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Nonvanishing at spatial extremity solutions of the defocusing nonlinear Schrödinger equation

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We show local existence of certain type of solutions for the Cauchy problem of the defocusing nonlinear Schrödinger equation with pure power nonlinearity, in various cases of open sets, unbounded or bounded. These solutions do not vanish at the boundary or at infinity. We also show, in certain cases, that these solutions are unique and global.

KEYWORDS
Cauchy problem, dark soliton, defocusing, extension operator, nonlinear Schrödinger equation, nonvanishing at spatial extremity solutions

1 | INTRODUCTION

The nonlinear Schrödinger (NLS) equation is a universal model describing the evolution of complex field envelopes in nonlinear dispersive media; it appears in a variety of physical contexts, ranging from optics to fluid dynamics and plasma physics, and it has attracted a huge interest from the rigorous mathematical analysis point of view, as well. The importance of the NLS model is not restrained to the case of conservative systems, but it is also associated to dissipative models. Many of the closely connected to the NLS equation pattern formation phenomena emanate via the genesis of localized structures with finite spatial support, or with sufficiently fast spatial decay, the so-called solitons. Among the various types of waves whose amplitude is modulated, there are two principal kinds of solitons, depending on the category of the nonlinearity; in the case of an attractive (or focusing) medium, the nonlinearity causes the formation of structures termed "bright solitons," while in the case of a repulsive (or defocusing) medium, the nonlinearity generates "dark solitons" (ie, nonlinear solitary waves having the form of localized dips in density, which decay off of a continuous-wave background; if the density of the dip tends to zero, the dark solitons are named "black," otherwise "grey").

Theoretical physical studies on dark solitons started in 1971, by the work of T. Tsuzuki[1] in the context of Bose-Einstein condensates. Two years later, in[2] V. E. Zakharov and A. B. Shabat demonstrated the complete integrability of the defocusing NLS equation using the inverse scattering transform (incidentally, the same authors had shown the integrability of the focusing NLS equation in[3]). The progress in the theory after that was very rapid and immense. As for experimental results, the progress was equally impressive: After the "early age" experiments of the 1970s, the "new age," which emerged in the middle of the first decade of the 21st century, is a period of spectacular progress. These led to a vast amount of literature. A detailed presentation of the physical studies (theoretical and experimental) and of the recent progress regarding the defocusing NLS is contained in Kevrekidis et al,[4] which incorporates an extensive bibliography.

Regarding the rigorous mathematical analysis of the NLS equation, the books of Bourgain,[5] Cazenave,[6] Cazenave and Haraux,[7] Sulem and Sulem,[8] and Tao[9] are classical by now. Moreover, the recent books Erdoğan and Tzirakis,[10] Fibich,[11] and Linares and Ponce[12] contribute substantially to the field. The reference lists in all these books are representative of the huge interest and amount of research work on the NLS equation.

In this work, we consider the n-dimensional defocusing NLS initial/“boundary” value problem

$$\begin{cases} \mathrm{i}v_{\mathrm{t}} + \Delta v - |v|^{\alpha} v = 0, \forall (\mathrm{t}, \mathrm{x}) \in \mathrm{J}^{*} \times \mathrm{U} \\ v = v_0, \text{ on } \{\mathrm{t} = 0\} \times \mathrm{U} \\ \text{not necessarily } v = 0 \text{ on } \mathrm{J} \times \partial \mathrm{U}, \text{ or } v \xrightarrow{|\mathrm{x}| \to \infty} 0 \text{ on } \mathrm{J} \times \mathrm{U}, \end{cases} \tag{1.1}$$

where $v : \mathrm{J} \times \mathrm{U} \to \mathbb{C}$, with $\mathrm{J} = [0, T]$, for $T > 0$, U an open set $\subseteq \mathbb{R}^{\mathrm{n}}$, and $\alpha > 0$. In the case that U is unbounded, we assume that v has a constant amplitude at infinity.

Since we are interested in all possible cases of open sets, U could be bounded (eg, a ball) or unbounded—with or without empty boundary (eg, $\mathbb{R}^{\mathrm{n}}_{+}$ or $\mathbb{R}^{\mathrm{n}}$, respectively). Let us recall that when $\mathrm{U} = \mathbb{R}^{\mathrm{n}}$, the existence of many such solutions (ie, dark solitons) is well-known.

Here we seek solutions of the form

$$v(\mathrm{t}, \mathrm{x}) = e^{\mathrm{irt}} (u(\mathrm{t}, \mathrm{x}) + \zeta(\mathrm{x})), \tag{1.2}$$

for $\mathrm{r} \in \mathbb{R}$ and u, ζ complex-valued functions over $\mathrm{J} \times \mathrm{U}$ and U, respectively. Assuming that u vanishes at the boundary and at infinity, but ζ, in contrast, survives, the problem (1.1) becomes

$$\begin{cases} \mathrm{i}u_{\mathrm{t}} + \Delta (u + \zeta) - \left(|u + \zeta|^{\alpha} + \mathrm{r} \right) (u + \zeta) = 0, \forall (\mathrm{t}, \mathrm{x}) \in \mathrm{J}^{*} \times \mathrm{U} \\ u = u_0, \text{on } \{\mathrm{t} = 0\} \times \mathrm{U} \\ u = 0, \text{on } \mathrm{J} \times \partial \mathrm{U} \text{ and } u \xrightarrow{|\mathrm{x}| \to \infty} 0, \text{ on } \mathrm{J} \times \mathrm{U}, \end{cases} \tag{1.3}$$

for given r, ζ and also $u_0 : \mathrm{U} \to \mathbb{C}$, which vanishes at the boundary and at infinity.

The problem (1.3) for $\mathrm{U} = \mathbb{R}^{\mathrm{n}}$ with $\mathrm{n} = 1, 2, 3$ and

$$\alpha = 2\tau, \text{for} \begin{cases} \tau \in \mathbb{N}^{*}, & \text{if } \mathrm{n} = 1, 2 \\ \tau = 1, & \text{if } \mathrm{n} = 3, \end{cases}$$

along with more general cases of nonlinearity, has been studied in Gallo.[13] There, it is shown that if $\mathrm{r} = -\rho^{\tau}$ with $\rho > 0$, as well as $\zeta \in \mathrm{C}^{\mathrm{k}+1}_{\mathrm{b}} (\mathbb{R}^{\mathrm{n}})$, $\mathrm{D}\zeta \in \mathrm{H}^{\mathrm{k}+1} (\mathbb{R}^{\mathrm{n}})$, with $\mathrm{k} = 1$ if $\mathrm{n} = 1$ and $\mathrm{k} = 2$ if $\mathrm{n} = 2, 3$, and additionally $\left(|\zeta|^2 - \rho \right) \in \mathrm{L}^2 (\mathbb{R}^{\mathrm{n}})$, then (1.3) is globally well posed.

In this work, we extend the above result, not only by weakening the assumptions, but also by considering more general cases of $\mathrm{U} \subseteq \mathbb{R}^{\mathrm{n}}$, other than the Euclidean space itself.

The present paper is organized as follows: Section 2 contains some necessary notation for the readers' convenience and also some preliminary results that will be used in the sequel. In Section 3, we rigorously formulate the problem and provide properties of the operators and the quantities that appear. Local existence, uniqueness and globality in bounded sets is considered in Section 4. In particular, for the case of bounded*U, we first show (see Theorem 1) local existence for every

$$\alpha \in \begin{cases} (0, \infty), & \text{if } n = 1, 2 \\ \left(0, \frac{4}{n-2}\right), & \text{otherwise}, \end{cases} \tag{1.4}$$

every $\mathrm{r} \in \mathbb{R}$ and every U, if $\zeta \in \mathrm{H}^1 (\mathrm{U}) \cap L^{\alpha+2} (\mathrm{U})$. A result on the uniqueness and globality of some of these solutions follows next (see Proposition 10). We also show (see Theorem 2) local existence for every $\alpha = 2\tau$, every $\mathrm{r} = -\rho^{\tau}$ with $\rho > 0$ and every U, if $\zeta \in \mathrm{X}^1 (\mathrm{U})$. We note that $\mathrm{X}^1 (\mathrm{U})$ stands for the Zhidkov space over U, defined as $\mathrm{X}^1 (\mathrm{U}) := \left\{ u \in \mathrm{L}^{\infty} (\mathrm{U}) \mid \mathrm{D}u \in \mathrm{L}^2 (\mathrm{U}) \right\}$ and equipped with its natural norm $\|\cdot\|_{\mathrm{X}^1(\mathrm{U})} := \|\cdot\|_{\mathrm{L}^{\infty}(\mathrm{U})} + \|\mathrm{D}\cdot\|_{\mathrm{L}^2(\mathrm{U})}$. The first version of such spaces over $\mathbb{R}$ is introduced in Zhidkov,[14] and a generalization for higher dimensions (along with certain modifications) is done in later studies.[13,15-17] In this work, however, we consider X^1 over any open set.

*We note that all of the results concerning the case of bounded U can also be applied to $\mathrm{H}^1_{\mathrm{per}} (\mathbb{R}^n)$.

Local existence in unbounded sets is studied in Section 5. We use the technique that appears in Gialelis[18] and is based on the extension-by-zero of certain approximations of solutions, each one of which is considered in a bigger bounded open set than the domain of the previous one, to extend Theorem 2 for any unbounded U (see Theorem 3), if $\zeta \in \mathrm{X}^1(\mathrm{U})$ and $\left(|\zeta|^2-\rho\right) \in \mathrm{L}^2(\mathrm{U})$. Uniqueness and globality in particular cases also is provided (see Proposition 11) for certain cases of α and U. Further, there are two appendices, A and B, containing some useful inequalities and cut-off functions, respectively, used in various points through the paper.

The choice of the case of the nonlinearity being of the pure power type is due to its numerous and classic applications in Physics and Nonlinear Science; very important examples of such nonlinearities in the NLS equation are the quintic ($\alpha = 4$) in the 1-dimensional case (which is related to Bose-Einstein condensation) and the cubic (or Kerr) ($\alpha = 2$), when n is equal to 2 or 3 (wave condensation in many areas of high theoretical and experimental interest, a classical one being Optics). Our results may be generalized to a wider class of nonlinearities; however, we do not consider such cases in the present work.

2 | PRELIMINARIES

We start with some notation used throughout the paper. Recall that $\mathrm{J} := [0, T]$, $T > 0$.

1. If $\mathrm{p}, r \in [1, \infty]$ and $\mathrm{k}, \mathrm{m} \in \mathbb{N}_0$, then we write

$$|\cdot|_{\mathrm{m,r,U}} := \|\cdot\|_{\mathrm{W^{m,r}(U)}}, \qquad |\cdot|_{\mathrm{-m,U}} := \|\cdot\|_{\mathrm{H^{-m}(U)}}$$
$$|\cdot|_{\mathrm{k,p,J;m,r,U}} := \|\cdot\|_{\mathrm{W^{k,p}(J;W^{m,r}(U))}}, \qquad |\cdot|_{\mathrm{k,p,J;-m,U}} := \|\cdot\|_{\mathrm{W^{k,p}(J;H^{-m}(U))}}.$$

   We omit $p = \infty$, $\mathrm{J} = [0, \infty)$, and $\mathrm{U} = \mathbb{R}^{\mathrm{n}}$ from the notation.
2. Let $\mathcal{F}(\mathrm{U}_1; \mathbb{C})$ be a function space over $\mathrm{U}_1 \subset \mathrm{U}_2 \subseteq \mathbb{R}^n$ and $f \in \mathcal{F}(\mathrm{U}_1)$. We denote by $\mathcal{E}_{\mathrm{U}_2} f$ its extension by zero in $U_2 \backslash U_1$ and $\mathcal{E}_{\mathrm{U}_2}\mathcal{F}(\mathrm{U}_1) := \left\{\mathcal{E}_{\mathrm{U}_2} f \mid f \in \mathcal{F}(\mathrm{U}_1)\right\}$. We omit $\mathrm{U}_2 = \mathbb{R}^n$ from these notations. Moreover, if $g \in \mathcal{F}(\mathrm{U}_2)$, we denote by $\mathcal{R}_{\mathrm{U}_1} g$ and $\mathcal{R}_{\mathrm{U}_1}\mathcal{F}(\mathrm{U}_2)$ the restriction of g in U_1 and the set of these restricted functions, respectively.
3. We write C and c for any nonnegative co+nstant factor and exponent, respectively. These constants may be explicitly calculated in terms of known quantities and may change from line to line and also within a certain line in a given computation. We also use the letter K for any increasing function $\mathrm{K} : [0, \infty)^{\mathrm{n}} \to [0, \infty)$. When J and U appear as subscripts in an element, they denote that this depends on them, while their absence designates independence.
4. If $u : \mathrm{J} \times \mathrm{U} \to \mathbb{C}$, with $u(\mathrm{t}, \cdot) \in \mathcal{F}(\mathrm{U}; \mathbb{C})$ for each $\mathrm{t} \in \mathrm{J}$, where $\mathcal{F}(\mathrm{U})$ is a function space over U, then, following the notation of, eg, Evans[19] and Temam,[20] we associate with u the mapping $\mathbf{u} : \mathrm{J} \to \mathcal{F}(\mathrm{U})$, defined by $[\mathbf{u}(\mathrm{t})](\mathrm{x}) := u(\mathrm{t}, \mathrm{x})$, for every $\mathrm{x} \in \mathrm{U}$ and $\mathrm{t} \in \mathrm{J}$.
5. We write $\alpha_{1,2,3,4} \geq 0$ such that

$$\alpha_1 \in \begin{cases} [0, \infty), & \text{if } n = 1, 2 \\ \left[0, \frac{4}{n-2}\right], & \text{otherwise,} \end{cases} \quad \alpha_2 \in \begin{cases} (0, \infty), & \text{if } n = 1, 2 \\ \left(0, \frac{4}{n-2}\right], & \text{otherwise} \end{cases}$$
$$\alpha_3 \text{ for every } \alpha \text{ as in (1.4) and } \alpha_4 = 2\tau, \text{ for } \tau \text{ as in Section 1.}$$

Corollary 1. *Let $\alpha > 0$ and $u, v \in L^{\alpha+2}(\mathrm{U})$. Then*

$$\int_{\mathrm{U}} |u|^{\alpha+1} |v| \, dx \leq |u|^{\alpha+1}_{0,\alpha+2,\mathrm{U}} |v|_{0,\alpha+2,\mathrm{U}}. \tag{2.1}$$

Proof. Use (A4) for $\mathrm{p} = \frac{\alpha+2}{\alpha+1}$ and $\mathrm{q} = \alpha + 2$. □

Corollary 2. *Let $\alpha > 0$ and also $u, v \in L^{\alpha+2}(\mathrm{U})$. Then*

$$\left||u|^{\alpha} u - |v|^{\alpha} v\right|_{0,\frac{\alpha+2}{\alpha+1},\mathrm{U}} \leq \mathrm{C}\left(|u|^{\mathrm{c}}_{0,\alpha+2,\mathrm{U}} + |v|^{\mathrm{c}}_{0,\alpha+2,\mathrm{U}}\right) |u - v|_{0,\alpha+2,\mathrm{U}}. \tag{2.2}$$

Proof. Direct application of (A2), (A4) for $p = \alpha + 1$ and $\mathrm{q} = \frac{\alpha+1}{\alpha}$ and (A1). □

Corollary 3. *Let $u \in \mathrm{H}_0^1(\mathrm{U})$. Then*

$$|u|_{0,\alpha_1+2,\mathrm{U}}^{\alpha_1+2} \leq C\,|u|_{1,2,\mathrm{U}}^{\frac{n\alpha_1}{2}}\,|u|_{0,2,\mathrm{U}}^{\frac{4-n\alpha_1}{2}+\alpha_1}. \tag{2.3}$$

If, in addition, $n = 2$ and $\tau \in [1,\infty)$, then

$$|u|_{0,2\tau,\mathrm{U}}^{2\tau} \leq C\,|u|_{1,2,\mathrm{U}}^{2(\tau-1)}\,|u|_{0,2,\mathrm{U}}^{2}. \tag{2.4}$$

Proof. The first inequality is direct from Theorem 7 (see also Remark 1) for $p = \alpha_1 + 2$, $r = q = 2$, $j = 0$, $\mathrm{m} = 1$ and $\theta = \frac{n\alpha_1}{2(\alpha_1+2)}$. As for the second one we set $\alpha_1 = 2(\tau-1)$ in (2.3). □

An known estimate of the constant in (2.4) is

$$C \leq (4\pi)^{(1-\tau)}\tau^{\tau}, \tag{2.5}$$

Proposition 1.

1. *Let $\mathcal{H}$ be a Hilbert space, as well as $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset L^{\infty}(\mathrm{J};\mathcal{H})$ and $\mathbf{u} : \mathrm{J} \to \mathcal{H}$ with $\mathbf{u}_k(\mathrm{t}) \rightharpoonup \mathbf{u}(\mathrm{t})$ in $\mathcal{H}$, for a.e. $t \in J$. If $\|\mathbf{u}_k\|_{L^{\infty}(\mathrm{J};\mathcal{H})} \leq C$ uniformly for all $k \in \mathbb{N}^*$, then $\mathbf{u} \in L^{\infty}(\mathrm{J};\mathcal{H})$ with $\|\mathbf{u}\|_{L^{\infty}(\mathrm{J};\mathcal{H})} \leq C$, where C is the same in both inequalities.*
2. *Let $\mathcal{F}$ be a Banach space with the Radon-Nikodym property with respect to the Lebesgue measure in $(\mathrm{J},\mathscr{B}(\mathrm{J}))$ and $\{\mathbf{u}_{\mathrm{k}}\}_{\mathrm{k}=1}^{\infty} \cup \{\mathbf{u}\} \subset \mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*)$ with $\mathbf{u}_{\mathrm{k}} \overset{*}{\rightharpoonup} \mathbf{u}$ in $\mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*)$.† If $\|\mathbf{u}_{\mathrm{k}}\|_{\mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*)} \leq \mathrm{C}$ uniformly for all $\mathrm{k} \in \mathbb{N}^*$, then $\|\mathbf{u}\|_{\mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*)} \leq \mathrm{C}$, where C is the same in both inequalities.*

Proof.

1. We derive that $\|\mathbf{u}(\mathrm{t})\|_{\mathcal{H}} \leq C$, for a.e. $\mathrm{t} \in \mathrm{J}$, from the (sequentially) weak lower semi-continuity of the norm. The result follows directly.
2. Let $v \in \mathcal{F}$ be such that $\|v\|_{\mathcal{F}} \leq 1$ and set $\mathbf{v} \in \mathrm{L}^1(\mathrm{J};\mathcal{F})$ the constant function with $\mathbf{v}(\mathrm{t}) := v$, for all $\mathrm{t} \in \mathrm{J}$. We have

$$\int_{\mathrm{s}}^{\mathrm{s+h}} \langle \mathbf{u}_{\mathrm{k}}, \mathbf{v}\rangle \mathrm{dt} \leq \mathrm{Ch}, \text{ for every } \mathrm{s} \in \mathrm{J}^{\mathrm{o}} \text{ and every sufficiently small } \mathrm{h} > 0.$$

Letting $\mathrm{k} \to \infty$, dividing both parts by h and then letting $\mathrm{h} \to 0$, we get $\langle \mathbf{u}(\mathrm{s}), v\rangle \leq \mathrm{C}$, for every $s \in J^{\mathrm{o}}$. Since v arbitrary, the proof is complete. □

Proposition 2. *Let $\alpha > 0$ and $r \in \mathbb{R}$, then*

$$V(x;\alpha,r) := \frac{1}{\alpha+2}x^{\alpha+2} + \frac{1}{2}rx^2 + \frac{\alpha}{\alpha+2}|r|^{\frac{\alpha+2}{\alpha}} \geq 0,\ \forall x \geq 0 \tag{2.6}$$

and also, for every $C_\alpha > \alpha + 2$ there exists an $A_\alpha > 0$, such that

$$x^{\alpha+2} \leq C_\alpha V(x),\ \forall x \geq A_\alpha. \tag{2.7}$$

Proof. For (2.6), if $r > 0$, the result is trivial. If $r < 0$ it is easy to show that $V(x) \geq V\left(|r|^{\frac{1}{\alpha}}\right) = 0$, for all $x \geq 0$. As for (2.7), we fix an arbitrary $C_\alpha > \alpha + 2$ and we set $f(x) = C_\alpha V(x) - x^{\alpha+2}$, for all $x \geq 0$. It is easy to show that $f(x) \geq f\left(\left(\frac{r}{C_\alpha-(\alpha+2)}\right)^{\frac{1}{\alpha}}\right)$, whereby the result follows since $f \xrightarrow{x\to\infty} \infty$. □

†That is, $\mathbf{u}_{\mathrm{k}} \overset{*}{\rightharpoonup} \mathbf{u}$ in $\sigma\left(\mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*), \mathrm{L}^1(\mathrm{J};\mathcal{F})\right)$. Note that $\mathrm{L}^{\infty}(\mathrm{J};\mathcal{F}^*) \cong \left(\mathrm{L}^1(\mathrm{J};\mathcal{F})\right)^*$ (see, eg, Diestel and Uhl[21], theorem 1,§IV.1).

Proposition 3. *Let $a, b \in \mathbb{C}$. Then*

$$a^{n+1} - a(n+1)b^n + nb^{n+1} = (a-b)^2\left(a^{n-1} + 2a^{n-2}b + \cdots + (n-1)ab^{n-2} + nb^{n-1}\right). \tag{2.8}$$

Proof. Direct application of the well-known identity

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right). \tag{2.9}$$

□

Proposition 4. *Let $\mathrm{U}_1 \subset \mathrm{U}_2 \subseteq \mathbb{R}^n$, $m \in \mathbb{N}_0$ and $\{u_k\}_{k=1}^{\infty} \cup \{u\} \subset \mathrm{H}^m(\mathrm{U}_2)$ such that $u_k \rightharpoonup u$ in $\mathrm{H}^m(\mathrm{U}_2)$. Then $\mathcal{R}_{\mathrm{U}_1}u_k \rightharpoonup \mathcal{R}_{\mathrm{U}_1}u$ in $\mathrm{H}^m(\mathrm{U}_1)$. The analogous result for L^p, with $p \in (1,\infty)$, instead of H^m also holds.*

Proof. We show the first result and in analogous fashion we get the second one. Let $v \in C_c^{\infty}(\mathrm{U}_1)$. Then, in view of the Riesz theorem, we have

$$\left\langle \mathcal{R}_{\mathrm{U}_1}u_k - \mathcal{R}_{\mathrm{U}_1}u, v\right\rangle = \sum_{|\beta|=0}^{m}\int_{\mathrm{U}_1} D^{\beta}\left(\mathcal{R}_{\mathrm{U}_1}u_k - \mathcal{R}_{\mathrm{U}_1}u\right)D^{\beta}\bar{v}dx = \sum_{|\beta|=0}^{m}\int_{\mathrm{U}_2} D^{\beta}(u_k-u)D^{\beta}\mathcal{E}_{\mathrm{U}_2}\bar{v}dx = \left\langle u_k - u, \mathcal{E}_{\mathrm{U}_2}v\right\rangle \to 0,$$

hence, the result follows from the density argument. □

3 | FORMULATION OF THE PROBLEM

Let $\alpha > 0$, $\zeta \in L^{\alpha+2}(\mathrm{U})$ and $r \in \mathbb{R}$. In view of (2.1) and the scaling invariant embedding $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow L^{\alpha+2}(\mathrm{U})$, we define $g : \mathrm{H}_0^1(\mathrm{U}) \to Y_\alpha := L^{\frac{\alpha+2}{\alpha+1}}(\mathrm{U}) + L^2(\mathrm{U}) \hookrightarrow \mathrm{H}^{-1}(\mathrm{U})$ to be the nonlinear and bounded operator such that

$$\langle g(u;\alpha,\zeta,r), v\rangle := \int_{\mathrm{U}} \left(|u+\zeta|^{\alpha} + r\right)(u+\zeta)\bar{v}dx, \text{ for } v \in \mathrm{H}_0^1(\mathrm{U}).$$

For the above operator we have the following estimate.

Proposition 5. *Let $u, v \in \mathrm{H}_0^1(\mathrm{U})$ and $\alpha = \alpha_2$. Then*

$$\|g(u) - g(v)\|_{Y_{\alpha_2}} \le K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{0,\alpha_2+2,\mathrm{U}}\right)\left(|u-v|_{0,\alpha_2+2,\mathrm{U}} + |u-v|_{0,2,\mathrm{U}}\right). \tag{3.1}$$

Proof. Applying (2.2) and the scaling invariant embedding $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow L^{\alpha_2+2}(\mathrm{U})$ we get us

$$\|g(u) - g(v)\|_{Y_{\alpha_2}} \le C\left(|u|_{1,2,\mathrm{U}}^c + |v|_{1,2,\mathrm{U}}^c + |\zeta|_{0,\alpha_2+2,\mathrm{U}}^c\right)|u-v|_{0,\alpha_2+2,\mathrm{U}} + C|u-v|_{0,2,\mathrm{U}}$$

and the result follows. □

Now, we further assume that $\zeta \in \mathrm{H}^1(\mathrm{U})$ and we define $\mathcal{N}[\cdot,\cdot] : \left(\mathrm{H}_0^1(\mathrm{U})\right)^2 \to \mathbb{C}$ to be the form which is associated with the operator $\Delta(\cdot+\zeta) - g$, such that $\mathcal{N}[u,v] := \langle \Delta(u+\zeta), v\rangle - \langle g(u), v\rangle$, for every $u, v \in \mathrm{H}_0^1(\mathrm{U})$.

We then restate the problem (1.3): we seek a solution $\mathbf{u}_{\mathrm{J}} \in L^{\infty}\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right) \cap W^{1,\infty}\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right)$ of

$$\begin{cases} i\left\langle \mathbf{u}_{\mathrm{J}}', v\right\rangle + \mathcal{N}[\mathbf{u}_{\mathrm{J}}, v] = 0, \ \forall v \in \mathrm{H}_0^1(\mathrm{U}), \text{ a.e. in } [\mathrm{J}] \\ \mathbf{u}_{\mathrm{J}}(0) = u_0. \end{cases} \tag{3.2}$$

We also provide an estimate for the form $\mathcal{N}$.

Proposition 6. *Let $u, v \in \mathrm{H}_0^1(\mathrm{U})$ and $\alpha = \alpha_2$. Then*

$$\left|\mathcal{N}[u,v]\right| \le K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{1,2,\mathrm{U}}, |\zeta|_{0,\alpha_2+2,\mathrm{U}}\right). \tag{3.3}$$

Proof. From (A4) ($p = q = 2$), (2.1) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha_2+2}(U)$, we get

$$\left|\mathcal{N}[u,v]\right| \leq C|D(u+\zeta)|_{0,2,U}|v|_{1,2,U} + C\,|u+\zeta|^c_{0,\alpha_2+2,U}\,|v|^c_{1,2,U},$$

hence the result follows. □

We further define the energy functional $\mathcal{E} : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\mathcal{E}(\cdot;\alpha,\zeta,r) := \frac{1}{2}\,|D(\cdot+\zeta)|^2_{0,2,U} + G(\cdot;\alpha,\zeta,r),$$

where $G : H_0^1(U) \to \mathbb{R}_+ \cup \{\infty\}$[‡], with

$$G(\cdot;\alpha,\zeta,r) := \int_U V(|\cdot+\zeta|;\alpha,r)\,dx.$$

For the functional G we have the following estimates.

Proposition 7. *Let $u, v \in H_0^1(U)$ and $\alpha = \alpha_2$. If $(G(u) - G(v)) \in \mathbb{R}$, then*

$$|G(u)-G(v)| \leq K\left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha_2+2,U}\right)\left(|u-v|_{0,\alpha_2+2,U} + |u-v|_{0,2,U}\right) \tag{3.4}$$

and

$$G(u) \leq K\left(|u|_{1,2,U}, |\zeta|_{1,2,U}, |\zeta|_{0,\alpha_2+2,U}, |U|\right). \tag{3.5}$$

Proof. From

$$G(u) - G(v) = \int_0^1 \frac{d}{ds}G(su+(1-s)v)\,ds = \int_0^1 \mathrm{Re}\,\langle g(su+(1-s)v), u-v\rangle\,ds, \tag{3.6}$$

(2.1) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha_2+2}(U)$ we get

$$|G(u)-G(v)| \leq C\left(|u|^c_{1,2,U} + |v|^c_{1,2,U} + |\zeta|^c_{0,\alpha_2+2,U} + |\zeta|^c_{1,2,U}\right)\left(|u-v|_{0,\alpha_2+2,U} + |u-v|_{0,2,U}\right).$$

As for the second estimate, we first notice that

$$G(0) = \int_U V(|\zeta|)\,dx = \frac{1}{\alpha_2+2}\,|\zeta|^{\alpha_2+2}_{0,\alpha_2+2,U} + \frac{1}{2}r\,|\zeta|^2_{0,2,U} + \frac{\alpha_2}{\alpha_2+2}|r|^{\frac{\alpha_2+2}{\alpha_2}}\,|U| \leq K\left(|\zeta|_{1,2,U}, |\zeta|_{0,\alpha_2+2,U}, |U|\right).$$

Then the result follows from the first and the triangle inequalities. □

3.1 | A special case of the operator

First, we assume that $\zeta \in L^{\alpha+2}(U) \cap L^\infty(U)$ and we extract two fine properties concerning the operator g.

Proposition 8. *Let $u, v \in H_0^1(U)$.*

i) If $n = 1$ and $\alpha > 0$, then $(g(u) - g(v)) \in L^2(U)$ with

$$|g(u)-g(v)|_{0,2,U} \leq K\left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{0,\infty,U}\right)|u-v|_{0,2,U}. \tag{3.7}$$

ii) If $n = 2$ and $\alpha > 0$, then $(g(u) - g(v)) \in L^2(U)$ with

$$|g(u)-g(v)|_{0,2,U} \leq K\left(|u|_{1,2,U}, |v|_{1,2,U}, |\zeta|_{0,\infty,U}\right)\left(|u-v|^{\frac{1}{2}}_{0,2,U} + |u-v|_{0,2,U}\right). \tag{3.8}$$

[‡]From (2.6) we get that G, hence $\mathcal{E}$ also, are positive-valued.

iii) If $n = 3$ and $\alpha = 2$, then

$$\|g(u) - g(v)\|_{Y_2} \leq K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}\right)\left(|u-v|_{0,4,\mathrm{U}} + |u-v|_{0,2,\mathrm{U}}\right) \tag{3.9}$$

and

$$\|g(u) - g(v)\|_{Y_2} \leq K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}\right)\left(|u-v|_{0,2,\mathrm{U}}^{\frac{1}{4}} + |u-v|_{0,2,\mathrm{U}}\right). \tag{3.10}$$

Proof. Let $n = 1, 2$. By simple application of (A2), we get

$$\int_{\mathrm{U}} |g(u) - g(v)|^2 dx \leq \int_{\mathrm{U}} C\left(|u|^{2\alpha} + |v|^{2\alpha}\right)|u-v|^2 dx + C\left(|\zeta|_{0,\infty,\mathrm{U}}^{c} + 1\right)|u-v|_{0,2,\mathrm{U}}^{2}.$$

For $n = 1$, we employ the scaling invariant embedding $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow L^\infty(\mathrm{U})$. For $n = 2$, we rewrite the above estimate as

$$\int_{\mathrm{U}} |g(u) - g(v)|^2 dx \leq \int_{\mathrm{U}} C\left(|u|^{2\alpha+1} + |v|^{2\alpha+1}\right)|u-v|\, dx + C\left(|\zeta|_{0,\infty,\mathrm{U}}^{c} + 1\right)|u-v|_{0,2,\mathrm{U}}^{2}$$

and we get the result from (A4) ($p = q = 2$) and the scaling invariant $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow L^\vartheta(\mathrm{U})$, for $\vartheta \in [2, \infty)$.

As for $n = 3$, the first estimate follows after simple calculations, from the scaling invariant $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow L^\vartheta(\mathrm{U})$, for $\vartheta \in [2, 6]$ and (2.2). The second one follows from the first and (2.3). □

We further notice that, by dealing as above, we also can have that

$$\|g(u) - g(v)\|_{L^{p_1}\left(\mathrm{J};L^{\frac{4}{3}}(\mathrm{U})\right)+L^{p_2}\left(\mathrm{J};L^2(\mathrm{U})\right)} \leq K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}\right)\left(|u-v|_{0,p_1,\mathrm{J};0,4,\mathrm{U}} + |u-v|_{0,p_2,\mathrm{J};0,2,\mathrm{U}}\right), \tag{3.11}$$

for every $u, v \in \mathrm{H}_0^1(\mathrm{U})$, and $p_1, p_2 \in [1, \infty]$, if $n = 3$ and $\alpha = 2$.

Proposition 9. *Let $u, v \in \mathrm{H}_0^1(\mathrm{U})$, $\alpha = \alpha_4$, $r = -\rho^\tau$, for an arbitrary $\rho > 0$ and $\left(|\zeta|^2 - \rho\right) \in L^2(\mathrm{U})$.*

i) If $n = 1, 2$, then g maps to $L^2(\mathrm{U})$ and

$$|g(u)|_{0,2,\mathrm{U}} \leq K\left(|u|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2 - \rho\right|_{0,2,\mathrm{U}}\right). \tag{3.12}$$

ii) If $n = 3$, then

$$\|g(u)\|_{Y_2} \leq K\left(|u|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2 - \rho\right|_{0,2,\mathrm{U}}\right). \tag{3.13}$$

Proof. We notice that $g(0) = \left(|\zeta|^{2\tau} - \rho^\tau\right)\zeta$, which belongs to $L^2(\mathrm{U})$. Indeed, by expanding via (2.9) we get $|g(0)|_{0,2,\mathrm{U}} \leq K\left(|\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2 - \rho\right|_{0,2,\mathrm{U}}\right)$. The results then follow from Proposition 8 and the triangle inequality. □

Let us now notice that ζ being in $L^{\alpha+2}(\mathrm{U})$ plays no essential role at any of the above results. Hence, for

$$\alpha = \alpha_4, r = r_s := -\rho^\tau \text{ for } \rho > 0 \text{ and } \zeta \in L^\infty(\mathrm{U}) \text{ with } \left(|\zeta|^2 - \rho\right) \in L^2(\mathrm{U}),$$

we define

$$g_s : \mathrm{H}_0^1(\mathrm{U}) \to \begin{cases} L^2(\mathrm{U}), & \text{if } n = 1, 2 \\ Y_2, & \text{if } n = 3, \end{cases} \text{ by } \langle g_s(u; \alpha_4, \zeta, r_s), v\rangle := \int_{\mathrm{U}} \left(|u+\zeta|^{\alpha_4} + r_s\right)(u+\zeta)\bar{v}\,dx, \text{ for } v \in \mathrm{H}_0^1(\mathrm{U}),$$

which satisfies the above estimates.

Now, we further assume that $\zeta \in X^1(\mathrm{U})$ and we define $\mathcal{N}_s[\cdot,\cdot] : \left(\mathrm{H}_0^1(\mathrm{U})\right)^2 \to \mathbb{C}$ to be the form which is associated with the operator $\Delta(\cdot + \zeta) - g_s$, such that $\mathcal{N}_s[u, v] := \langle \Delta(u+\zeta), v\rangle - \langle g_s(u), v\rangle$, for every $u, v \in \mathrm{H}_0^1(\mathrm{U})$. We note that apart from belonging to $\mathcal{L}\left(\mathrm{H}^1(\mathrm{U}); \mathrm{H}^{-1}(\mathrm{U})\right)$, $\Delta \in \mathcal{L}\left(X^1(\mathrm{U}); \mathrm{H}^{-1}(\mathrm{U})\right)$ also,[§] with its usual definition. From (3.12) and (A4) (for $p = q = 2$), we derive the following estimate

[§]Recall that if N_1 and N_2 are normed linear spaces, $\mathcal{L}(N_1; N_2)$ denotes the set of all linear and continuous maps $\mathcal{T} : N_1 \to N_2$.

$$\left|\mathcal{N}_s[u,v]\right| \leq K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right) \tag{3.14}$$

for every $u, v \in \mathrm{H}_0^1(\mathrm{U})$.

We also define the respective energy functional $\mathcal{E}_s : \mathrm{H}_0^1(\mathrm{U}) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\mathcal{E}_s(\cdot;\alpha_4,\zeta,r_s) := \frac{1}{2}\left(|D(\cdot+\zeta)|_{0,2,\mathrm{U}}^2 + G_s(\cdot;\alpha_4,\zeta,r_s)\right),$$

where $G_s : \mathrm{H}_0^1(\mathrm{U}) \to \mathbb{R}_+ \cup \{\infty\}$, with

$$G_s(\cdot;\alpha_4,\zeta,r_s) := \int_{\mathrm{U}} V(|\cdot+\zeta|;\alpha_4,r_s)\,dx,$$

for which we have

$$|G_s(u)-G_s(v)| \leq K\left(|u|_{1,2,\mathrm{U}}, |v|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right)|u-v|_{0,2,\mathrm{U}}^c, \tag{3.15}$$

from (3.6), (3.12), as well as (3.13) and the (2.3) if $n = 3$. Moreover, $G_s(0) < K\left(|\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right)$, which is obtained easily from (2.8). Hence, from (3.15) and the triangle inequality we get us

$$G_s(u) \leq K\left(|u|_{1,2,\mathrm{U}}, |\zeta|_{0,\infty,\mathrm{U}}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right), \tag{3.16}$$

for every $u \in \mathrm{H}_0^1(\mathrm{U})$ and so $\mathcal{E}_s, G_s : \mathrm{H}_0^1(\mathrm{U}) \to \mathbb{R}_+$.

4 | SOLUTIONS IN BOUNDED SETS

4.1 | Existence for $\mathbf{r} \in \mathbb{R}$

Here, we assume that $\mathrm{U} \subset \mathbb{R}^n$ is bounded.

Theorem 1. *Let $u_0 \in \mathrm{H}_0^1(\mathrm{U})$ and $\alpha = \alpha_3$. Then for every $T > 0$, there exists a solution $\mathbf{u}_{\mathrm{J}} \in L^\infty\left(\mathrm{J};\mathrm{H}_0^1(\mathrm{U})\right) \cap W^{1,\infty}\left(\mathrm{J};\mathrm{H}^{-1}(\mathrm{U})\right)$ of (3.2), such that*

$$|\mathbf{u}_{\mathrm{J}}|_{0,\mathrm{J};1,2,\mathrm{U}} + \left|\mathbf{u}_{\mathrm{J}}'\right|_{0,\mathrm{J};-1,\mathrm{U}} \leq \mathcal{K} := \begin{cases} K\left(|u_0|_{1,2,\mathrm{U}}, |\zeta|_{1,2,\mathrm{U}}, |\zeta|_{0,\alpha_3+2,\mathrm{U}}\right), & \text{if } r \geq 0 \\ K_{\mathrm{U}}\left(|u_0|_{1,2,\mathrm{U}}, |\zeta|_{1,2,\mathrm{U}}, |\zeta|_{0,\alpha_3+2,\mathrm{U}}\right), & \text{if } r < 0. \end{cases} \tag{4.1}$$

Proof.

Step 1: We make use of the standard Faedo-Galerkin method. It holds true that $\mathrm{H}_0^1(\mathrm{U}) \hookrightarrow\hookrightarrow \mathrm{L}^2(\mathrm{U})$; hence, there exists a countable subset of $\mathrm{H}_0^1(\mathrm{U}) \cap C^\infty(\mathrm{U})$, which is an orthogonal basis of $\mathrm{L}^2(\mathrm{U})$, eg, the complete set of eigenfunctions for the operator $-\Delta$ in $\mathrm{H}_0^1(\mathrm{U})$.[¶] Let $\{w_{\mathrm{k}}\}_{\mathrm{k}=1}^\infty \subset \mathrm{H}_0^1(\mathrm{U}) \cap C^\infty(\mathrm{U})$ be that basis, appropriately normalized so that $\{w_{\mathrm{k}}\}_{\mathrm{k}=1}^\infty$ be an orthonormal basis of $\mathrm{L}^2(\mathrm{U})$. Fixing any $m \in \mathbb{N}^*$, we define $\mathbf{d}_m \in C^\infty(\mathrm{J}_m;\mathbb{C}^m)$, with $\mathbf{d}_m(\mathrm{t}) := \left[d_m^1(\mathrm{t}), \cdots, d_m^m(\mathrm{t})\right]^{\mathrm{T}}$, to be the unique maximal solution (i.e. $\mathrm{J_m}$ with $0 \in \mathrm{J}_m^\circ$ is the maximal open interval on which the solution is defined) of the initial-value problem

$$\begin{cases} \mathbf{d}_m'(\mathrm{t}) = F_m(\mathbf{d}_m(\mathrm{t})), \ \forall \mathrm{t} \in \mathrm{J}_m^* \\ \mathbf{d}_m(0) = \left[(u_0,w_1), \cdots, (u_0,w_m)\right]^{\mathrm{T}}, \end{cases}$$

where $F_m \in C^\infty\left(\mathbb{R}^{2m};\mathbb{C}^m\right)$ with

$$F_m^k(\mathbf{z}) := i\mathcal{N}\left[\sum_{l=1}^m z_l w_l, w_k\right], \ \text{for all } \mathbf{z} \in \mathbb{C}^m, \ \text{with } \mathbf{z} := [z_1, \cdots, z_m]^{\mathrm{T}}, \ \text{and all } k \in \{1, \cdots, m\}.$$

¶This specific subset is an orthogonal basis of both $\mathrm{H}_0^1(\mathrm{U})$ and $\mathrm{L}^2(\mathrm{U})$.

Now, we define $\mathbf{u}_m \in C^\infty\left(\mathrm{J}_m; \mathrm{H}_0^1(\mathrm{U}) \cap C^\infty(\mathrm{U})\right)$, with

$$\mathbf{u}_m(\mathrm{t}) := \sum_{k=1}^{m} d_m^k(\mathrm{t})\, w_k.$$

It is then trivial to verify that

$$i\left(\mathbf{u}_m', w_k\right) + \mathcal{N}[\mathbf{u}_m, w_k] = 0, \text{ everywhere in } \mathrm{J}_m \text{ and for all } k \in \{1, \cdots, m\}. \tag{4.2}$$

Note that $u_{0\mathrm{m}} := u_\mathrm{m}(0, \cdot) = \mathbf{u}_\mathrm{m}(0) \to u_0$ in $\mathrm{L}^2(\mathrm{U})$ and $|u_{0\mathrm{m}}|_{0,2,\mathrm{U}} \leq |u_0|_{0,2,\mathrm{U}}$. Furthermore, $|u_{0\mathrm{m}}|_{1,2,\mathrm{U}} \leq |u_0|_{1,2,\mathrm{U}}$. Indeed, since $\sum_{\mathrm{k}=1}^{\mathrm{m}} a_\mathrm{k} w_\mathrm{k} \in \mathrm{span}\{w_\mathrm{k}\}_{\mathrm{k}=1}^{\mathrm{m}}$ for some $\{a_\mathrm{k}\}_{\mathrm{k}=1}^{\mathrm{m}} \subset \mathbb{C}$ we have that $(\Delta u_{0\mathrm{m}}, u_{0\mathrm{m}}) = (\Delta u_{0\mathrm{m}}, u_0)$; hence, we get

$$|\mathrm{D}u_{0\mathrm{m}}|_{0,2,\mathrm{U}}^2 = -\langle \Delta u_{0\mathrm{m}}, u_{0\mathrm{m}}\rangle = (\mathrm{D}u_{0\mathrm{m}}, \mathrm{D}u_0) \leq \frac{1}{2}|\mathrm{D}u_{0\mathrm{m}}|_{0,2,\mathrm{U}}^2 + \frac{1}{2}|\mathrm{D}u_0|_{0,2,\mathrm{U}}^2.$$

Therefore, $|\mathrm{D}u_{0\mathrm{m}}|_{0,2,\mathrm{U}} \leq |\mathrm{D}u_0|_{0,2,\mathrm{U}}$.

Step 2: We multiply the variational equation in (4.2) by $-\overline{d_\mathrm{m}^{\mathrm{k}\prime}}(\mathrm{t})$, sum for $\mathrm{k} = 1, \dots, m$, and take real parts of both sides, and thus obtain

$$\frac{d}{d\mathrm{t}}\mathcal{E}(\mathbf{u}_m) = 0, \text{ that is } \mathcal{E}(\mathbf{u}_m) \leq \mathcal{E}(u_0), \tag{4.3}$$

hence, if $r \geq 0$ we have that $|\mathbf{u}_m|_{1,2,\mathrm{U}} \leq \mathcal{K}$ and thus $\mathrm{J}_m \equiv \mathbb{R}$. Since $m \in \mathbb{N}^*$ is arbitrary, we get $|\mathbf{u}_m|_{1,2,\mathrm{U}} \leq \mathcal{K}$, for all $m \in \mathbb{N}^*$. Hence, we conclude that $\{\mathbf{u}_m\}_{m=1}^\infty$ is uniformly bounded in $L^\infty\left(\mathbb{R}; \mathrm{H}_0^1(\mathrm{U})\right)$, with

$$|\mathbf{u}_m|_{0;1,2,\mathrm{U}} \leq \mathcal{K}, \ \forall m \in \mathbb{N}^*. \tag{4.4}$$

If $r < 0$, from (3.5) we have that $|Du_m|_{0,2,\mathrm{U}} \leq \mathcal{K}$ and thus $\mathrm{J}_m \equiv \mathbb{R}$. Therefore, from the Poincaré inequality, we also get $|\mathbf{u}_m|_{0,2,\mathrm{U}} \leq \mathcal{K}$ and thus (4.4) follows for K_U instead of K.

Step 3: We fix an arbitrary $v \in \mathrm{H}_0^1(\mathrm{U})$ with $|v|_{1,2,\mathrm{U}} \leq 1$ and write $v = \mathcal{P}v \oplus (\mathrm{I} - \mathcal{P})v$, where $\mathcal{P}$ is the projection in $\mathrm{span}\{w_\mathrm{k}\}_{\mathrm{k}=1}^{\mathrm{m}}$. Since $\mathbf{u}_\mathrm{m}' \in \mathrm{span}\{w_\mathrm{k}\}_{\mathrm{k}=1}^{\mathrm{m}}$ and $\mathcal{N}\left[h, g\right]$ is (conjugate) linear for g, from the variational equation in (4.2), we get that

$$\langle \mathbf{u}_\mathrm{m}', v\rangle = \left(\mathbf{u}_\mathrm{m}', v\right) = \left(\mathbf{u}_\mathrm{m}', \mathcal{P}v\right) = \mathrm{i}\mathcal{N}[\mathbf{u}_\mathrm{m}, \mathcal{P}v].$$

Applying (3.3), we derive $\left|\langle \mathbf{u}_\mathrm{m}', v\rangle\right| \leq \mathcal{K}$. Hence, $\left\{\mathbf{u}_\mathrm{m}'\right\}_{\mathrm{m}=1}^\infty$ is uniformly bounded in $\mathrm{L}^\infty\left(\mathbb{R}; \mathrm{H}^{-1}(\mathrm{U})\right)$, with

$$\left|\mathbf{u}_\mathrm{m}'\right|_{0;-1,\mathrm{U}} \leq \mathcal{K}, \forall \mathrm{m} \in \mathbb{N}^*. \tag{4.5}$$

Step 4: Let $T > 0$. From (4.4), (4.5), Theorem 1.3.14 i) in[6] and Proposition 1 i), there exist a subsequence $\left\{\mathbf{u}_{m_l}\right\}_{l=1}^\infty \subseteq \{\mathbf{u}_m\}_{m=1}^\infty$ and a function $\mathbf{u}_\mathrm{J} \in L^\infty\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right) \cap W^{1,\infty}\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right)$, such that

$$\mathbf{u}_{m_l}(\mathrm{t}) \rightharpoonup \mathbf{u}_\mathrm{J}(\mathrm{t}) \text{ in } \mathrm{H}_0^1(\mathrm{U}), \text{ for every t} \in \mathrm{J} \text{ and also } |\mathbf{u}_\mathrm{J}|_{0,\mathrm{J};1,2,\mathrm{U}} \leq \mathcal{K}. \tag{4.6}$$

$\mathrm{H}^{-1}(\mathrm{U})$ is separable since $\mathrm{H}_0^1(\mathrm{U})$ is separable, hence by the Dunford-Pettis theorem (see, e.g.,[21 Theorem 1, section III.3]) we have $L^\infty\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right) \cong \left(L^1\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right)\right)^*$. In virtue of the above, from (4.5), the Banach-Alaoglu-Bourbaki theorem (see, e.g.,[22 Theorem 3.16] and 1 Proposition 1 ii), there exist a subsequence of $\left\{\mathbf{u}_{\mathrm{m}_\mathrm{l}}\right\}_{\mathrm{l}=1}^\infty$, which we still denote as such and a function $\mathbf{h} \in L^\infty\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right)$, such that

$$\mathbf{u}_{m_l}' \overset{*}{\rightharpoonup} \mathbf{h} \text{ in } L^\infty\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right) \text{ and also } |\mathbf{h}|_{0,\mathrm{J};-1,\mathrm{U}} \leq \mathcal{K}. \tag{4.7}$$

From the convergence in (4.6), Lemma 1.1, Chapter 3 in,[20] along with the Leibniz rule, we can derive that

$$\int_{\mathrm{J}} \left(\mathbf{u}'_{m_l}, \psi v\right) dt \to \int_{\mathrm{J}} \langle \mathbf{u}'_{\mathrm{J}}, \psi v \rangle dt, \quad \forall \psi \in C^1_c(\mathrm{J}), v \in \mathrm{H}^1_0(\mathrm{U}),$$

hence $\mathbf{h} \equiv \mathbf{u}'_{\mathrm{J}}$.

Step 5α: Since U is bounded, $\mathrm{H}^1_0(\mathrm{U}) \hookrightarrow\hookrightarrow L^2(\mathrm{U}) \hookrightarrow \mathrm{H}^{-1}(\mathrm{U})$ (see Remark 1). Hence, from (4.4), (4.5) and the Aubin-Lions-Simon lemma (see,[23 Theorem II.5.16]), there exist a subsequence of $\left\{\mathbf{u}_{m_l}\right\}_{l=1}^{\infty}$, which we still denote as such and a function $\mathbf{y} \in C\left(\mathrm{J}; L^2(\mathrm{U})\right)$, such that

$$\mathbf{u}_{m_l} \to \mathbf{y} \text{ in } C\left(\mathrm{J}; L^2(\mathrm{U})\right). \tag{4.8}$$

From the convergence in (4.6), we deduce that $\mathbf{y} \equiv \mathbf{u}_{\mathrm{J}}$.

Step 5β: Since $\alpha_3 \neq \frac{4}{n-2}$ for $n \geq 3$, from (4.4), (4.8) and (2.3) we have

$$\mathbf{u}_{m_l} \to \mathbf{u}_{\mathrm{J}} \text{ in } C\left(\mathrm{J}; L^{\alpha_3+2}(\mathrm{U})\right). \tag{4.9}$$

Step 5γ: From (3.1), (4.4), the bound in (4.6), (4.8) and (4.9) we get

$$g\left(\mathbf{u}_{m_l}\right) \to g\left(\mathbf{u}_{\mathrm{J}}\right) \text{ in } C\left(\mathrm{J}; Y_{\alpha_3}\right). \tag{4.10}$$

Step 6: Let now $\psi \in C^{\infty}_c(\mathrm{J})$ and fix $N \in \mathbb{N}^*$. We choose m_l such that $N \leq \mathrm{m}_l$ and $v \in \operatorname{span}\{w_k\}_{k=1}^N$, hence, by the linearity of the inner product, we get from (4.2) that

$$\int_{\mathrm{J}} i\left(\mathbf{u}'_{m_l}, \psi v\right) + \mathcal{N}\left[\mathbf{u}_{m_l}, \psi v\right] dt = 0.$$

We then pass to the weak, $*$-weak and strong limits (since $\psi v \in L^1\left(\mathrm{J}; \mathrm{H}^1_0(\mathrm{U})\right)$) to get

$$\int_{\mathrm{J}} i\left\langle \mathbf{u}'_{\mathrm{J}}, \psi v\right\rangle + \mathcal{N}\left[\mathbf{u}_{\mathrm{J}}, \psi v\right] dt = 0.$$

Since ψ is arbitrary, $\mathbf{u}_{\mathrm{J}}$ satisfies the variational equation in (3.2) for every $v \in \operatorname{span}\{w_k\}_{k=1}^N$. By the linear and continuous dependence on v, we get the desired result, after letting $N \to \infty$.

For the initial condition, we fix an arbitrary $\mathrm{t}_0 \in \mathrm{J}^*$. Let $v \in \mathrm{H}^1_0(\mathrm{U})$ be arbitrary and $\phi \in C^1(\mathrm{J})$ such that $\phi(0) \neq 0$ and $\phi(\mathrm{t}_0) = 0$. We then have from,[20] Lemma 1.1, Chapter 3, along with the Leibniz rule, that

$$\int_0^{\mathrm{t}_0} \left(\mathbf{u}'_{m_l}, \phi v\right) dt = -\int_0^{\mathrm{t}_0} \left(\mathbf{u}_{m_l}, \phi' v\right) dt - \left(u_{0m_l}, \phi(0) v\right),$$

$$\int_0^{\mathrm{t}_0} \left\langle \mathbf{u}'_{\mathrm{J}}, \phi v\right\rangle dt = -\int_0^{\mathrm{t}_0} \left(\mathbf{u}_{\mathrm{J}}, \phi' v\right) dt - \left(\mathbf{u}_{\mathrm{J}}(0), \phi(0) v\right).$$

Passing to the $*$-weak limits in the first equality, using that $u_{0\mathrm{m}} \to u_0$ in $L^2(\mathrm{U})$ and the fact that $v \in \mathrm{H}^1_0(\mathrm{U})$ is arbitrary, we derive that $\mathbf{u}_{\mathrm{J}}(0) = u_0$. □

4.2 | Uniqueness and globality

It is obvious that the uniqueness of the extracted local solutions implies the "globality" of those solutions.

Proposition 10. *Let $\mathbf{u}_J$ be as in Theorem 1 and $\zeta \in L^{\infty}(\mathrm{U})$. If $n = 1$, or $n = 2$ and $\alpha \in (0, 2]$, then $\mathbf{u}_J$ is unique everywhere in J.*

Proof. Let $u_{0,1} = u_{0,2}$ and $\mathbf{u}_{J,1}, \mathbf{u}_{J,2}$ be the corresponding solutions. Setting $\mathbf{w} := \mathbf{u}_{J,1} - \mathbf{u}_{J,2}$, we have

$$i\mathbf{w}' + \Delta\mathbf{w} - \left(g\left(\mathbf{u}_{J,1}\right) - g\left(\mathbf{u}_{J,2}\right)\right) \overset{H^{-1}(U)}{=} 0, \text{ a.e. in J.} \tag{4.11}$$

We apply the functional of (4.11) on $\mathbf{w}$, and take the imaginary parts of both parts to get us

$$\frac{d}{dt}|\mathbf{w}|^2_{0,2,U} \le C\left|\left\langle g\left(\mathbf{u}_{J,1}\right) - g\left(\mathbf{u}_{J,2}\right), \mathbf{w}\right\rangle\right|, \text{ a.e. in J.} \tag{4.12}$$

If $n = 1$, from (3.7) and (4.1) we deduce that

$$\frac{d}{dt}|\mathbf{w}|^2_{0,2,U} - C|\mathbf{w}|^2_{0,2,U} \le 0,$$

hence $\mathbf{w} \equiv 0$ everywhere in J, from Grönwall's inequality, since $\mathbf{w}(0) = 0$ and $\mathbf{w} \in C\left(J; L^2(U)\right)$.

If $n = 2$, we get, from (4.12) and the fact that $\zeta \in L^\infty(U)$,

$$|\mathbf{w}|^2_{0,2,U} \le C\int_0^t \left|\left\langle g\left(\mathbf{u}_{J,1}\right) - g\left(\mathbf{u}_{J,2}\right), \mathbf{w}\right\rangle\right| ds \le C\int_0^t \left(|\mathbf{w}|^2_{0,2,U} + \int_U \left(\left|\mathbf{u}_{J,1}\right|^2 + \left|\mathbf{u}_{J,2}\right|^2\right)|\mathbf{w}|^2 dx\right) ds,$$

for $t \in J^*$. In order to estimate the spatial integral, let $p > 2$. Then

$$\begin{aligned}\int_U \left|\mathbf{u}_{J,1}\right|^2|\mathbf{w}|^2 dx &= \int_U \left(\left|\mathbf{u}_{J,1}\right|^p|\mathbf{w}|^2\right)^{\frac{2}{p}}|\mathbf{w}|^{\frac{2p-4}{p}} dx \le \\ &\overset{(A4)}{\le} \left(\int_U \left|\mathbf{u}_{J,1}\right|^p|\mathbf{w}|^2 dx\right)^{\frac{2}{p}} |\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U} \overset{(A4)}{\le} \left(\int_U \left|\mathbf{u}_{J,1}\right|^{2p} dx\right)^{\frac{1}{p}} |\mathbf{w}|^{\frac{4}{p}}_{0,4,U}|\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U}.\end{aligned} \tag{4.13}$$

Applying (2.4) and (2.5), we get, from the scaling invariant embedding $H^1_0(U) \hookrightarrow L^4(U)$ and (4.1), that

$$\int_U \left|\mathbf{u}_{J,1}\right|^2|\mathbf{w}|^2 dx \le Cp|\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U}.$$

By repeating the above argument for the second term inside the parenthesis, we deduce, for p sufficiently large such that $|\mathbf{w}|^2_{0,2,U} \le p|\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U}$, that

$$|\mathbf{w}|^2_{0,2,U} \le Cp\int_0^t |\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U} ds, \quad \forall t \in J^*.$$

Therefore,

$$\int_0^t |\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U} ds \le (Ct)^{\frac{p}{2}}, \quad \forall t \in J^*.$$

Choosing $t_0 \in J^*$ sufficiently small, we have from Fatou's lemma that

$$\int_0^{t_0} |\mathbf{w}|^2_{0,2,U} dt \le \liminf_{p\to\infty}\int_0^{t_0} |\mathbf{w}|^{\frac{2p-4}{p}}_{0,2,U} ds \le 0,$$

which implies that $\mathbf{w} \equiv 0$ on $[0, t_0]$. By repeating the above argument as many times as needed in order to cover J^*, we get us uniqueness. ☐

4.3 | Special solutions

Theorem 2. *If we replace $\mathcal{N}$ with $\mathcal{N}_s$ in (3.2), then for every $T > 0$, there exists a solution $\mathbf{u}_J \in L^\infty\left(J; H^1_0(U)\right) \cap W^{1,\infty}\left(J; H^{-1}(U)\right)$ of (3.2), such that*

$$|\mathbf{u}_{\mathrm{J}}|_{0,\mathrm{J};1,2,\mathrm{U}} + |\mathbf{u}'_{\mathrm{J}}|_{0,\mathrm{J};-1,\mathrm{U}} \le K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right). \tag{4.14}$$

Proof. We use the proof of Theorem 1 as a pattern. Throughout the proof, we set g_s, $\mathcal{N}_s$, $\mathcal{E}_s$ and G_s instead of g, $\mathcal{N}$, $\mathcal{E}$ and G, respectively. The proof goes as the aforementioned one, with the following modifications:

Step 2: From (4.3) and (3.16) we get

$$|Du_m|_{0,2,\mathrm{U}} \le K\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right), \quad \forall m \in \mathbb{N}^* \tag{4.15}$$

and thus $\mathrm{J}_m \equiv \mathbb{R}$. Then, we multiply the variational equation in (4.2) by $\overline{d_{\mathrm{m}}^{\mathrm{k}}}(\mathrm{t})$, sum for $k = 1, \dots, \mathrm{m}$ and take imaginary parts of both sides, and thus obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}_m|^2_{0,2,\mathrm{U}} - \mathrm{Im}\,(D\zeta, Du_m) - \mathrm{Im}\left(\left(|\mathbf{u}_m+\zeta|^{\alpha_4}+r_s\right)(\mathbf{u}_m+\zeta), \mathbf{u}_m\right) = 0. \tag{4.16}$$

Applying (4.15) and expanding in view of (2.9), we deduce

$$\frac{d}{dt}|\mathbf{u}_m|^2_{0,2,\mathrm{U}} - K\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right)\left(1+|\mathbf{u}_m|^{\alpha_4+1}_{0,\alpha_4+1,\mathrm{U}}\right) \le 0. \tag{4.17}$$

In order to estimate the non constant term inside the parenthesis, we imitate the technique which has already been developed for the proof of Lemma 3.3 in.[13] We set $B = \left(A_{\alpha_4} + |\zeta|_{0,\infty,\mathrm{U}} + 1\right)^2$, where A_{α_4} is as in Proposition 2, $Q := \left\{x \in \mathrm{U} \,|\, |\mathbf{u}_m+\zeta| \le \sqrt{B}\right\}$ and $R := Q^{\complement} \cap \mathrm{U}$. Then

$$\begin{aligned}
|\mathbf{u}_m|^{\alpha_4+1}_{0,\alpha_4+1,\mathrm{U}} &= \int_Q |\mathbf{u}_m|^2|\mathbf{u}_m|^{\alpha_4-1}dx + \int_R |\mathbf{u}_m|^{\alpha_4+1}dx \le \\
&\overset{(A1)}{\le} \int_{\left\{x\in\mathrm{U}\,|\,|\mathbf{u}_m|\le\sqrt{B}+|\zeta|_{0,\infty,\mathrm{U}}\right\}} |\mathbf{u}_m|^2|\mathbf{u}_m|^{\alpha_4-1}dx + C\int_R |\zeta|^{\alpha_4+1} + |\mathbf{u}_m+\zeta|^{\alpha_4+1}dx \le \\
&\overset{(2.7)}{\le} \left(\sqrt{B}+|\zeta|_{0,\infty,\mathrm{U}}\right)^c |\mathbf{u}_m|^2_{0,2,\mathrm{U}} + C\,|\zeta|^c_{0,\infty,\mathrm{U}}\int_R dx + CG_s(\mathbf{u}_m) \le \\
&\le \left(\sqrt{B}+|\zeta|_{0,\infty,\mathrm{U}}\right)^c |\mathbf{u}_m|^2_{0,2,\mathrm{U}} + \frac{C\,|\zeta|^c_{0,\infty,\mathrm{U}}}{\left(\sqrt{B}-|\zeta|_{0,\infty,\mathrm{U}}\right)^c}|\mathbf{u}_m|^2_{0,2,\mathrm{U}} + CG_s(\mathbf{u}_m) \le \\
&\overset{(3.16)}{\le} K\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right)\left(1+|\mathbf{u}_m|^2_{0,2,\mathrm{U}}\right).
\end{aligned} \tag{4.18}$$

Let $T > 0$. From (4.17) and (4.18), we derive that

$$|\mathbf{u}_m|_{0,2,\mathrm{U}} \le K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right) \text{ in J}, \quad \forall m \in \mathbb{N}^*. \tag{4.19}$$

From (4.15) and (4.19) we conclude that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is uniformly bounded in $C\left(\mathrm{J}; \mathrm{H}^1_0(\mathrm{U})\right)$, with

$$|\mathbf{u}_m|_{0,\mathrm{J};1,2,\mathrm{U}} \le K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right), \quad \forall m \in \mathbb{N}^*. \tag{4.20}$$

Step 3: We make use of (3.14) instead of (3.3) to get that $\left\{\mathbf{u}'_m\right\}_{m=1}^{\infty}$ is uniformly bounded in $C\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right)$, with

$$|\mathbf{u}'_m|_{0,\mathrm{J};-1,\mathrm{U}} \le K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right), \quad \forall m \in \mathbb{N}^*. \tag{4.21}$$

Step 4 and step 5: We omit the first sentence in *STEP* 4. We refer to (4.20) instead of (4.4), to (4.21) instead of (4.5). We also set $K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2-\rho\right|_{0,2,\mathrm{U}}\right)$ in place of $\mathcal{K}$. We omit *STEP* 5β. In *STEP* 5γ we refer to (3.7), (3.8) and (3.10) instead of (3.1), we omit the reference in (4.9) and also we replace Y_{α_3} with $L^2(\mathrm{U})$ if $n = 1, 2$ and Y_2 if $n = 3$. □

5 | SOLUTIONS IN UNBOUNDED SETS

Theorem 3. *Let* $\mathrm{U} \subseteq \mathbb{R}^n$ *to be unbounded and* $u_0 \in \mathrm{H}_0^1(\mathrm{U})$. *Then the conclusion of Theorem 2 still holds.*

Proof.

Step 1: Since U open, we fix an arbitrary $B_\varrho(x_0) \subset \mathrm{U}$. Let $u_{0,k} := \mathcal{R}_{\mathrm{U}}\eta_k u_0$, for all $k \in \mathbb{N}^*$, where $\{\eta_k\}_{k=1}^\infty$ as in Appendix B. Hence, for all $k \in \mathbb{N}^*$, we have that

$$\left|u_{0,k}\right|_{1,2,\mathrm{U}} \leq C|u_0|_{1,2,\mathrm{U}}. \tag{5.1}$$

We also notice that

$$u_{0,k} = 0, \text{ in } B_{a_k}(x_0)^{\complement} \cap \mathrm{U},$$

hence, by fixing a $\delta = \delta(\varrho, a_1)$ such that $\delta < a_1 - \varrho$ and by setting $B_k := B_{a_k+\delta}(x_0) \cap \mathrm{U}$, for every $k \in \mathbb{N}^*$, we obtain that $\left\{\mathcal{R}_{B_k} u_{0,k}\right\}_{k=1}^\infty \subset \mathrm{H}_0^1(B_k)$ (see also Lemma 9.5 in[22]). Moreover,

$$u_{0,k} \to u_0 \text{ in } L^2(\mathrm{U}). \tag{5.2}$$

Indeed,

$$\left|u_{0,k} - u_0\right|_{0,2,\mathrm{U}} = |(\eta_k - 1)u_0|_{0,2,\mathrm{U}} \leq |u_0|_{0,2,B_{a_{k-1}}(x_0)^{\complement}\cap\mathrm{U}} \to 0.$$

Step 2α: Fixing any $k \in \mathbb{N}^*$, we consider (3.2) (with $\mathcal{N}_{\mathrm{s}}$ instead of $\mathcal{N}$) in $U = B_k$, where we take $\mathcal{R}_{B_k}u_{0,k}$ as our initial datum. Let $T > 0$. From the proof of Theorem 2 it follows that there exist $\left\{\mathbf{u}_m^k\right\}_{m=1}^\infty \subset C^\infty\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U}) \cap C^\infty(\mathrm{U})\right)$, such that

$$\left|\mathbf{u}_m^k\right|_{0,\mathrm{J};1,2,B_k} + \left|\mathbf{u}_m^{k\prime}\right|_{0,\mathrm{J};-1,B_k} \leq K_{\mathrm{J}}\left(\left|u_{0,k}\right|_{1,2,B_k}, \|\zeta\|_{X^1(B_k)}, \left||\zeta|^2 - \rho\right|_{0,2,B_k}\right), \quad \forall m \in \mathbb{N}^*. \tag{5.3}$$

From (5.1) and (5.3) we deduce that

$$\left|\mathbf{u}_m^k\right|_{0,\mathrm{J};1,2,B_k} + \left|\mathbf{u}_m^{k\prime}\right|_{0,\mathrm{J};-1,B_k} \leq K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2 - \rho\right|_{0,2,\mathrm{U}}\right), \quad \forall m \in \mathbb{N}^*. \tag{5.4}$$

For convenience, we set $\mathcal{K} := K_{\mathrm{J}}\left(|u_0|_{1,2,\mathrm{U}}, \|\zeta\|_{X^1(\mathrm{U})}, \left||\zeta|^2 - \rho\right|_{0,2,\mathrm{U}}\right)$.

Step 2β: From the fact that the local regularity of the eigenfunctions at the boundary depends on the local smoothness of the boundary and also that $\partial B_k \backslash \partial U \in C^\infty$, we get that $\mathbf{u}_m^k, \mathbf{u}_m^{k\prime} \in C^\infty(\partial B_k \backslash \partial \mathrm{U})$, with

$$\mathcal{R}_{\partial B_k\backslash\partial\mathrm{U}}\mathbf{u}_m^k = \mathcal{R}_{\partial B_k\backslash\partial\mathrm{U}}\mathbf{u}_m^{k\prime} = 0, \quad \forall m \in \mathbb{N}^*.$$

Therefore, the extensions by zero $\mathbf{v}_m^k := \mathcal{E}_{\mathrm{U}}\mathbf{u}_m^k$, for all $m \in \mathbb{N}^*$, are continuous in $\partial B_k \backslash \partial U$ and thus $\left\{\mathbf{v}_m^k\right\}_{m=1}^\infty \subset C^\infty\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right)$. Evidently,

$$\left|\mathbf{v}_m^k\right|_{0,\mathrm{J};1,2,\mathrm{U}} = \left|\mathbf{u}_m^k\right|_{0,\mathrm{J};1,2,B_k} \text{ and } \left|\mathbf{v}_m^{k\prime}\right|_{0,\mathrm{J};-1,\mathrm{U}} = \left|\mathbf{u}_m^{k\prime}\right|_{0,\mathrm{J};-1,B_k},$$

hence, from (5.4), we get that

$$\left|\mathbf{v}_m^k\right|_{0,\mathrm{J};1,2,\mathrm{U}} + \left|\mathbf{v}_m^{k\prime}\right|_{0,\mathrm{J};-1,\mathrm{U}} \leq \mathcal{K}, \quad \forall m \in \mathbb{N}^*.$$

Step 2γ: Since $k \in \mathbb{N}^*$ is arbitrary, $\left\{\mathbf{v}_m^k\right\}_{k,m=1}^\infty \subset C^\infty\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right)$ with

$$\left|\mathbf{v}_m^k\right|_{0,\mathrm{J};1,2,\mathrm{U}} + \left|\mathbf{v}_m^{k\prime}\right|_{0,\mathrm{J};-1,\mathrm{U}} \leq \mathcal{K}, \quad \forall k, m \in \mathbb{N}^*, \tag{5.5}$$

and we intend to pass to the limits $k, \mathrm{m} \to \infty$.

Step 3α: We fix the diagonal subsequence $\left\{\mathbf{v}_m^m\right\}_{m=1}^{\infty}$. Dealing as in *STEP* 4 of the proof of Theorem 1, there exist a subsequence $\left\{\mathbf{v}_{m_l}^{m_l}\right\}_{l=1}^{\infty} \subseteq \left\{\mathbf{v}_m^m\right\}_{m=1}^{\infty}$ and a function $\mathbf{u}_{\mathrm{J}} \in L^{\infty}\left(\mathrm{J}; \mathrm{H}_0^1(\mathrm{U})\right) \cap W^{1,\infty}\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right)$, such that

$$\begin{cases} \mathbf{v}_{m_l}^{m_l}(\mathrm{t}) \rightharpoonup \mathbf{u}_{\mathrm{J}}(\mathrm{t}) & \text{in } \mathrm{H}_0^1(\mathrm{U}), \text{ for every } \mathrm{t} \in \mathrm{J} \\ \mathbf{v}_{m_l}^{m_l\prime} \overset{*}{\rightharpoonup} \mathbf{u}_{\mathrm{J}}' & \text{in } L^{\infty}\left(\mathrm{J}; \mathrm{H}^{-1}(\mathrm{U})\right), \end{cases} \quad \text{and also } |\mathbf{u}_{\mathrm{J}}|_{0,\mathrm{J};1,2,\mathrm{U}} + \left|\mathbf{u}_{\mathrm{J}}'\right|_{0,\mathrm{J};-1,\mathrm{U}} \leq \mathcal{K}. \tag{5.6}$$

Step 3β: From (3.7), (3.8), (3.10), (5.5) and Lemma 3.3.6 in[6] we deduce that $\left\{g_s\left(\mathbf{v}_{m_l}^{m_l}\right)\right\}_{l=1}^{\infty}$ is bounded in $C^{0,\frac{1}{2}}\left(\mathrm{J}; L^2(\mathrm{U})\right)$ if $n = 1, 2$, or in $C^{0,\frac{1}{2}}(\mathrm{J}; Y_2)$ if $n = 3$. Hence, from Proposition 1.1.2 in the same reference book, there exist a subsequence of $\left\{\mathbf{v}_{m_l}^{m_l}\right\}_{l=1}^{\infty}$, which we still denote as such, and a function $\mathbf{f} \in C\left(\mathrm{J}; L^2(\mathrm{U})\right)$ if $n = 1, 2$, or $\mathbf{f} \in C(\mathrm{J}; Y_2)$ if $n = 3$, such that

$$g_s\left(\mathbf{v}_{m_l}^{m_l}(\mathrm{t})\right) \rightharpoonup \mathbf{f}(\mathrm{t}) \text{ in } \begin{cases} L^2(\mathrm{U}), & \text{if } n = 1, 2 \\ Y_2, & \text{if } n = 3, \end{cases} \quad \text{for every } \mathrm{t} \in \mathrm{J}. \tag{5.7}$$

We then deal as in *STEP* 6 of the proof of Theorem 1 to get

$$i\mathbf{u}_{\mathrm{J}}' + \Delta(\mathbf{u}_{\mathrm{J}} + \zeta) + \mathbf{f} \overset{\mathrm{H}^{-1}(\mathrm{U})}{=} 0, \text{ a.e. in J}. \tag{5.8}$$

Step 3γ: Let Ω be any bounded $\subset$ U, such that $\mathrm{H}^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, e.g. a ball (see Remark 1). From (5.7) and Proposition 4 we have

$$g_s\left(\mathcal{R}_{\Omega}\mathbf{v}_{m_l}^{m_l}(\mathrm{t})\right) = \mathcal{R}_{\Omega}g_s\left(\mathbf{v}_{m_l}^{m_l}(\mathrm{t})\right) \rightharpoonup \mathcal{R}_{\Omega}\mathbf{f}(\mathrm{t}) \text{ in } \begin{cases} L^2(\mathrm{U}), & \text{if } n = 1, 2 \\ Y_2, & \text{if } n = 3, \end{cases} \quad \text{for every } \mathrm{t} \in \mathrm{J}. \tag{5.9}$$

On the other hand, from the first convergence in (5.6),

$$\mathcal{R}_{\Omega}\mathbf{v}_{m_l}^{m_l}(\mathrm{t}) \rightharpoonup \mathcal{R}_{\Omega}\mathbf{u}_{\mathrm{J}}(\mathrm{t}) \text{ in } \mathrm{H}_0^1(\Omega), \text{ for every } \mathrm{t} \in \mathrm{J},$$

hence, from the weak lower semi-continuity of the H^1-norm and the aforementioned compact embedding, we obtain that there exist a subsequence of $\left\{\mathbf{v}_{m_l}^{m_l}\right\}_{l=1}^{\infty}$, which we still denote as such, for which we have

$$\mathcal{R}_{\Omega}\mathbf{v}_{m_l}^{m_l}(\mathrm{t}) \to \mathcal{R}_{\Omega}\mathbf{u}_{\mathrm{J}}(\mathrm{t}) \text{ in } L^2(\Omega), \text{ for every } \mathrm{t} \in \mathrm{J}.$$

From (3.7), (3.8), (3.10) and the latter convergence we get

$$g_s\left(\mathcal{R}_{\Omega}\mathbf{v}_{m_l}^{m_l}(\mathrm{t})\right) \to g_s(\mathcal{R}_{\Omega}\mathbf{u}_{\mathrm{J}}(\mathrm{t})) = \mathcal{R}_{\Omega}g_s(\mathbf{u}_{\mathrm{J}}(\mathrm{t})) \text{ in } \begin{cases} L^2(\mathrm{U}), & \text{if } n = 1, 2 \\ Y_2, & \text{if } n = 3, \end{cases} \quad \text{for every } \mathrm{t} \in \mathrm{J}. \tag{5.10}$$

From (5.9) and (5.10) we derive $\mathcal{R}_{\Omega}g_s(\mathbf{u}_T) \equiv \mathcal{R}_{\Omega}\mathbf{f}$ and since Ω is arbitrary, $g_s(\mathbf{u}_{\mathrm{J}}) \equiv \mathbf{f}$, hence (5.8) becomes

$$i\mathbf{u}_{\mathrm{J}}' + \Delta(\mathbf{u}_{\mathrm{J}} + \zeta) + g_s(\mathbf{u}_{\mathrm{J}}) \overset{\mathrm{H}^{-1}(\mathrm{U})}{=} 0, \text{ a.e. in J}.$$

Step 4: As far as the initial condition is concerned, we employ (5.1) when we deal as in *STEP* 6. $\mathbf{u}_{\mathrm{J}}(0) = u_0$ then follows. □

5.1 | Uniqueness and Globality

Again, it suffices to show uniqueness to also gain globality. Here, we make use of the estimate (3.11) for $n = 3$.

Proposition 11. *Let $\mathbf{u}_J$ be a solution as above. If $n = 1$, or $n = 2$ and $\alpha_4 = 2$, or $n = 3$ and $\mathrm{U} = \mathbb{R}^3$, then $\mathbf{u}_J$ is unique everywhere in J.*

Proof. For the first two cases the proof is exactly as of Proposition 10. For the third case, let $\mathbf{w}$ be as in the aforementioned Proposition. We first note that $\mathbf{w}$ takes the form (see, e.g.,[6])

$$\mathbf{w} = i\int_0^{\mathrm{t}} \mathcal{T}(\mathrm{t}-s)\left(g_s\left(\mathbf{u}_{\mathrm{J},1}\right) - g_s\left(\mathbf{u}_{\mathrm{J},2}\right)\right) ds, \text{ a.e. in J}, \tag{5.11}$$

where $\mathcal{T}(\mathrm{t}) \in \mathcal{L}\left(L^{\frac{p}{p-1}}\left(\mathbb{R}^3\right); L^p\left(\mathbb{R}^3\right)\right)$ for $p \in [2,\infty]$ and $\mathrm{t} \in \mathrm{J}^*$, with

$$\mathcal{T}(\mathrm{t})v = \left(\frac{1}{4\pi i\mathrm{t}}\right)^{\frac{3}{2}} e^{\frac{i|\cdot|^2}{4\mathrm{t}}} * v, \quad \forall v \in L^{\frac{p}{p-1}}\left(\mathbb{R}^3\right) \text{ and } \|\mathcal{T}(\mathrm{t})\|_{\mathcal{L}\left(L^{\frac{p}{p-1}}(\mathbb{R}^3);L^p(\mathbb{R}^3)\right)} \leq (4\pi\mathrm{t})^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

Let $\mathrm{t}_0 \in \mathrm{J}^*$ and $\mathrm{J}_0 = [0,\mathrm{t}_0]$. Now, the pairs $\left(\frac{8}{3},4\right)$ and $(\infty,2)$ are admissible.[#] From (5.11), (3.11) for $p_1 = \frac{8}{5}$ and $p_2 = 1$, as well as the Strichartz estimate (see, e.g.,[6 Theorem 2.3.3,] or,[9 Theorem 2.3]), we have

$$|\mathbf{w}|_{0,\infty,\mathrm{J}_0;0,2} \leq C\left(|\mathbf{w}|_{0,\frac{8}{5},\mathrm{J}_0;0,4} + |\mathbf{w}|_{0,1,\mathrm{J}_0;0,2}\right)$$
$$|\mathbf{w}|_{0,\frac{8}{3},\mathrm{J}_0;0,4} \leq C\left(|\mathbf{w}|_{0,\frac{8}{5},\mathrm{J}_0;0,4} + |\mathbf{w}|_{0,1,\mathrm{J}_0;0,2}\right).$$

Applying (A4) (for $(p,q) = \left(\frac{5}{3},\frac{5}{2}\right)$ and also $(p,q) = (\infty,1)$), the above estimates yield

$$|\mathbf{w}|_{0,\infty,\mathrm{J}_0;0,2} \leq C\mathrm{t}_0^c\left(|\mathbf{w}|_{0,\frac{8}{3},\mathrm{J}_0;0,4} + |\mathbf{w}|_{0,\infty,\mathrm{J}_0;0,2}\right)$$
$$|\mathbf{w}|_{0,\frac{8}{3},\mathrm{J}_0;0,4} \leq C\mathrm{t}_0^c\left(|\mathbf{w}|_{0,\frac{8}{3},\mathrm{J}_0;0,4} + |\mathbf{w}|_{0,\infty,\mathrm{J}_0;0,2}\right).$$

For sufficiently small t_0, we then get

$$|\mathbf{w}|_{0,\infty,\mathrm{J}_0;0,2} + |\mathbf{w}|_{0,\frac{8}{3},\mathrm{J}_0;0,4} = 0,$$

therefore, $\mathbf{w} = 0$ in $[0,\mathrm{t}_0]$. Uniqueness follows by repeating the above argument as many times as needed in order to cover J. □

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[#] A pair $(p,q) \in [2,\infty]^2$ is called admissible if $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$ and $(p,q,n) \neq (2,\infty,2)$.

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APPENDIX A: USEFUL INEQUALITIES

We first state two elementary inequalities.

Theorem 4. *Let* $p > 0$*,* $\alpha \geq 0$ *and* $z_1, z_2 \in \mathbb{C}$*. Then*

$$|z_1 + z_2|^{\mathrm{p}} \leq \mathrm{C}\left(|z_1|^{\mathrm{p}} + |z_2|^{\mathrm{p}}\right) \tag{A1}$$

and

$$\left||z_1|^{\alpha} z_1 - |z_2|^{\alpha} z_2\right| \leq \mathrm{C}\,|z_1 - z_2|\left(|z_1|^{\alpha} + |z_2|^{\alpha}\right). \tag{A2}$$

We also recall Young inequality with constant ϵ and Hölder inequality.

Theorem 5. *Let* $\mathrm{a}, \mathrm{b} \in [0, \infty)$ *and* $\mathrm{p}, \mathrm{q} \in (1, \infty)$*, such that* $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$*. Then*

$$\mathrm{ab} \leq \epsilon \mathrm{a}^{\mathrm{p}} + \mathrm{C}\mathrm{b}^{\mathrm{q}}, \forall \epsilon > 0, \textit{where } \mathrm{C} = \frac{1}{(\epsilon \mathrm{p})^{\frac{\mathrm{q}}{\mathrm{p}}}\mathrm{q}}. \tag{A3}$$

Theorem 6. *Let* $\mathrm{p}, \mathrm{q} \in [1, \infty]$, *such that* $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$, $u \in \mathrm{L}^{\mathrm{p}}(\mathrm{U})$ *and* $v \in \mathrm{L}^{\mathrm{q}}(\mathrm{U})$. *Then*

$$\int_{\mathrm{U}} |uv|\, dx \leq |u|_{0,\mathrm{p},\mathrm{U}} |v|_{0,\mathrm{q},\mathrm{U}}. \tag{A4}$$

The following result is a version of the Gagliardo-Nirenberg interpolation inequality taken from.[10]

Theorem 7. *Let* $q, r \in [1, \infty]$ *and* $j, m \in \mathbb{N}_0$ *such that* $j < m$. *Then*

$$\sum_{|\beta|=j} \left|D^{\beta} u\right|_{0,p} \leq C \left(\sum_{|\beta|=m} \left|D^{\beta} u\right|_{0,r} \right)^{\theta} |u|_{0,q}^{1-\theta}, \quad \forall u \in C_c^m\left(\mathbb{R}^n\right), \tag{A5}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}, \quad \forall \theta \in \left[\frac{j}{m}, 1\right],$$

where C is a constant depending only on n, m, j, q, r and θ.

There is an exception:

$$\textit{If } m - j - \frac{n}{r} \in \mathbb{N}_0, \ \textit{then (A5) holds only for all } \theta \in \left[\frac{j}{m}, 1\right).$$

Remark 1. The following Sobolev embeddings are true (see, eg, Brezis[22])

$$\begin{aligned}
&\mathrm{W}^{\mathrm{m,p}}\left(\mathbb{R}^{\mathrm{n}}\right) \hookrightarrow \mathrm{L}^{\mathrm{q}}\left(\mathbb{R}^{\mathrm{n}}\right), \text{where } \frac{1}{\mathrm{q}} = \frac{1}{\mathrm{p}} - \frac{\mathrm{m}}{\mathrm{n}} \text{ with } \mathrm{mp} < \mathrm{n},\\
&\mathrm{W}^{\mathrm{m,p}}\left(\mathbb{R}^{\mathrm{n}}\right) \hookrightarrow \mathrm{L}^{\mathrm{q}}\left(\mathbb{R}^{\mathrm{n}}\right), \text{where } \mathrm{q} \in [\mathrm{p}, \infty) \text{ with } \mathrm{mp} = \mathrm{n},\\
&\mathrm{W}^{\mathrm{m,p}}\left(\mathbb{R}^{\mathrm{n}}\right) \hookrightarrow \mathrm{L}^{\infty}\left(\mathbb{R}^{\mathrm{n}}\right), \text{with } \mathrm{mp} > \mathrm{n}.
\end{aligned}$$

It is then easy to see that the following embeddings

$$\begin{aligned}
&\mathrm{W}_0^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\mathrm{q}}(\mathrm{U}), \text{where } \frac{1}{\mathrm{q}} = \frac{1}{\mathrm{p}} - \frac{\mathrm{m}}{\mathrm{n}} \text{ with } \mathrm{mp} < \mathrm{n},\\
&\mathrm{W}_0^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\mathrm{q}}(\mathrm{U}), \text{where } \mathrm{q} \in [\mathrm{p}, \infty) \text{ with } \mathrm{mp} = \mathrm{n},\\
&\mathrm{W}_0^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\infty}(\mathrm{U}), \text{with } \mathrm{mp} > \mathrm{n}
\end{aligned}$$

are also true for every $\mathrm{U} \subseteq \mathbb{R}^{\mathrm{n}}$. These embeddings are, additionally, scaling invariant, since, for every inequality of the corresponding embedding, we have $\mathrm{C}_{\mathrm{U}} = \mathrm{C}_{\mathbb{R}^{\mathrm{n}}} = \mathrm{C}$ for every $\mathrm{U} \subseteq \mathbb{R}^{\mathrm{n}}$. Indeed, we only have to notice that

$$\mathcal{E}\mathrm{C}_{\mathrm{c}}^{\mathrm{m}}(\mathrm{U}) \subset \mathrm{C}_{\mathrm{c}}^{\mathrm{m}}\left(\mathbb{R}^{\mathrm{n}}\right) \text{ and } \left|\mathrm{D}^{\beta} u\right|_{0,\mathrm{p},\mathrm{U}} = \left|D^{\beta} \mathcal{E} u\right|_{0,\mathrm{p}}, \forall \begin{cases} u \in \mathrm{C}_{\mathrm{c}}^{\mathrm{m}}(\mathrm{U}) \\ \text{multi-index } \beta \text{ such that } 0 \leq |\beta| \leq m \\ \mathrm{p} \in [1, \infty], \end{cases}$$

(see also Adams and Fournier[24]). By the use of the above argumentation, we see that Theorem 7 is also true for every $u \in W_0^{m,p}(\mathrm{U})$ and also (A5) is scaling invariant in the aforementioned space.

We note that the embeddings

$$\begin{aligned}
&\mathrm{W}^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\mathrm{q}}(\mathrm{U}), \ \text{where } \frac{1}{\mathrm{q}} = \frac{1}{\mathrm{p}} - \frac{\mathrm{m}}{\mathrm{n}} \text{ with } \mathrm{mp} < \mathrm{n},\\
&\mathrm{W}^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\mathrm{q}}(\mathrm{U}), \text{where } \mathrm{q} \in [\mathrm{p}, \infty) \text{ with } \mathrm{mp} = \mathrm{n},\\
&\mathrm{W}^{\mathrm{m,p}}(\mathrm{U}) \hookrightarrow \mathrm{L}^{\infty}(\mathrm{U}), \text{with } \mathrm{mp} > \mathrm{n},
\end{aligned}$$

are true for appropriate choices of $\mathrm{U} \subseteq \mathbb{R}^{\mathrm{n}}$. Possible such choices are as follows: (1) $\mathbb{R}_+^{\mathrm{n}}$, (2) any U that satisfies the cone condition, (3) any bounded U with a locally Lipschitz boundary, (4) any Lipschitz domain, etc (see textbooks[22,24,25] for definitions and more examples/counterexamples). Evidently, these embeddings and the corresponding inequalities depend on the choice of U. Moreover, for the above special cases of $\mathrm{U} \subseteq \mathbb{R}^{n}$, the (compact) Rellich-Kondrachov embeddings

$$
\begin{aligned}
&W^{1,p}(\mathrm{U}) \hookrightarrow\hookrightarrow L^q(\mathrm{U}), \text{ where } q \in [1, p^*) \text{ and } \frac{1}{q^*} = \frac{1}{p} - \frac{1}{n} \text{ with } p < n,\\
&W^{1,p}(\mathrm{U}) \hookrightarrow\hookrightarrow L^q(\mathrm{U}), \text{ where } q \in [p, \infty) \text{ with } p = n,\\
&W^{1,p}(\mathrm{U}) \hookrightarrow\hookrightarrow C(\bar{U}), \text{ with } p > n,
\end{aligned}
$$

are true if, in addition, U is bounded. On the contrary, if we replace $W^{1,p}(\mathrm{U})$ with $W_0^{1,p}(\mathrm{U})$, there is no restriction on the choice of U, except for being bounded. The latter follows from the fact that we only need the aforementioned continuous embeddings and the boundedness of U, in order to prove the compact ones.

APPENDIX B: CUT-OFF FUNCTIONS

Let $f \in \mathrm{C}^\infty(\mathbb{R})$ with

$$
f(\mathrm{t}) := \begin{cases} e^{-\frac{1}{\mathrm{t}}}, & \mathrm{t} > 0 \\ 0, & \mathrm{t} \leq 0, \end{cases}
$$

$\mathrm{B}_\varrho(\mathrm{x}_0) \subset \mathbb{R}^\mathrm{n}$ fixed and $\{a_\mathrm{k}\}_{\mathrm{k}=1}^\infty \subset \mathbb{R}_+$ increasing, such that $a_\mathrm{k} > \varrho$ for all $\mathrm{k} \in \mathbb{N}^*$ and $a_\mathrm{k} \nearrow \infty$. We define $\{\eta_k\}_{k=1}^\infty \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^\mathrm{n})$ by

$$
\eta_k(x; x_0, a_{k-1}, a_k) := \frac{f(a_\mathrm{k} - |\mathrm{x} - \mathrm{x}_0|)}{f(|\mathrm{x} - \mathrm{x}_0| - a_{\mathrm{k}-1}) + f(a_\mathrm{k} - |\mathrm{x} - \mathrm{x}_0|)}, \forall \mathrm{x} \in \mathbb{R}^\mathrm{n}, \mathrm{k} \in \mathbb{N}^* \backslash \{1\}
$$

and

$$
\eta_1\left(x; B_\varrho(x_0), a_1\right) := \frac{f(a_1 - |\mathrm{x} - \mathrm{x}_0|)}{f(|\mathrm{x} - \mathrm{x}_0| - \varrho) + f(a_1 - |\mathrm{x} - \mathrm{x}_0|)}, \forall \mathrm{x} \in \mathbb{R}^\mathrm{n}.
$$

It is trivial to show that

$$
\eta_k(x) = \begin{cases} 1, & \mathrm{x} \in \overline{\mathrm{B}_{a_{\mathrm{k}-1}}(\mathrm{x}_0)} \\ 0, & \mathrm{x} \in \mathbb{R}^\mathrm{n} \backslash \mathrm{B}_{a_\mathrm{k}}(\mathrm{x}_0), \end{cases} \quad \forall \mathrm{k} \in \mathbb{N}^* \backslash \{1\} \text{ and } \eta_1(x) = \begin{cases} 1, & \mathrm{x} \in \overline{\mathrm{B}_\varrho(\mathrm{x}_0)} \\ 0, & \mathrm{x} \in \mathbb{R}^\mathrm{n} \backslash \mathrm{B}_{a_1}(\mathrm{x}_0), \end{cases}
$$

and that, also, if $a_{\mathrm{k}+1} - a_\mathrm{k} = a_1 - \varrho = C$ for all $\mathrm{k} \in \mathbb{N}^*$, where C is independent of k, then $\left| \mathrm{D}^\beta \eta_{\mathrm{B}_\varrho(\mathrm{x}_0),\mathrm{k}} \right|_{0,\infty} \leq \mathrm{C}_\mathrm{m}$, for some $\{\mathrm{C}_\mathrm{m}\}_{\mathrm{m}=0}^\infty \subset \mathbb{R}_+$, uniformly for all $\mathrm{k} \in \mathbb{N}^*$ and every β such that $|\beta| = \mathrm{m}$. In particular, $C_0 = 1$.