overline{J};\, L^2(U) \cap L^{\alpha+2}(U)\big)$. In view of **Proposition** 7, we deduce that $\mathbf{y} \equiv \mathbf{u}_J$. Moreover, from (2.10), (3.1), the latter convergence, and (3.13), we obtain that $\big|\mathbf{u}_{J,m_l}\big|_{1,2,U} \to |\mathbf{u}_J|_{1,2,U}$ uniformly in $\overline{J}$. Hence, from Proposition 1.3.14 (iii) in [4] we get that $\mathbf{u}_{J,m_l} \to \mathbf{u}_J$ in $C\big(\overline{J};\, H_0^1(U)\big)$. Since $\{u_{0,m}\}_{m=1}^{\infty}$ is arbitrary we deduce that for every $\{u_{0,m}\}_{m=1}^{\infty} \subset H_0^1(U)$ such that $u_{0,m} \to u_0$ in $H_0^1(U)$, there exists a subsequence $\{u_{0,m_l}\}_{l=1}^{\infty} \subset \{u_{0,m}\}_{m=1}^{\infty}$ such that $\mathbf{u}_{J,m_l} \to \mathbf{u}_J$ in $C\big(\overline{J};\, H_0^1(U)\big)$. Hence, $\mathbf{u}_{J,m} \to \mathbf{u}_J$ in $C\big(\overline{J};\, H_0^1(U)\big)$ and since $u_0 \in H_0^1(U)$ is arbitrary we conclude that the map $u_0 \mapsto \mathbf{u}_J$ is continuous.

3.4 Regularity

Here, we provide a regularity result, which is useful for the next section. We do not intend to exhaust the whole subject, thus, we only show weak H^2-regularity for a particular type of the extracted solutions.

Theorem 2 *Let* $\mathbf{u}_J$ *be the unique, energy conserving, continuously dependent on the initial datum, strong* H_0^1*-solution of (2.6) in* $U = \bigcup_{j=1}^{\infty} U_j$*, for pairwise disjoint* *open (and bounded) intervals* U_j *with* $\left|U_j\right| \geq \delta > 0$*, for every* $j \in \mathbb{N}$*. If* $u_0 = H_0^1(U) \cap H^2(U)$*,* $\zeta \in H^4(U)$ *and* α *is as in (1.3), then* $\mathbf{u}_J \in L^\infty\big(J_+;\, H_0^1(U) \cap H^2(U)\big) \cap W^{1,\infty}\big(J_+;\, L^2(U)\big)$*, with*

$$|\mathbf{u}_J|_{0,J_+;2,2,U} + \left|\mathbf{u}_J{}'\right|_{0,J_+;0,2,U} \leq \widehat{\mathcal{K}}, \tag{3.14}$$

where $\widehat{\mathcal{K}} := K_{J_+,U}\big(|u_0|_{2,2,U},\, |\zeta|_{4,2,U}\big)$*.*

Proof STEP 1: Let $\{\mathbf{u}_m\}_{m=1}^\infty$ be as in the proof of **Theorem** 1. We have that $|u_{0m}|_{2,2,U} \leq |u_0|_{2,2,U}$. Indeed, since $D^4 u_{0m} \in \operatorname{span}\{w_l\}_{l=1}^m$ we have $\big(D^4 u_{0m},\, u_{0m}\big) = \big(D^4 u_{0m},\, u_0\big)$, hence we get that

$$\begin{aligned} \left|D^2 u_{0m}\right|_{0,2,U}^2 = \big(D^4 u_{0m},\, u_{0m}\big) = \big(D^2 u_{0m},\, D^2 u_0\big) \leq \\ \leq \frac{1}{2}\left|D^2 u_{0m}\right|_{0,2,U}^2 + \frac{1}{2}\left|D^2 u_0\right|_{0,2,U}^2, \end{aligned}$$

therefore $\left|D^2 u_{0m}\right|_{0,2,U} \leq \left|D^2 u_0\right|_{0,2,U}$.

STEP 2: We multiply the variational equation in (3.3) by $\lambda_l^2 \overline{d_m^l}(t)$, where λ_l the lth eigenvalue of $-D^2$ on $H_0^1(U)$, sum for $l = 1, \ldots, m$ and take imaginary parts of both sides to find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|D^2\mathbf{u}_m\right|_{0,2,U}^2 - \operatorname{Im}\big(D^4\zeta,\, D^2\mathbf{u}_m\big) - \\ - \operatorname{Im}\big(D^2\big[\big(|\mathbf{u}_m+\zeta|^{2\tau} + r\big)\,(\mathbf{u}_m+\zeta)\big],\, D^2\mathbf{u}_m\big) = 0. \end{aligned} \tag{3.15}$$

Note that

$$\begin{aligned} &\big(D^2\big(|\mathbf{u}_m+\zeta|^{2\tau}\,(\mathbf{u}_m+\zeta)\big),\, D^2\mathbf{u}_m\big) = \\ &= \left(\sum_{q_1+\cdots+q_{2\tau+1}=2}\prod_{l=1}^{\tau+1} D^{q_l}(\mathbf{u}_m+\zeta)\prod_{l=\tau+2}^{2\tau+1} D^{q_l}\overline{(\mathbf{u}_m+\zeta)},\, D^2\mathbf{u}_m\right), \end{aligned}$$

hence we deduce, by application of (2.1) for $p_1 = p_2 = 2$ that

$$\begin{aligned} &\left|\big(D^2\big(|\mathbf{u}_m+\zeta|^{2\tau}\,(\mathbf{u}_m+\zeta)\big),\, D^2\mathbf{u}_m\big)\right| \leq \\ &\leq \left|\sum_{q_1+\cdots+q_{2\tau+1}=2}\prod_{l=1}^{\tau+1} D^{q_l}(\mathbf{u}_m+\zeta)\prod_{l=\tau+2}^{2\tau+1} D^{q_l}\overline{(\mathbf{u}_m+\zeta)}\right|_{0,2,U} \times \\ &\times \left|D^2\mathbf{u}_m\right|_{0,2,U} \end{aligned}$$

and again for $p_l = \frac{2}{q_l}$, where $l = 1, \ldots, 2\tau + 1$, to get

$$\left|\left(D^2\left(|\mathbf{u}_m+\zeta|^{2\tau}(\mathbf{u}_m+\zeta)\right), D^2\mathbf{u}_m\right)\right| \le$$
$$\le \sum_{q_1+\cdots+q_{2\tau+1}=2} \prod_{l=1}^{2\tau+1} \left|D^{q_l}(\mathbf{u}_m+\zeta)\right|_{0,\frac{4}{q_l},U} \left|D^2(\mathbf{u}_m+\zeta)\right|_{0,2,U}.$$

Moreover, from the Gagliardo-Nirenberg inequality, in view of Theorem 8.6 in [3],[3] we have

$$\left|D^j u\right|_{0,\frac{4}{j},I} \le K\left(\frac{1}{|I|}\right) \left|D^2 u\right|_{0,2,I}^{\frac{j}{2}} |u|_{0,\infty,I}^{\frac{2-j}{2}}, \text{ for } j = 0, 1, 2,$$

I being an interval (bounded or not) and $u \in H^2(I)$. Hence

$$\left|D^j u\right|_{0,\frac{4}{j},U} \le C(\delta) \left|D^2 u\right|_{0,2,U}^{\frac{j}{2}} |u|_{0,\infty,U}^{\frac{2-j}{2}}, \text{ for } j = 0, 1, 2.$$

From the above inequality, the embedding $H^1(U) \hookrightarrow L^\infty(U)$ and (3.1), we then have

$$\begin{aligned} &\left|\left(D^2\left(|\mathbf{u}_m+\zeta|^{2\tau}(\mathbf{u}_m+\zeta)\right), D^2\mathbf{u}_m\right)\right| \le \\ &\le C|(\mathbf{u}_m+\zeta)|_{0,\infty,U}^{2\tau} \left|D^2(\mathbf{u}_m+\zeta)\right|_{0,2,U}^2 \le \\ &K_U\left(|u_0|_{1,2,U}, |\zeta|_{2,2,U}\right)\left(1+\left|D^2\mathbf{u}_m\right|_{0,2,U}^2\right). \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), we derive

$$\left|D^2\mathbf{u}_m\right|_{0,2,U}^2 \le K_{J_+,U}\left(|u_0|_{2,2,U}, |\zeta|_{4,2,U}\right),$$

everywhere in J_+, from which, along with the estimates of **Theorem** 1, we obtain that $\{\mathbf{u}_m\}_{m=1}^\infty$ is bounded in $C\left(\overline{J_+}; H^2(U)\right)$, with

$$|\mathbf{u}_m|_{0,J_+;2,2,U} \le \widehat{\mathcal{K}}, \ \forall m \in \mathbb{N}. \tag{3.17}$$

STEP 3: We set $m = m_l$. Obviously, $C\left(\overline{J_+}; H^2(U)\right) \hookrightarrow L^2\left(J_+; H^2(U)\right)$. Therefore, applying Theorem 3, Sect. D.4 in [6] and **Proposition** 6 3., we get from (3.17) that there exist a subsequence of $\left\{\mathbf{u}_{m_l}\right\}_{l=1}^\infty$, which we still denote as such and a function $\mathbf{v} \in L^\infty\left(J_+; H^2(U)\right)$, such that

[3]We can modify the reflection technique used for the proof of this result, in order to cover the case of the extension of H^2-functions. In particular, we can apply the reflection technique used for Theorem 5.19 in [1].

$$\mathbf{u}_{m_l} \rightharpoonup \mathbf{v} \text{ in } L^2\big(J_+;\, H^2(U)\big) \text{ and also } |\mathbf{v}|_{0,J_+;2,2,U} \leq \widehat{\mathcal{K}}. \tag{3.18}$$

From (3.7), we can easily derive that $\mathbf{v} \equiv \mathbf{u}_J$.

STEP 4: From the embedding $H^1(U) \hookrightarrow L^\infty(U)$ and (2.14), we get that $g(\mathbf{u}_J(\cdot))$ maps to $L^2(U)$. Hence, from (3.18) and the variational equation in $H_0^1(U)$, we deduce that $\mathbf{u}_J$ solves the equation in $L^2(U)$ and

$$\big|\mathbf{u}_J'\big|_{0,J_+;0,2,U} \leq \widehat{\mathcal{K}}. \tag{3.19}$$

3.5 A Special Case of Solutions

We notice that problem (2.16) allows us to consider ζ which do not vanish at infinity, if U is unbounded, hence the $\cdot_s$-formulation is crucial for those sets. Before we proceed with the study of the unbounded case, we provide the next result.

Theorem 3 *Let $\mathbf{u}_J$ be the unique, energy conserving, continuously dependent on the initial datum, strong H_0^1-solution of (2.16). Then*

$$|\mathbf{u}_J|_{0,J_+;1,2,U} + \big|\mathbf{u}_J'\big|_{0,J_+;-1,U} \leq \mathcal{K}, \tag{3.20}$$

where $\mathcal{K} := K_{J_+}\Big(|u_0|_{1,2,U},\, \|\zeta\|_{X^1(U)},\, \big||\zeta|^2 - \rho\big|_{0,2,U}\Big)$.

Moreover, if $U = \bigcup_{j=1}^{\infty} U_j$, for pairwise disjoint open (and bounded) intervals U_j with $\big|U_j\big| \geq \delta > 0$, for every $j \in \mathbb{N}$, $u_0 \in H_0^1(U) \cap H^2(U)$ and $\zeta \in X^4(U)$, then

$$|\mathbf{u}_J|_{0,J_+;2,2,U} + \big|\mathbf{u}_J'\big|_{0,J_+;0,2,U} \leq \widehat{\mathcal{K}}, \tag{3.21}$$

where $\widehat{\mathcal{K}} := K_{J_+}\Big(|u_0|_{2,2,U},\, \|\zeta\|_{X^4(U)},\, \big||\zeta|^2 - \rho\big|_{0,2,U}\Big)$.

Proof Let $\{\mathbf{u}_m\}_{m=1}^{\infty}$ be as in the proof of **Theorem** 1. From (3.4) and (2.19) we get

$$|D\mathbf{u}_m|_{0,2,U} \leq K\Big(|u_0|_{1,2,U},\, \|\zeta\|_{X^1(U)},\, \big||\zeta|^2 - \rho\big|_{0,2,U}\Big),\ \forall m \in \mathbb{N}. \tag{3.22}$$

Then, we multiply the variational equation in (3.3) by $\overline{d_m^k}(t)$, sum for $k = 1, \ldots, m$ and take imaginary parts of both sides, and thus obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\mathbf{u}_m|_{0,2,U}^2 - \mathrm{Im}(D\zeta,\, D\mathbf{u}_m) - \\ &- \mathrm{Im}\Big(\big(|\mathbf{u}_m + \zeta|^{2\tau} - \rho^\tau\big)\,(\mathbf{u}_m + \zeta)\,,\, \mathbf{u}_m\Big) = 0. \end{aligned} \tag{3.23}$$

Applying (3.22) and expanding in view of (2.15), we deduce

$$\frac{d}{dt}|\mathbf{u}_m|_{0,2,U}^2 - K\Big(|u_0|_{1,2,U},\, \|\zeta\|_{X^1(U)},\, \big||\zeta|^2-\rho\big|_{0,2,U}\Big)\times \\ \times\big(1+|\mathbf{u}_m|_{0,2\tau+1,U}^{2\tau+1}\big) \le 0. \tag{3.24}$$

In order to estimate the non constant term inside the parenthesis, we imitate the technique which has already been developed for the proof of Lemma 3.3 in [8]. We set $B=\big(A_{2\tau}+|\zeta|_{0,\infty,U}+1\big)^2$, where $A_{2\tau}$ is as in (2.9), $Q:=\Big\{x\in U \;\Big|\; |\mathbf{u}_m+\zeta|\le\sqrt{B}\Big\}$ and $R:=Q^{\mathrm{c}}\cap U$. Then

$$\begin{aligned}
|\mathbf{u}_m|_{0,2\tau+1,U}^{2\tau+1} &= \int_Q |\mathbf{u}_m|^2|\mathbf{u}_m|^{2\tau-1}dx + \int_R |\mathbf{u}_m|^{2\tau+1}dx \le \\
&\le \int_{\left\{x\in U \,\middle|\, |\mathbf{u}_m|\le\sqrt{B}+|\zeta|_{0,\infty,U}\right\}} |\mathbf{u}_m|^2|\mathbf{u}_m|^{2\tau-1}dx + \\
&\quad + C\int_R |\zeta|^{2\tau+1} + |\mathbf{u}_m+\zeta|^{2\tau+1}dx \le \\
&\overset{(2.9)}{\le} \Big(\sqrt{B}+|\zeta|_{0,\infty,U}\Big)^c |\mathbf{u}_m|_{0,2,U}^2 + C|\zeta|_{0,\infty,U}^c\int_R dx + CG_s(\mathbf{u}_m) \le \\
&\le \Big(\sqrt{B}+|\zeta|_{0,\infty,U}\Big)^c |\mathbf{u}_m|_{0,2,U}^2 + \\
&\quad + \frac{C|\zeta|_{0,\infty,U}^c}{\big(\sqrt{B}-|\zeta|_{0,\infty,U}\big)^c}|\mathbf{u}_m|_{0,2,U}^2 + CG_s(\mathbf{u}_m) \le \\
&\overset{(2.19)}{\le} K\Big(|u_0|_{1,2,U},\, \|\zeta\|_{X^1(U)},\, \big||\zeta|^2-\rho\big|_{0,2,U}\Big)\big(1+|\mathbf{u}_m|_{0,2,U}^2\big).
\end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we derive that

$$|\mathbf{u}_m|_{0,2,U} \le \mathscr{K} \text{ in } J_+,\ \forall m\in\mathbb{N}. \tag{3.26}$$

From (3.22) and (3.26) we conclude that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is uniformly bounded in $C\big(\overline{J_+};\, H_0^1(U)\big)$, with

$$|\mathbf{u}_m|_{0,J_+;1,2,U} \le \mathscr{K},\ \forall m\in\mathbb{N}. \tag{3.27}$$

In addition, we make use of (2.17) to get that $\big\{\mathbf{u}_m{}'\big\}_{m=1}^{\infty}$ is uniformly bounded in $C\big(\overline{J_+};\, H^{-1}(U)\big)$, with

$$\big|\mathbf{u}_m'\big|_{0,J_+;-1,U} \le \mathscr{K},\ \forall m\in\mathbb{N}. \tag{3.28}$$

We then deal as in **Theorem** 1 in order to obtain (3.20).

As far as the estimate (3.21) is concerned, we deal exactly as in **Theorem** 2, employing (3.20) instead of (3.1), as well as the scaling invariant embedding $H_0^1(U)\hookrightarrow L^\infty(U)$ instead of the scaling dependent $H^1(U)\hookrightarrow L^\infty(U)$.

4 Solutions in Unbounded Sets

Here, we assume $U \subseteq \mathbb{R}$ to be unbounded.

Theorem 4 *Let $u_0 \in H_0^1(U)$. Then there exists a unique and global solution $\mathbf{u} \in L^\infty_{loc}\big(0, \infty;\, H_0^1(U)\big) \cap W^{1,\infty}_{loc}\big(0, \infty;\, H^{-1}(U)\big)$ of (2.16) in the positive time ray, such that*

$$\left|\mathbf{u}_{J_+}\right|_{0,J_+;1,2,U} + \left|\mathbf{u}'_{J_+}\right|_{0,J_+;-1,U} \le \mathscr{K},\ \ \forall J_+, \tag{4.1}$$

where $\mathbf{u}_{J_+} := \mathcal{R}_{J_+}\mathbf{u}$ and also

$$E(\mathbf{u}) \le E(u_0)\,,\ \ \textit{everywhere in } \mathbb{R}_+. \tag{4.2}$$

Moreover, if $U = \bigcup_{j=1}^\infty U_j$, for pairwise disjoint open (bounded or not) intervals U_j with $\left|U_j\right| \ge \delta > 0$, for every $j \in \mathbb{N}$., $u_0 \in H_0^1(U) \cap H^2(U)$ and $\zeta \in X^4(U)$, then $\mathbf{u} \in L^\infty_{loc}\big(0, \infty;\, H_0^1(U) \cap H^2(U)\big) \cap W^{1,\infty}_{loc}\big(0, \infty;\, L^2(U)\big)$, with

$$\left|\mathbf{u}_{J_+}\right|_{0,J_+;2,2,U} + \left|\mathbf{u}'_{J_+}\right|_{0,J_+;0,2,U} \le \widehat{\mathscr{K}},\ \ \forall J_+. \tag{4.3}$$

and also

$$E(\mathbf{u}) = E(u_0)\,,\ \ \textit{everywhere in } \mathbb{R}_+. \tag{4.4}$$

Proof We only show local existence in $H_0^1(U)$. H^2-regularity follows analogously. Equation (4.4) is a direct result of the fact that the regular solution $\mathbf{u}$ satisfies the equation in $L^2(U)$. We get uniqueness and globality exactly as in **Proposition** 7.

STEP 1: Since U is open, we fix an arbitrary $B_\varrho(x_0) \subset U$. Let $u_{0,k} := \mathcal{R}_U \eta_k u_0$, for all $k \in \mathbb{N}$, where $\{\eta_k\}_{k=1}^\infty$ is defined as follows: let $f \in C^\infty(\mathbb{R})$ with

$$f(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \le 0, \end{cases}$$

and $\{a_k\}_{k=1}^\infty \subset \mathbb{R}_+$ increasing, such that $a_k > \varrho$ for all $k \in \mathbb{N}$ and $a_k \nearrow \infty$. We define $\{\eta_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{R})$ by

$$\eta_k(x;\, x_0, a_{k-1}, a_k) := \frac{f\,(a_k - |x - x_0|)}{f\,(|x - x_0| - a_{k-1}) + f\,(a_k - |x - x_0|)},\ \ \forall\, x \in \mathbb{R},\, k \in \mathbb{N} \setminus \{1\}$$

and

$$\eta_1\big(x;\, B_\varrho(x_0)\,,\, a_1\big) := \frac{f(a_1 - |x - x_0|)}{f(|x - x_0| - \varrho) + f(a_1 - |x - x_0|)},\ \ \forall x \in \mathbb{R}.$$

It is trivial to show that

$$\eta_k(x) = \begin{cases} 1, & x \in \overline{B_{a_{k-1}}(x_0)} \\ 0, & x \in B_{a_k}(x_0)^{\mathsf{c}}, \end{cases} \quad \forall k \in \mathbb{N} \setminus \{1\} \text{ and } \eta_1(x) = \begin{cases} 1, & x \in \overline{B_{\varrho}(x_0)} \\ 0, & x \in B_{a_1}(x_0)^{\mathsf{c}}. \end{cases}$$

If, in addition, $a_{k+1} - a_k = a_1 - \varrho = C$ uniformly for all $k \in \mathbb{N}$ (i.e. C is independent of k), then $\left|D^{\beta}\eta_k\right|_{0,\infty} \leq C_m$, for some $\{C_m\}_{m=0}^{\infty} \subset \mathbb{R}_+$, uniformly for all $k \in \mathbb{N}$ and every multi-index β such that $|\beta| = m$. In particular, $C_0 = 1$.

Therefore, for all $k \in \mathbb{N}$, we have that

$$\left|u_{0,k}\right|_{1,2,U} \leq C|u_0|_{1,2,U}. \tag{4.5}$$

We also notice that $u_{0,k} = 0$, in $B_{a_k}(x_0)^{\mathsf{c}} \cap U$, hence, by setting $B_k := B_{a_k}(x_0) \cap U$, for every $k \in \mathbb{N}$, we obtain that $\left\{\mathcal{R}_{B_k} u_{0,k}\right\}_{k=1}^{\infty} \subset H_0^1(B_k)$. Moreover,

$$u_{0,k} \to u_0 \text{ in } L^2(U). \tag{4.6}$$

Indeed,

$$\left|u_{0,k} - u_0\right|_{0,2,U} = |(\eta_k - 1)u_0|_{0,2,U} \leq |u_0|_{0,2,B_{a_{k-1}}(x_0)^{\mathsf{c}} \cap U} \to 0.$$

STEP 2α: Let J_+ be arbitrary. Fixing any $k \in \mathbb{N}$, we consider (2.16) in $U = B_k$, where we take $\mathcal{R}_{B_k} u_{0,k}$ as our initial datum and we set $\mathbf{u}^k \in L^{\infty}\left(J_+;\, H_0^1(B_k)\right) \cap W^{1,\infty}\left(J_+;\, H^{-1}(B_k)\right)$ to be the solution that **Theorem** 3 provides. From its proof, it follows that there exist $\left\{\mathbf{u}_m^k\right\}_{m=1}^{\infty} \subset C^{\infty}\left(\overline{J_+};\, H_0^1(U) \cap C^{\infty}(\overline{U})\right)$, such that

$$\begin{aligned} &\left|\mathbf{u}_m^k\right|_{0,J_+;1,2,B_k} + \left|\mathbf{u}_m^{k\,\prime}\right|_{0,J_+;-1,B_k} \leq \\ &\leq K_{J_+}\left(\left|u_{0,k}\right|_{1,2,B_k}, \|\zeta\|_{X^1(B_k)}, \left||\zeta|^2 - \rho\right|_{0,2,B_k}\right), \ \forall m \in \mathbb{N} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \mathbf{u}_m^k(t) &\rightharpoonup \mathbf{u}^k(t) \ \text{ in } H_0^1(B_k), \ \text{ for every } t \in \overline{J_+}, \\ \mathbf{u}_m^{k\,\prime} &\overset{*}{\rightharpoonup} \mathbf{u}^{k\prime} \text{ in } L^{\infty}\left(J_+;\, H^{-1}(B_k)\right). \end{aligned} \tag{4.8}$$

From (4.5), (4.7) and the increasing property of K we deduce that

$$\left|\mathbf{u}_m^k\right|_{0,J_+;1,2,B_k} + \left|\mathbf{u}_m^{k\,\prime}\right|_{0,J_+;-1,B_k} \leq \mathscr{K}, \ \forall m \in \mathbb{N}. \tag{4.9}$$

STEP 2β: Since $\mathcal{R}_{\partial B_k \setminus \partial U} \mathbf{u}_m^k = 0$, the extensions by zero $\mathbf{v}_m^k := \mathcal{E}_U \mathbf{u}_m^k$, for all $m \in \mathbb{N}$,[4] are continuous in $\partial B_k \setminus \partial U$ and thus $\left\{\mathbf{v}_m^k\right\}_{m=1}^{\infty} \subset C^{\infty}\left(\overline{J_+};\, H_0^1(U)\right)$. Evidently,

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[4]For the H^2-regularity, we define $\mathbf{v}_m^k := \eta_k \mathcal{E}_U \mathbf{u}_m^k$, for all $m \in \mathbb{N}$.

$$\left|\mathbf{v}_m^k\right|_{0,J_+;1,2,U} = \left|\mathbf{u}_m^k\right|_{0,J_+;1,2,B_k}$$
$$\left|\mathbf{v}_m^{k\,\prime}\right|_{0,J_+;-1,U} = \left|\mathbf{u}_m^{k\,\prime}\right|_{0,J_+;-1,B_k},$$

hence, from (4.9), we get that

$$\left|\mathbf{v}_m^k\right|_{0,J_+;1,2,U} + \left|\mathbf{v}_m^{k\,\prime}\right|_{0,J_+;-1,U} \leq \mathscr{K}, \ \forall m \in \mathbb{N}.$$

STEP 2γ: Dealing as in *STEP* 4 of the proof of **Theorem** 3, there exist a subsequence $\{\mathbf{v}_{m_l}^k\}_{l=1}^\infty \subseteq \{\mathbf{v}_m^k\}_{m=1}^\infty$ and a function $\mathbf{v}^k \in L^\infty(J_+;\, H_0^1(U)) \cap W^{1,\infty}(J_+;\, H^{-1}(U))$, such that

$$\begin{aligned} \mathbf{v}_{m_l}^k(t) &\rightharpoonup \mathbf{v}^k(t) \ \text{ in } H_0^1(U), \ \text{ for every } t \in \overline{J_+}, \\ \mathbf{v}_{m_l}^{k\,\prime} &\overset{*}{\rightharpoonup} \mathbf{v}^{k\prime} \text{ in } L^\infty(J_+;\, H^{-1}(U)), \\ \left|\mathbf{v}^k\right|_{0,J_+;1,2,U} &+ \left|\mathbf{v}^{k\prime}\right|_{0,J_+;-1,U} \leq \widetilde{\mathcal{K}}. \end{aligned} \tag{4.10}$$

Since $k \in \mathbb{N}$ is arbitrary, $\{\mathbf{v}^k\}_{k=1}^\infty \subset L^\infty(J_+;\, H_0^1(U)) \cap W^{1,\infty}(J_+;\, H^{-1}(U))$ and the above estimate is satisfied for each $k \in \mathbb{N}$.

STEP 3α: Dealing again as before, there exist a subsequence $\{\mathbf{v}^{k_l}\}_{l=1}^\infty \subseteq \{\mathbf{v}^k\}_{k=1}^\infty$ and a function $\mathbf{u}_{J_+} \in L^\infty(J_+;\, H_0^1(U)) \cap W^{1,\infty}(J_+;\, H^{-1}(U))$, such that

$$\begin{aligned} \mathbf{v}^{k_l}(t) &\rightharpoonup \mathbf{u}_{J_+}(t) \ \text{ in } H_0^1(U), \ \text{ for every } t \in \overline{J_+}, \\ \mathbf{v}^{k_l\prime} &\overset{*}{\rightharpoonup} \mathbf{u}_{J_+}' \ \text{ in } L^\infty(J_+;\, H^{-1}(U)), \\ \left|\mathbf{u}_{J_+}\right|_{J_+;1,2,U} &+ \left|\mathbf{u}_{J_+}'\right|_{J_+;-1,U} \leq \widetilde{\mathcal{K}}. \end{aligned} \tag{4.11}$$

STEP 3β: From (2.13), the estimate in (4.10) and Lemma 3.3.6 in [4] we deduce that $\{g_s(\mathbf{v}^{k_l})\}_{l=1}^\infty$ is bounded in $C^{0,\frac{1}{2}}(\overline{J_+};\, L^2(U))$. Hence, from Proposition 1.1.2 in [4], there exist a subsequence of $\{\mathbf{v}^{k_l}\}_{l=1}^\infty$, which we still denote as such, and a function $\mathbf{f} \in C(\overline{J_+};\, L^2(U))$, such that

$$g_s(\mathbf{v}^{k_l}(t)) \rightharpoonup \mathbf{f}(t) \ \text{ in } L^2(U), \ \forall t \in \overline{J_+}. \tag{4.12}$$

STEP 4α: Let Ω be any bounded open interval $\subset U$. For $k \in \mathbb{N}$ big enough so that $\Omega \subseteq B_k$, we have

$$\begin{aligned} \left(\mathbf{v}^k, \mathcal{E}_U v\right) = \left(\mathbf{u}^k, \mathcal{E}_{B_k} v\right), \ \ &\left(g_s(\mathbf{v}^k), \mathcal{E}_U v\right) = \left(g_s(\mathbf{u}^k), \mathcal{E}_{B_k} v\right) \\ \text{and } \left\langle \mathbf{v}^{k\prime}, \mathcal{E}_U v\right\rangle &= \left\langle \mathbf{u}^{k\prime}, \mathcal{E}_{B_k} v\right\rangle, \end{aligned} \tag{4.13}$$

for every $v \in C_c^\infty(\Omega)$. Indeed, for the first equality, we get from (4.10)

$$\int_U \mathbf{v}^k_{m_l} \mathcal{E}_U \bar{v} dx \to \int_U \mathbf{v}^k \mathcal{E}_U \bar{v} dx$$

and from (4.8)

$$\int_U \mathbf{v}^k_{m_l} \mathcal{E}_U \bar{v} dx = \int_{B_k} \mathcal{R}_{B_k} \mathbf{v}^k_{m_l} \mathcal{E}_{B_k} \bar{v} dx \to \int_{B_k} \mathbf{u}^k \mathcal{E}_{B_k} \bar{v} dx.$$

The second equality follows similarly. The third equality follows from the first one and Lemma 1.1, Chap. 3, in [11]. Now, since $\mathbf{u}^k$ is a solution in B_k,

$$i \left\langle \mathbf{u}^{k\prime}, \mathcal{E}_{B_k} v \right\rangle + \mathcal{N}_s\left[\mathbf{u}^k, \mathcal{E}_{B_k} v\right] = 0, \ \forall v \in C_c^\infty(\Omega), \text{ a.e. in } J_+,$$

hence, from (4.13),

$$i \left\langle \mathbf{v}^{k\prime}, \mathcal{E}_U v \right\rangle + \mathcal{N}_s\left[\mathbf{v}^k, \mathcal{E}_U v\right] = 0, \ \forall v \in C_c^\infty(\Omega), \text{ a.e. in } J_+. \tag{4.14}$$

STEP 4β: From the first convergence in (4.11), the weak lower semi-continuity of the H^1-norm and the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we obtain that there exist a subsequence of $\left\{\mathbf{v}^{k_l}\right\}_{l=1}^\infty$, which we still denote as such, for which we have

$$\mathbf{v}^{k_l}(t) \to \mathbf{u}_{J_+}(t) \text{ in } L^2(\Omega), \ \forall t \in \overline{J_+}. \tag{4.15}$$

We set $k = k_l$ in (4.14) and we pass to the limit $l \to \infty$. From (4.11), (4.12) and (4.15), we deduce that

$$\int_{J_+} \left(i \left\langle \mathbf{u}'_{J_+}, \mathcal{E}_U v \right\rangle + \left\langle \Delta \mathbf{u}_{J_+}, \mathcal{E}_U v \right\rangle + \left\langle \mathbf{f}, \mathcal{E}_U v \right\rangle \right) \overline{\psi} dt = 0,$$

for every $v \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(J_+)$, hence

$$i \left\langle \mathbf{u}'_{J_+}, \mathcal{E}_U v \right\rangle + \left\langle \Delta \mathbf{u}_{J_+}, \mathcal{E}_U v \right\rangle + \left\langle \mathbf{f}, \mathcal{E}_U v \right\rangle = 0, \ \forall v \in C_c^\infty(\Omega), \text{ a.e. in } J_+. \tag{4.16}$$

STEP 4γ: From (4.12) and **Lemma** 3 we have

$$g_s\left(\mathcal{R}_\Omega \mathbf{v}^{k_l}(t)\right) = \mathcal{R}_\Omega g_s\left(\mathbf{v}^{k_l}(t)\right) \rightharpoonup \mathcal{R}_\Omega \mathbf{f}(t) \text{ in } L^2(U), \ \forall t \in \overline{J_+}. \tag{4.17}$$

On the other hand, from (4.15) and **Lemma** 3,

$$\mathcal{R}_\Omega \mathbf{v}^{k_l}(t) \to \mathcal{R}_\Omega \mathbf{u}_{J_+}(t) \text{ in } L^2(\Omega), \ \forall t \in \overline{J_+}.$$

From (2.13) we get

$$g_s\left(\mathcal{R}_\Omega \mathbf{v}^{k_l}(t)\right) \to g_s\left(\mathcal{R}_\Omega \mathbf{u}_{J_+}(t)\right) = \mathcal{R}_\Omega g_s\left(\mathbf{u}_{J_+}(t)\right) \text{ in } L^2(U)\,,\ \forall t \in \overline{J_+}. \quad (4.18)$$

From (4.17) and (4.18) we derive $\mathcal{R}_\Omega g_s(\mathbf{u}) \equiv \mathcal{R}_\Omega \mathbf{f}$ and so (4.16) gets the form

$$i\left\langle \mathbf{u}'_{J_+}, \mathcal{E}_U v\right\rangle + \mathcal{N}_s\left[\mathbf{u}_{J_+}, \mathcal{E}_U v\right] = 0,\ \forall v \in C_c^\infty(\Omega)\,,\ \text{a.e. in } J_+.$$

Since Ω is arbitrary, $\mathbf{u}_{J_+}$ satisfies the variational equation in (2.16).

STEP 5: As far as the initial condition is concerned, let t_0, v, ϕ be as in *STEP* 6β of the proof of **Theorem** 3. Then

$$\begin{aligned} \int_0^{t_0} \left(\mathbf{v}_m^{k\,\prime}, \phi v\right) dt &= -\int_0^{t_0} \left(\mathbf{v}_m^k, \phi' v\right) dt - \left(\mathbf{v}_m^k(0)\,, \phi(0)\, v\right), \\ \int_0^{t_0} \left\langle \mathbf{u}'_{J_+}, \phi v\right\rangle dt &= -\int_0^{t_0} \left(\mathbf{u}_{J_+}, \phi' v\right) dt - \left(\mathbf{u}_{J_+}(0)\,, \phi(0)\, v\right). \end{aligned} \quad (4.19)$$

Moreover, $\left(\mathbf{v}_m^k(0)\,, \phi(0)\, v\right) = \left(\mathbf{u}_m^k(0)\,, \phi(0)\, \mathcal{R}_{B_k} v\right)$, hence, by setting $m = m_l$ and letting $l \to 0$, we get

$$\int_0^{t_0} \left\langle \mathbf{v}^{k\prime}, \phi v\right\rangle dt = -\int_0^{t_0} \left(\mathbf{v}^k, \phi' v\right) dt - \left(\mathcal{R}_{B_k} u_{0k}, \phi(0)\, \mathcal{R}_{B_k} v\right).$$

Since $\left(\mathcal{R}_{B_k} u_{0k}, \phi(0)\, \mathcal{R}_{B_k} v\right) = \left(u_{0k}, \phi(0)\, v\right)$, we set $k = k_l$ and we pass to the limit as $l \to \infty$, applying (4.6), to get

$$\int_0^{t_0} \left\langle \mathbf{u}'_{J_+}, \phi v\right\rangle dt = -\int_0^{t_0} \left(\mathbf{u}_{J_+}, \phi' v\right) dt - \left(u_0, \phi(0)\, v\right). \quad (4.20)$$

From the second equation in (4.19) and (4.20), we conclude to $\mathbf{u}_{J_+}(0) = u_0$.

In fact, for $U = \mathbb{R}$, we need a weaker assumption on ζ, in order for the H^2-regularity result of **Theorem** 4 to hold. Indeed, in view of **Theorem** 4, (2.13) and (2.14), the following is a direct application of Theorem 5.3.1 and Remark 5.3.2 in [4].

Theorem 5 *If $v_0 \in H^2(\mathbb{R})$ and $\zeta \in X^2(\mathbb{R})$, then there exists a unique and global solution $\mathbf{u} \in L^\infty_{loc}\left(0, \infty; H_0^1(U) \cap H^2(U)\right) \cap W^{1,\infty}_{loc}\left(0, \infty; L^2(U)\right)$ of (2.16), with $E(\mathbf{u}) = E(u_0)$ everywhere in $\mathbb{R}_+$.*

Proposition 9 *Let $U = \bigcup_{j=1}^\infty U_j$, for pairwise disjoint open (bounded or not) intervals U_j with $\left|U_j\right| \geq \delta > 0$, for every $j \in \mathbb{N}$, $u_0 \in H_0^1(U)$,*

$$\zeta \in \begin{cases} X^2(U)\,, & \text{if } U = \mathbb{R} \\ X^4(U)\,, & \text{otherwise} \end{cases}$$

and **u** *the corresponding weak* H_0^1*-solution of (2.16). Then the energy of* **u** *is conserved,* **u** *is a strong* H_0^1*-solution, continuously dependent on the initial datum.*

Proof It suffices to show that the energy is conserved. Let J_+ be arbitrary and $\{u_{0,m}\}_{m=1}^{\infty} \subset C_c^{\infty}(U)$ be such that $u_{0,m} \to u_0$ in $H_0^1(U)$. We write as $\mathbf{u}_{J_+}$ and $\mathbf{u}_{J_+,m}$, for $m \in \mathbb{N}$, the unique corresponding solutions of the problem (2.16). In view of **Theorems** 4 and **5**, we have that $\mathbf{u}_{J_+,m}$ satisfies the differential equation in $L^2(U)$, for all $m \in \mathbb{N}$, hence, we can easily derive that the energy of every $\mathbf{u}_{J_+,m}$ is conserved, i.e.

$$E\left(\mathbf{u}_{J_+,m}\right) = E\left(u_{0,m}\right), \ \forall m \in \mathbb{N}, \ \text{everywhere in } J_+. \tag{4.21}$$

Moreover, we have that $\mathbf{u}_{J_+,m} \to \mathbf{u}_{J_+}$ in $C\left(\overline{J_+}; L^2(U)\right)$. Indeed, dealing as in the proof of **Proposition** 7, we have that

$$|\mathbf{w}_m|_{0,2,U}^2 \leq C \int_0^t |\mathbf{w}_m|_{0,2,U}^2 ds, \ \forall t \in J_+,$$

where $\mathbf{w}_m := \mathbf{u}_{J_+,m} - \mathbf{u}_{J_+}$, for all $m \in \mathbb{N}$, therefore, the convergence follows from the Grönwall inequality and the fact that $u_{0,m} \to u_0$ in $L^2(U)$. Now, we deal as in the proof of **Theorem** 1 from *STEP* 4α to *STEP* 6β, minding to exclude *STEP* 5 and apply the above extracted convergence as well as (2.13), instead. Hence, there exist a subsequence $\{\mathbf{u}_{J_+,m_l}\}_{l=1}^{\infty} \subseteq \{\mathbf{u}_{J_+,m}\}_{m=m_0}^{\infty}$ and a function $\mathbf{y} \in L^{\infty}\left(J_+; H_0^1(U)\right) \cap W^{1,\infty}\left(J_+; \mathrm{H}^{-1}(\mathrm{U})\right)$, such that $\mathbf{y}$ solves the problem (2.6) and also $\mathbf{u}_{J_+,m_l} \to \mathbf{y}$ in $C\left(\overline{J_+}; L^2(U)\right)$. From the uniqueness of the solution, we deduce that $\mathbf{y} \equiv \mathbf{u}_{J_+}$. Moreover, from (2.10), (4.1), the latter convergence and (4.21), we obtain that $\left|\mathbf{u}_{J_+,m_l}\right|_{1,2,U} \to \left|\mathbf{u}_{J_+}\right|_{1,2,U}$ uniformly in $\overline{J_+}$. Applying the aforementioned convergences, we then easily get from (4.21) that $E\left(\mathbf{u}_{J_+}\right) = E(u_0)$, everywhere in J_+.

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