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Regularity of nonvanishing – at infinity or at the boundary – solutions of the defocusing nonlinear Shrödinger equation

Nikolaos Gialelis[a], Nikos I. Karachalios[b], and Ioannis G. Stratis[a]

[a]Department of Mathematics, National and Kapodistrian University of Athens, Athens, Greece;
[b]Department of Mathematics, University of the Aegean, Karlovassi, Greece

ABSTRACT
Considering the defocusing nonlinear Schrödinger equation (NLSE) in generic (bounded or unbounded) open sets $U \subseteq \mathbb{R}^n$ for $n = 1$, 2, and 3, we prove the regularity of weak, non-vanishing solutions at infinity or at the boundary of U. Our approach is based on suitably defined extension operators, along with *a priori* estimates for regular functions, under certain assumptions on the smoothness of the boundary. The results cover physically significant classes of solutions, as dark-solitons and compacton waveforms, when the notion of such solutions is extended in higher-dimensional set-ups.

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1. Introduction

The defocusing nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial v}{\partial t} - \Delta v + |v|^{2\tau} v = 0, \tau \in \mathbb{N}, \tag{1.1}$$

is one of the universal mathematical models for wave propagation in nonlinear dispersive media. It appears in a plethora of physical contexts ranging from nonlinear optics and plasma physics, to the description of Bose–Einstein condensates (BEC), and has attracted extensive efforts from the physical [22] and mathematical viewpoint [2, 5, 6, 10, 13, 25, 30, 31]. This interest is due to its fascinating complex phenomenology associated with the existence of localized waveforms supported on the top of a stable continuous background. In $1D$ set-ups, one of the most famous such waveforms is the dark-soliton: when $\tau = 1$ (cubic $1D$ defocusing NLSE), the simplest expression of the dark-soliton solution has the form

CONTACT Ioannis G. Stratis istratis@math.uoa.gr Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-15784 Athens, Greece.

$$v(t,x)=e^{it}\tanh\frac{x}{\sqrt{2}}. \tag{1.2}$$

In $2D$ and $3D$ set-ups, the aforementioned solutions have highly nontrivial generalizations (if compared with the $1D$ case), described by the so-called vortices in $2D$ and vortex rings in $3D$, with a particular relevance to the emergence of complex spatiotemporal structures in fluid mechanics and the physics of BECs.

The aim of this article is to discuss global-in-time existence and regularity properties of such solutions for defocusing NLSE (1.1) in higher-dimensional settings. First, we comment on some key works on the problem which motivated the present explorations, and then, we discuss the novelties and extensions presented herein. The principal works considering the existence of soliton-like solutions on the top of a finite background for the NLSE equation on $\mathbb{R}$, are [33–35]. The natural phase spaces proved to be the so-called Zhidkov spaces $X^m(\mathbb{R})$, with m being a natural number (their definition is stated in the next section). In particular, Zhidkov spaces properly capture the description of non-decaying solitonic structures satisfying non-vanishing boundary conditions. They may cover the existence of solutions whose density $|u|^2$ but not the solution itself is localized on the backround, such as "density dips" corresponding to solutions u with a kink-like topological structure, as the dark solitons. Obviously, the behaviour of such waveforms cannot be described by the standard Sobolev spaces $H^s(\mathbb{R})$, or even by their affine counterparts. The above fundamental papers established the local well-posedeness of the NLSE in $X^m(\mathbb{R})$. Representative references for the extensions of the problem on $\mathbb{R}^n$, are the contributions [15–17] and [21]. In [15] it is proved that the Cauchy problem for the NLSE is locally well-posed in $X^m(\mathbb{R}^n)$ for $m>n/2$. Furthermore, the justification of the natural conservation laws for $n\leq 2$, implies global existence in $X^1(\mathbb{R}^n)$, and additionally, justifies the stability results for dark solitons given in [24]. The energy conservation for any dimension $n\geq 2$, for initial conditions $u_0\in X^m(\mathbb{R}^n), m>n/2$, was first established in [21], extending accordingly, the global well-posedeness in $X^2(\mathbb{R}^2)$. For extensions in higher dimensions $n\leq 3$ (and for a general class of nonlinearities) [16], the problem is restricted to the non-vanishing boundary condition for the density, thus the problem is discussed in affine Sobolev spaces. The global well-posedeness of the Gross–Pitaevskii (GPE) equation in $u_0+H^1(\mathbb{R}^n)$, $n=2$, 3, for non-vanishing initial conditions u_0 at infinity, is proved in [17].

A first difference between our explorations in this article and the existing results so far, is that the problem will be considered in an arbitrary open set of $\mathbb{R}^n$, $n=1$, 2, and 3. For instance, we shall seek solutions expressed in the form

$$v(t,x)=e^{i\rho^{2\tau}t}w(t,x),\ \text{ with }\ w(t,x)=u(t,x)+\zeta(x), \tag{1.3}$$

for $\rho>0$ as well as

$$\begin{cases}\tau\in\mathbb{N}, & \text{if } n=1,2,\\ \tau=1, & \text{if } n=3.\end{cases} \tag{1.4}$$

Then, we consider (1.1) for $x\in U\subseteq\mathbb{R}^n, n=1,2,3,$ where U is open, and $t\in J_0\subseteq\mathbb{R}$, an open interval containing $t_0=0$. In (1.3), we assume that $u:J_0\times\bar{U}\to\mathbb{C}$ and $\zeta:\bar{U}\to\mathbb{C}$ are sufficiently smooth. Substituting the *ansatz* (1.3) in (1.1), we actually seek solutions of the problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} - \Delta(u+\zeta) + \left(|u+\zeta|^{2\tau} - \rho^{2\tau}\right)(u+\zeta) = 0, \ \text{in } J_0^* \times U, \\ u = u_0, \ \text{in } \{t=0\} \times \bar{U}, \\ u = 0, \ \text{in } J_0 \times \partial U, \ \text{and } u \overset{|x|\to\infty}{\to} 0, \ \text{in } J_0 \times \bar{U}, \end{cases} \tag{1.5}$$

where $J_0^* = J_0 \setminus \{0\}$.

Note that the last condition in (1.5) covers the following cases:

1. $U = \mathbb{R}^n$. In this case, we shall require that $u(t,x)$ vanishes as $|x| \to \infty$, so that accordingly $v(t, x)$ shares the same decaying behavior with $\zeta(x)$ on a constant background of intensity $\rho > 0$. This case corresponds to solutions decaying as $|x| \to \infty$, on the top of a non-vanishing background, and generalizes the notion of dark-solitons in the higher-dimensional setting (see e.g., [15, 16, 21]).
2. $U \subset \mathbb{R}^n$, unbounded. In this case $u=0$ on $J_0 \times \partial U$, with $u(t,x) \to 0$ as $|x| \to \infty$ in $J_0 \times \bar{U}$.
3. U bounded. In this case $u=0$ on $J_0 \times \partial U$.

The last two cases are of physical significance: motivated by the class of compacton solutions of the NLSE, [28, 29], seeking solutions of the form (1.3) in a generic U, generalizes the notion of non-vanishing compactons in higher-dimensional settings. Recall that a compact wave is a solitary wave with a compact support, outside which it vanishes identically. A robust[1] compact wave is called a compacton. There is great variety of dark compactons, for example in 1D, such as cuspons (cusp-alike density dips), compact kinks (kinkons), or dark compactons (smooth density dips), see [27] for a description. To highlight further the role of the terms appearing in the NLSE of the problem (1.5), let us note the following: substitution of the first expression $v(t,x) = e^{i\rho^{2\tau}t}w(t,x)$ gives the following GPE type equation for w

$$i\frac{\partial w}{\partial t} - \Delta w + \left(|w|^{2\tau} - \rho^{2\tau}\right)w = 0, \tag{1.6}$$

see e.g., [17] for $\rho = 1$. Then, in the case of the $1D$ cubic NLSE ($n = 1, \tau = 1$), the dark soliton solution (1.2) corresponds to $\rho = 1$. Thus, the constant ρ represents the amplitude of the finite background in the multi-dimensional setting, as mentioned above. The term $-\rho^{2\tau}(u+\zeta)$ in (1.5) appears after inserting the expression (1.3) for w in (1.6).

In the above-generalized set-up for the domains U, the main results of [16] are extended in [19], by weakening the regularity assumptions on ζ. Furthermore, motivated by the variety of dark compactons with respect to their regularity, a first study for the H^2-regularity of solutions of the problem (1.5) when $n = 1$, was considered in [20].

In this article, we extend the latter regularity result for both cases of the spatial domains discussed above, in the following cases of nonlinearity exponents and spatial dimensions:

[1]A robust wave is an orbitally stable solitary wave that almost preserves its shape (in an appropriate norm) after interacting with another solitary wave or, more generally, with another (arbitrary) localized disturbance, [29].

$$\begin{cases} \tau \in \mathbb{N}, & \text{if } n = 1, \\ \tau = 1, & \text{if } n = 2. \end{cases} \tag{1.7}$$

The classical approach for the weak solvability, which allows for the derivation of estimates for the energy of the solutions, is based on the "regularized nonlinearity" technique, presented in [5]. Here, we introduce an alternative approach, which not only permits the derivation of such estimates, but it is also applied to derive the aforementioned regularity results. In more detail, this approach can be described in the following two key points:

1. *Local-in time weak solutions and their extension to global.* First, we use an energy method to establish local-in time existence of weak solutions for the problem (1.5); see also [5, Chapter 3] for the standard Cauchy problem for the NLS equation in the general domain. However, as underlined above, the energy method is drastically modified so that to extend the local existence results in the general domains U including $\mathbb{R}^N$, and not only in $\mathbb{R}^N$ which is the case of [15, 16]: Instead of the "regularized nonlinearity" technique, the Galerkin approximations are combined with an approximative domain expansion scheme for the original domain U. This is achieved by introducing suitable extension/restriction operators and cutoff functions. The existence of global in time solutions is associated with the uniqueness of local-in time ones for arbitrary time intervals shown above (see also [5, Theorem 3.3.9] for a related argument). For the case of general bounded or unbounded domains, we are applying suitable versions of either the Sobolev ($n = 1$), or the Trudinger/Gagliardo–Nirenberg ($n = 2$, [26]) inequalities in order to establish uniqueness for arbitrary time intervals J_0 as above, and thus, their continuation for all $t \in \mathbb{R}$. In the case where $U = \mathbb{R}^n$, $n = 1, 2, 3$, Strichartz (dispersive) estimates are replacing the above inequalities, as in the proof of [19, Proposition 11].
2. *Regularity of solutions.* We derive suitable estimates of the extension operators defined on sets with adequately smooth boundaries along with certain *a priori* estimates for regular functions in order to estimate the high-order weak derivatives of the nonlinearity. These estimates are proved by a combination of multivariate Faá di Bruno formula and Gagliardo–Nirenberg type inequalities. These estimates are used to extend the regularity results in the case of bounded sets and in the case of the whole of $\mathbb{R}^n$, generalizing the results of [21], which cover only the case $U = \mathbb{R}^n$. An additional feature is that in the case of bounded domains we provide information on the dependence of the elliptic regularity estimates from boundary characteristics. It should also be remarked that while all regular solutions are proved to be unique and global, the regularity proofs are independent of uniqueness.

In summary, the above results extend the existence of non-vanishing, global-in time solutions, for the case of bounded or unbounded multidimensional domains, covering a wide class of physically relevant waveforms, while they associate their regularity, shown so far only for $U = \mathbb{R}^n$, with the properties of their boundary. The proposed approach, being fairly generic, can be potentially applied to other relevant higher-dimensional

NLS or GPE counterparts as the sublinear NLS [37], or models incorporating dissipative effects [18, 22]. The energy method approach could also be extended for establishing the existence and regularity of non-vanishing solutions beyond NLS equations; we refer to [37] for compacton solutions of the complex sublinear Klein–Gordon equation.

This article is organized as follows: In Section 2, after the proof of some auxiliary results relevant to the properties of operators defined by the nonlinear term, we weakly formulate the problem (1.5). For this purpose, we assume that ζ belongs to a Zhidkov space, and impose suitable restrictions on ζ, in order to control its behavior at infinity, or on the boundary of U. The conclusions of Subsection 2.3 are useful for the extension of the local existence result in bounded sets (see Theorem 3.1), also in unbounded ones (see Theorem 3.2). These theorems are the main results of Section 3 and are used in conjunction with Sobolev or Trudinger-type or dispersive-type estimates, to establish uniqueness and global-in time existence. The questions about regularity are answered in Section 4. Therein, we derive the estimates of the extension operators (Theorem A.1) and combine them with *a priori* estimates for regular functions (see Proposition A.9) as mentioned above. This way, we estimate the derivatives of the nonlinearity (see Corollary 4.2 for $n = 1$ and Corollary 4.3 for $n = 2$), employing the multivariate Faá di Bruno formula and Gagliardo–Nirenberg type inequalities. The regularity result in bounded sets is proved in Theorem 4.1 and Corollary 4.4, while for the whole of $\mathbb{R}^n$ in Theorem 4.2.

Throughout the article, we systematically employ suitable extensions of some known results which are included (for the sake of brevity without proof) in Appendix A.

Some notation.

We denote by C any generic positive constant, as well as, any increasing function $C : [0,\infty]^m \to (0,\infty]$, for some $m \in \mathbb{N}$. The presence of the subscript $\cdot_w$ to a differential operator for "space"-variables indicates that we consider the operator with the weak (i.e., distributional) sense, while its absence indicates differentiations in the classical sense. In what follows, U, U_j, $j \in \mathbb{N}$, are arbitrary open subsets of $\mathbb{R}^n$. Also, $\mathcal{X}(U)$ stands for a space of functions defined on U. If $u \in \mathcal{X}(U;\mathbb{C})$ and also every derivative – in some sense $\mathcal{S}$ – of the kth order ($k \in \mathbb{N}_0$), i.e., every $D^\alpha_{\mathcal{S}} u$, with $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = k$, exists, then $\nabla^k_{\mathcal{S}} u$ stands for the vector having as components those derivatives. Following the notation of, e.g., [12] and [32], if $u : J \times U \to \mathbb{C}$, with $u(t,\cdot) \in \mathcal{X}$ for each $t \in J$, then we associate with u the mapping $\mathbf{u} : J \to \mathcal{X}$, defined by $[\mathbf{u}(t)](x) := u(t,x)$, for every $x \in U$ and $t \in J$. For the weak derivative (when it exists) of the "time"-variable of a function-space-valued function $\mathbf{u}$, we simply write $\mathbf{u}'$.

2. Properties of operators and energy functionals – weak formulation of the problem

As in [16], the problem (1.5) is naturally formulated in the class of Zhidkov spaces: for every $m \in \mathbb{N}, X^m(U)$ will stand for the Zhidkov space over U, which is the Banach space

$$X^m(U) := \left\{ u \in L^\infty(U) \mid \nabla^k_w u \in L^2(U), \text{ for } k = 1, \dots, m \right\},$$

endowed with the norm

$$||u||_{X^m(U)} = ||u||_{L^\infty(U)} + \sum_{1\le k\le m} ||\nabla_w^k u||_{L^2(U)}.$$

We remark that, in this article, we will consider the spaces X^m over any open set $U \subseteq \mathbb{R}^n$, $n=1, 2, 3$, and not only in $\mathbb{R}^n$. The standard example of a function which is an element of these spaces is $\tanh x$; it belongs in $\cap_{m=1}^{\infty} X^m(\mathbb{R})$. We note that these spaces considered over $\mathbb{R}$ were first introduced in [33]. Their generalizations in $\mathbb{R}^n, n \ge 1$ (along with the suitable modifications), were first introduced in [15, 17, 36] and [16].

2.1. Basic continuity properties of operators

We proceed first by recalling some useful versions of the Gagliardo–Nirenberg inequalities, and, then by describing Lipschitz continuity properties of the nonlinear operators involved in the problem (1.5). The former is stated in the following lemma.

Lemma 2.1. *If*

$$\alpha \in \begin{cases} (0,\infty), & \text{if } n = 1,2, \\ \left(0, \dfrac{4}{n-2}\right), & \text{otherwise,} \end{cases}$$

then

$$||u||_{L^{\alpha+2}(\mathbb{R}^n)} \le C||\nabla_w u||_{L^2(\mathbb{R}^n)}^{\frac{n\alpha}{2(\alpha+2)}} ||u||_{L^2(\mathbb{R}^n)}^{1-\frac{n\alpha}{2(\alpha+2)}}, \ \forall u \in C_c^\infty(\mathbb{R}^n),$$

or else

$$||u||_{L^{\alpha+2}(U)} \le C||\nabla_w u||_{L^2(U)}^{\frac{n\alpha}{2(\alpha+2)}} ||u||_{L^2(U)}^{1-\frac{n\alpha}{2(\alpha+2)}}, \ \forall u \in H_0^1(U). \tag{2.1}$$

Proof. This is a version of the Gagliardo–Nirenberg interpolation inequality (see, e.g., Theorem 1.3.7 in [5]). □

The first continuity property for operators refers to the power-law nonlinearity considered as a mapping on $L^p(U)$-spaces, for suitable exponents p.

Lemma 2.2. *If* $\alpha \in [0,\infty)$, *then*

$$\left\| |u|^\alpha u - |v|^\alpha v \right\|_{L^{\frac{\alpha+2}{\alpha+1}}(U)} \le C\left(\|u\|_{L^{\alpha+2}(U)}^\alpha + \|v\|_{L^{\alpha+2}(U)}^\alpha\right)\|u-v\|_{L^{\alpha+2}(U)}, \ \forall u,v \in L^{\alpha+2}(U). \tag{2.2}$$

Proof. It is a direct application of the elementary inequality

$$\left||z_1|^q z_1 - |z_2|^q z_2\right| \le C_q|z_1-z_2|\left(|z_1|^q + |z_2|^q\right), \ \forall\ z_1, z_2 \in \mathbb{C}, \ \forall q \in [0,\infty), \tag{2.3}$$

the Hölder inequality for $p_1 = \alpha+1$ and $p_2 = \frac{\alpha+1}{\alpha}$, and the elementary inequality

$$|z_1+z_2|^q \le C_q\left(|z_1|^q + |z_2|^q\right), \ \forall\ z_1, z_2 \in \mathbb{C}, \ \forall q \in [0,\infty). \tag{2.4}$$

□

The next two propositions establish the well definiteness and continuity properties of the nonlinear term

$$g(u;\rho,\tau,\zeta) := \left(|u+\zeta|^{2\tau}-\rho^{2\tau}\right)\left(\bar{u}+\bar{\zeta}\right),$$

of the NLS (1.5), when considered as a nonlinear operator $g:\mathcal{X}\to\mathcal{Y}$, with $\mathcal{X},\mathcal{Y}$ being suitable functional spaces for our purposes. For brevity, in most of the cases, we will simply write $g(u)$ for the above nonlinear term. We assume that ρ and τ are as in (1.4), and that $\zeta\in L^\infty(U)$.

Proposition 2.1. *Let $u,v \in H_0^1(U)$.*

1. *If $n=1$, then*

$$\|g(u)-g(v)\|_{L^2(U)} \le C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\|u-v\|_{L^2(U)}. \tag{2.5}$$

2. *If $n=2$, then*

$$\begin{aligned}\|g(u)-g(v)\|_{L^2(U)} \le C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ \times\Big(\|u-v\|_{L^4(U)}+\|u-v\|_{L^2(U)}\Big)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\|g(u)-g(v)\|_{L^2(U)} \le C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ \times\Big(\|u-v\|^{\frac{1}{2}}_{L^2(U)}+\|u-v\|_{L^2(U)}\Big).\end{aligned} \tag{2.7}$$

3. *If $n=3$, then*

$$\begin{aligned}\|g(u)-g(v)\|_{L^{\frac{4}{3}}(U)+L^2(U)} \le C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ \times\Big(\|u-v\|_{L^4(U)}+\|u-v\|_{L^2(U)}\Big)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\|g(u)-g(v)\|_{L^{\frac{4}{3}}(U)+L^2(U)} \le C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ \times\Big(\|u-v\|^{\frac{1}{4}}_{L^2(U)}+\|u-v\|_{L^2(U)}\Big).\end{aligned} \tag{2.9}$$

Proof. Let $n=1,2$. Using the inequality (2.3), we get

$$\int_U |g(u)-g(v)|^2dx \le C\int_U \left(|u|^{4\tau}+|v|^{4\tau}\right)|u-v|^2dx + C\Big(\|\zeta\|^{4\tau}_{L^\infty(U)}+1\Big)\|u-v\|^2_{L^2(U)}.$$

For $n=1$, we employ the scaling invariant embedding $H_0^1(U)\hookrightarrow L^\infty(U)$ to derive (2.5). For $n=2$, we get (2.6) by applying Hölder's inequality ($p_1=p_2=2$) and the scaling invariant embedding $H_0^1(U)\hookrightarrow L^\vartheta(U)$, for $\vartheta\in[2,\infty)$. Then, the inequality (2.7) follows from (2.6) and (2.1). For $n=3$, we note first that

$$g(u)-g(v) := I_1+I_2,$$

where

$$I_1 = |u|^2\bar{u} - |v|^2\bar{v} \text{ and } I_2 = 2\bar{\zeta}\left(|u|^2 - |v|^2\right) + \zeta(\bar{u}^2 - \bar{v}^2) + \left(2|\zeta|^2 - \rho^2\right)(u - v) + \bar{\zeta}^2(u - v).$$

By using (2.2), we deduce that $I_1 \in L^{\frac{4}{3}}(U)$, satisfying the inequality

$$\|I_1\|_{L^{\frac{4}{3}}(U)} \le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}\Big)\|u - v\|_{L^4(U)}.$$

For the term I_2, we apply again Hölder's inequality ($p_1 = p_2 = 2$) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^\vartheta(U)$ for $\vartheta \in [2, 6]$, to get that

$$\int_U |I_2|^2 dx \le C\Big(\|\zeta\|_{L^\infty(U)}\Big)\int_U |u - v|^2\left(|u|^2 + |v|^2\right) + |u - v|^2 dx \le$$
$$\le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}\Big)\Big(\|u - v\|_{L^4(U)}^2 + \|u - v\|_{L^2(U)}^2\Big).$$

Hence, (2.8) follows by a combination of the above inequalities for I_1 and I_2. The proof of (2.9) is a consequence of (2.8) combined with the Gagliardo–Nirenberg inequality (2.1). □

A useful corollary of Proposition 2.1, following in particular from (2.8), is the inequality

$$\|g(u) - g(v)\|_{L^{p_1}\left(J;L^{\frac{4}{3}}(U)\right) + L^{p_2}(J;L^2(U))} \le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}\Big) \times \times \Big(\|u - v\|_{L^{p_1}(J;L^4(U))} + \|u - v\|_{L^{p_2}(J;L^2(U))}\Big), \tag{2.10}$$

which holds for every $u, v \in H_0^1(U)$ and $p_1, p_2 \in [1, \infty]$, when $n = 3$.

Proposition 2.2. *Let $u, v \in H_0^1(U)$, and assume that $(|\zeta| - \rho) \in L^2(U)$.*

1. *If $n = 1, 2$, then*

$$\|g(u)\|_{L^2(U)} \le C\Big(\|u\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big). \tag{2.11}$$

2. *If $n = 3$, then*

$$\|g(u)\|_{L^{\frac{4}{3}}(U) + L^2(U)} \le C\Big(\|u\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big). \tag{2.12}$$

Proof. First, we prove that $g(0) = (|\zeta|^{2\tau} - \rho^{2\tau})\bar{\zeta} \in L^2(U)$. Indeed, by using the identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}), \tag{2.13}$$

we get

$$\|g(0)\|_{L^2(U)} \le C\Big(\|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big).$$

Then, the claimed inequalities follow from Proposition 2.1. □

Summarizing the above results, for ρ and τ as in (1.4) and $\zeta \in L^\infty(U)$ with $(|\zeta| - \rho) \in L^2(U)$, we deduce that

$$g : H_0^1(U) \to \begin{cases} L^2(U), & \text{if } n = 1,2 \\ L^{\frac{4}{3}}(U) + L^2(U), & \text{if } n = 3 \end{cases} \hookrightarrow H^{-1}(U).$$

2.2. Definition of weak solutions and energy functionals

We proceed to the definition of weak solutions and associated energy functionals. We start by further assuming that $\zeta \in X^1(U)$, and by defining the form $\mathcal{N}[\cdot,\cdot] : H_0^1(U) \times H_0^1(U) \to \mathbb{C}$, as

$$\mathcal{N}[u,v] := \langle \Delta_w(u+\zeta), v\rangle + \langle g(u), v\rangle, \ \forall \ u,v \ \in H_0^1(U).$$

Then, the problem (1.5) is weakly formulated as follows: for every $u_0 \in H_0^1(U)$, a weak solution of the problem (1.5) is a function

$$\mathbf{u} \in L^\infty\big(J_0; H_0^1(U)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U)\big),$$

satisfying the formula

$$\begin{cases} \langle i\mathbf{u}', v\rangle + \mathcal{N}[\mathbf{u}, v] = 0, \ \text{for every } v \in H_0^1(U), \ \text{a.e. in } J_0, \\ \mathbf{u}(0) = u_0. \end{cases} \tag{2.14}$$

From (2.11), (2.12) and Hölder's inequality, we get the following estimate for $\mathcal{N}[u,v]$:

$$\big|\mathcal{N}[u,v]\big| \le C\Big(\|u\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big)\|v\|_{H^1(U)}, \ \forall \ u,v \ \in H_0^1(U). \tag{2.15}$$

For the above weak solutions, we shall consider the respective energy functional $E : H_0^1(U) \to [0,\infty]$,

$$E(\cdot;\rho,\tau,\zeta) := \frac{1}{2}\|\nabla_w(\cdot+\zeta)\|_{L^2(U)}^2 + G(\cdot;\rho,\tau,\zeta).$$

In the definition of E, the functional $G : H_0^1(U) \to [0,\infty]$ is given by

$$G(\cdot;\rho,\tau,\zeta) := \int_U V(|\cdot+\zeta|;\rho,\tau)dx,$$

where $V : [0,\infty) \to [0,\infty)$ is defined as

$$V(x;\rho,\tau) := \frac{1}{2\tau+2}x^{2\tau+2} - \frac{1}{2}\rho^{2\tau}x^2 + \frac{\tau}{2\tau+2}\rho^{2\tau+2}. \tag{2.16}$$

It is straightforward to check that for every constant $C_\tau > 2\tau+2$, we have

$$x^{2\tau+2} \le C_\tau V(x), \ \forall x \ge \left(\frac{C_\tau \rho^{2\tau}}{C_\tau - (2\tau+2)}\right)^{\frac{1}{2\tau}} > \rho. \tag{2.17}$$

The functional G satisfies several estimates given in the following proposition.

Proposition 2.3. *Let $u, v \ \in H_0^1(U)$.*

1. *If $n = 1, 2$, then*

$$|G(u) - G(v)| \le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big)\times \\ \times\|u - v\|_{L^2(U)}. \tag{2.18}$$

2. *If $n = 3$, then*

$$|G(u) - G(v)| \le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big)\times \\ \times\Big(\|u - v\|_{L^4(U)} + \|u - v\|_{L^2(U)}\Big), \tag{2.19}$$

$$|G(u) - G(v)| \le C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big)\times \\ \times\Big(\|u - v\|^{\frac{1}{4}}_{L^2(U)} + \|u - v\|_{L^2(U)}\Big), \tag{2.20}$$

as well as

$$G(u) \le C\Big(\|u\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big). \tag{2.21}$$

Proof. For every fixed $x \in U$ we use the mean value theorem to write

$$\begin{aligned} &V(|u(x) + \zeta(x)|) - V(|v(x) + \zeta(x)|) \\ &\quad= \int_0^1 |su(x) + (1-s)v(x) + \zeta(x)|^{2\tau}\mathrm{Re}\big[\big(su(x) + (1-s)v(x) + \zeta(x)\big)(\bar{u}(x) - \bar{v}(x))\big]ds- \\ &\qquad - \int_0^1 \rho^{2\tau}\mathrm{Re}\big[\big(su(x) + (1-s)v(x) + \zeta(x)\big)(\bar{u}(x) - \bar{v}(x))\big]ds = \\ &\quad= \mathrm{Re}\bigg(\int_0^1 \Big(|su(x) + (1-s)v(x) + \zeta(x)|^{2\tau} - \rho^{2\tau}\Big)\big(su(x) + (1-s)v(x) + \zeta(x)\big)(\bar{u}(x) - \bar{v}(x))ds\bigg). \end{aligned}$$

Hence, we have that for all $x \in U$,

$$\begin{aligned} &V(|u + \zeta|) - V(|v + \zeta|) \\ &\qquad= \mathrm{Re}\bigg(\int_0^1 \Big(|su + (1-s)v + \zeta|^{2\tau} - \rho^{2\tau}\Big)(su + (1-s)v + \zeta)(\bar{u} - \bar{v})ds\bigg). \end{aligned}$$

Using the identity (2.13), we expand the power-like terms in the right-hand side, getting

$$\begin{aligned} V(|u + \zeta|) - V(|v + \zeta|) = \mathrm{Re}\bigg(\int_0^1 &\Big(|su + (1-s)v + \zeta|^2 - \rho^2\Big)(su + (1-s)v + \zeta)(\bar{u} - \bar{v})\times \\ &\times (|su + (1-s)v + \zeta|^{2(\tau-1)} + \rho^2|su + (1-s)v + \zeta|^{2(\tau-2)} + \ldots + \\ &+ \rho^{2(\tau-2)}|su + (1-s)v + \zeta|^2 + \rho^{2(\tau-1)})ds\bigg). \end{aligned}$$

Setting $w = su + (1-s)v$, and further expanding the term $|w + \zeta|^2$, we derive the inequalities

$$
\begin{aligned}
\left|\left(|w+\zeta|^2-\rho^2\right)(w+\zeta)\right| &= \left|\left(|w|^2+2\mathrm{Re}\left(\bar{\zeta}w\right)+|\zeta|^2-\rho^2\right)(w+\zeta)\right| \leq \\
&\leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|w|^2+|w|+||\zeta|-\rho|\right)(|w|+|\zeta|) \leq \\
&\leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|w|^3+|w|^2+|w|+||\zeta|-\rho|\right) \\
&\leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|w|^3+||\zeta|-\rho|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left||w+\zeta|^{2(\tau-1)}+\rho^2|w+\zeta|^{2(\tau-2)}+\ldots+\rho^{2(\tau-2)}|w+\zeta|^2+\rho^{2(\tau-1)}\right| \leq C\left||w+\zeta|^{2(\tau-1)}+1\right| \leq \\
&\quad \leq C\left||w|^{2(\tau-1)}+|\zeta|^{2(\tau-1)}+1\right| \leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|w|^{2(\tau-1)}+1\right).
\end{aligned}
$$

Then, we have that

$$
\begin{aligned}
&\left|\left(|w+\zeta|^2-\rho^2\right)(w+\zeta)\right|\left||w+\zeta|^{2(\tau-1)}+\rho^2|w+\zeta|^{2(\tau-2)}+\ldots+\rho^{2(\tau-2)}|w+\zeta|^2+\rho^{2(\tau-1)}\right| \leq \\
&\quad \leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|w|^{2\tau+1}+||\zeta|-\rho|\right) \leq \\
&\quad \leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(s^{2\tau+1}|u|^{2\tau+1}+(1-s)^{2\tau+1}|v|^{2\tau+1}+||\zeta|-\rho|\right).
\end{aligned}
$$

The latter, implies the estimate for V

$$
|V(|u+\zeta|)-V(|v+\zeta|)| \leq C\left(\|\zeta\|_{L^\infty(U)}\right)\left(|u|^{2\tau+1}+|v|^{2\tau+1}+||\zeta|-\rho|\right)|u-v|.
$$

In turn, the functional G satisfies

$$
|G(u)-G(v)| \leq C\left(\|\zeta\|_{L^\infty(U)}\right)\int_U\left(|u|^{2\tau+1}+|v|^{2\tau+1}+||\zeta|-\rho|\right)|u-v|dx.
$$

To conclude with the claimed estimates, we work similarly as in Propositions 2.1 and 2.2. For $n=\ 1,2$, we employ Hölder's inequality for $p_1=p_2=2$ and the scaling invariant embedding $H_0^1(U)\hookrightarrow L^{4\tau+2}(U)$ to get (2.18). For $n=3$ and $\tau=1$, we get (2.19), again from Hölder's inequality applied once for $p_1=\frac{4}{3}$ and $p_2=4$, then for $p_1=p_2=2$, and by using the scaling invariant embedding $H_0^1(U)\hookrightarrow L^4(U)$. The estimate (2.20) follows from (2.19), along with (2.1).

For the proof of (2.21), it suffices to show that

$$
G(0) \leq C\left(\|\zeta\|_{L^\infty(U)}, \||\zeta|-\rho\|_{L^2(U)}\right).
$$

To this end, we first observe that

$$
G(0) \leq \int_U V(|\zeta|)dx = \int_U \frac{1}{2(\tau+1)}|\zeta|^{2(\tau+1)} - \frac{1}{2}\rho^{2\tau}|\zeta|^2 + \frac{\tau}{2(\tau+1)}\rho^{2(\tau+1)}dx,
$$

and next, we employ the identity

$$
a^{n+1}-a(n+1)b^n+nb^{n+1}=(a-b)^2\left(a^{n-1}+2a^{n-2}b+\ldots+(n-1)ab^{n-2}+nb^{n-1}\right),
$$

to obtain the desired estimate. □

From Proposition 2.3, it follows that indeed, the functionals $E, G \ : H_0^1(U) \to [0,\infty)$ are well defined.

Remark 2.1. *An alternative proof for Proposition 2.3 can be given by verifying that g is the Gateaux derivative of G.*

2.3. Properties of non-linear operators on domain restrictions

Here, we make a note concerning the definition of operators and functionals discussed previously, on restrictions of functions considered on subsets of the original domain. For the definition, notation, and properties of these restrictions we refer to Appendix A and its Subsections A.1, A.3, and A.4.

The scaling invariant Sobolev embeddings are essential for the definition of the operator g, as well as the functionals E, G on $H_0^1(U)$, for every U. Hence, in virtue of Corollary A.1, by defining these operators and functionals for every $u \in H_0^1(U)$ for some arbitrary U, we can also consider them defined for every $((\mathcal{R}(U,V))u) \in H^1(V)$ for every open $V \subseteq U$ (note that we have $((\mathcal{R}(U,V))u) \in H^1(V)$ for every open $V \subseteq U$ from Proposition A.1). This means that we do not need to impose any regularity assumptions on ∂V in order to consider the scaling dependent Sobolev embeddings of Corollary A.2. However, this is not true for the bounds derived by using (2.1), for which we need to employ the results on restriction operators stated in Theorem A.1.

Thus, we get, for every U and every $u, v \ \in H_0^1(U)$, that

$$g \circ (\mathcal{R}(U,V)) : H_0^1(U) \to \begin{cases} L^2(V), & \text{if } n = \ 1,2 \\ L^{\frac{4}{3}}(V) + L^2(V), & \text{if } n = 3, \end{cases} \quad \text{for every open } V \subseteq U, \tag{2.22}$$

and

$$E \circ (\mathcal{R}(U,V)), G \circ (\mathcal{R}(U,V)) \ : H_0^1(U) \to [0,\infty), \text{ for every open } V \subseteq U, \tag{2.23}$$

are well defined, and satisfy

$$\begin{aligned} &\|\big(g \circ (\mathcal{R}(U,V))\big)(u) - \big(g \circ (\mathcal{R}(U,V))\big)(u)(v)\|_{L^2(V)} \leq \\ &\leq C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}\Big) \times \\ &\times \|(\mathcal{R}(U,V))u - (\mathcal{R}(U,V))v\|_{L^2(V)}, \text{ if } n = 1, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\|\big(g \circ (\mathcal{R}(U,V))\big)(u) - \big(g \circ (\mathcal{R}(U,V))\big)(u)(v)\|_{L^2(V)} \leq \\ &\leq C\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^\infty(U)}\Big) \times \\ &\times \Big(\|(\mathcal{R}(U,V))u - (\mathcal{R}(U,V))v\|_{L^4(V)} + \|(\mathcal{R}(U,V))u - (\mathcal{R}(U,V))v\|_{L^2(V)}\Big), \text{ if } n = 2, \end{aligned} \tag{2.25}$$

$$\begin{aligned}&\|\big(g\circ(\mathcal{R}(U,V))\big)(u)-\big(g\circ(\mathcal{R}(U,V))\big)(u)(v)\|_{L^{\frac{4}{3}}(V)+L^2(V)}\leq\\&\quad\leq C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\&\qquad\times\Big(\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^4(V)}+\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)}\Big),\ \text{if}\ n=3,\end{aligned}\tag{2.26}$$

as well as

$$\begin{aligned}&|(G\circ(\mathcal{R}(U,V)))(u)-(G\circ(\mathcal{R}(U,V)))(v)|\leq\\&\leq C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times\\&\times\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)},\ \text{if}\ n=\ 1,2,\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}&|(G\circ(\mathcal{R}(U,V)))(u)-(G\circ(\mathcal{R}(U,V)))(v)|\leq\\&\quad\leq C\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times\\&\quad\times\Big(\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^4(V)}+\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)}\Big),\ \text{if}\ n=3,\end{aligned}\tag{2.28}$$

for every open $V\subseteq U$.

3. Weak solutions

In this section, we prove the existence of weak solutions for the problem (1.5), by suitably implementing the Faedo-Galerkin method. Before we proceed, we state and prove some preliminary lemmata.

Lemma 3.1. *For every $f\in H^{-1}(U)$ there exists $\{f_j\}_{j=0}^n\subset L^2(U)$ such that*

$$\langle f,v\rangle=\int_U v\overline{f_0}+\sum_{j=1}^n(\partial^j v)\overline{f_j}dx,\ \ \forall v\in H_0^1(U),$$

and, in particular, we have

$$(v,f)=\langle f,v\rangle,\ \ \forall v\in H_0^1(U),\ \ \forall f\in L^2(U).$$

Proof. The first result follows from a direct application of the complex version of Riesz–Fréchet representation theorem (see, e.g., Proposition 11.27 in [3]). The second is a direct consequence of the first one. □

Lemma 3.2. *Let J be bounded, $\mathcal{X}_1$ be a Banach space and $\mathcal{X}_2$ be a Banach space with the Radon–Nikodym property with respect to the Lebesgue measure in $(J,\mathscr{B}(J))$.*

1. *Let $\{\mathbf{u}_k\}_k\subset L^\infty(J;\mathcal{X}_1)$ and $\mathbf{u}:J\to\mathcal{X}_1$ with $\mathbf{u}_k(t)\rightharpoonup\mathbf{u}(t)$ in $\mathcal{X}_1$, for a.e. $t\in J$. If $\|\mathbf{u}_k\|_{L^\infty(J;\mathcal{X}_1)}\leq C$ uniformly for every k, then $\mathbf{u}\in L^\infty(J;\mathcal{X}_1)$ with $\|\mathbf{u}\|_{L^\infty(J;\mathcal{X}_1)}\leq C$, where C is the same in both inequalities.*

2. *Let* $\{\mathbf{u}_k\}_k \cup \{\mathbf{u}\} \subset L^\infty(J;\mathcal{X}_2^*)$ *with* $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ *in* $L^\infty(J;\mathcal{X}_2^*)$[2]. *If* $\|\mathbf{u}_k\|_{L^\infty(J;\mathcal{X}_2^*)} \le C$ *uniformly for every k, then* $\|\mathbf{u}\|_{L^\infty(J;\mathcal{X}_2^*)} \le C$, *where C is the same in both inequalities.*

Proof. For Point 1., we derive that $\|\mathbf{u}(t)\|_{\mathcal{X}_1} \le C$, for a.e. $t \in J$, from the (sequentially) weak lower semi-continuity of the norm, hence the result follows. As for Point 2, let $v \in \mathcal{X}_2$ be such that $\|v\|_{\mathcal{X}_2} \le 1$ and set $\mathbf{v} : J \to \mathcal{X}_2$ the constant function with $\mathbf{v}(t) := v$, for all $t \in J$. We have

$$\int_s^{s+h} \langle \mathbf{u}_k, \mathbf{v} \rangle dt \le Ch, \text{ for a.e. } s \in J^\circ \text{ and every sufficiently small } h > 0.$$

Considering the limit $\mathbf{u}_k \rightharpoonup *\mathbf{u}$ in $L^\infty(J;\mathcal{X}_2^*)$, dividing both parts by h and then letting $h \searrow 0$, we get, from the Lebesgue differentiation theorem, that $\langle \mathbf{u}(s), v \rangle \le C$, for a.e. $s \in J^\circ$. Since v arbitrary, the proof is complete. □

Lemma 3.3. *Let* $z_0 \in \mathbb{C}^n$ *and* $z : J_0 \to \mathbb{C}^n$ *be the unique, maximal solution of the initial-value problem*

$$\begin{cases} z'(t) = iF(z(t)), \ \forall t \in J_0^* \\ z(0) = z_0 \end{cases},$$

for an appropriate function F (e.g., locally Lipschitz). If $z_0 \in \mathbb{R}^n$ *and* $\bar{F}(z) = F(\bar{z})$, *then* J_0 *is symmetric around 0 and also* $z(t) = \bar{z}(-t)$, *for all* $t \in J_0$.

Proof. We define $-J_0 := \{t \in \mathbb{R} \mid -t \in J_0\}$ and also $y : -J_0 \to \mathbb{C}^n$ with $y(t) := \bar{z}(-t)$, for all $t \in -J_0$. Since $z_0 \in \mathbb{R}^n$ and $\bar{F}(z) = F(\bar{z})$, we can easily see that y solves the above problem (in $-J_0$). Hence $-J_0 \subseteq J_0$, since z is the maximal solution. Therefore, J_0 is symmetric around 0. We can now define the function $x : J_0 \to \mathbb{C}^n$ as $x(t) := \bar{z}(-t)$, for all $t \in J_0$ and we deduce that x also solves the problem (in J_0). Hence, $\bar{z}(-t) = x(t) = z(t)$, for all $t \in J_0$, since z is unique. □

Lemma 3.4. *Let* $m \in \mathbb{N}, p \in [1,\infty]$, U_1, U_2, $\phi \in C_c^\infty(U_1)$ *and* $u \in W_0^{m,p}(U_2)$. *If we set*

$$\varphi := (\mathcal{R}(U_1, U_1 \cap U_2))\phi \text{ and } v := (\mathcal{R}(U_2, U_1 \cap U_2))u,$$

then

$$(\varphi v) \in W_0^{m,p}(U_1 \cap U_2), \text{ with } \|\varphi v\|_{W^{m,p}(U_1 \cap U_2)} \le C\Big(\|\phi\|_{C_B^m(U_1)}\Big)\|u\|_{W^{m,p}(U_2)}.$$

Proof. We assume that $U_1 \cap U_2 \neq \emptyset$, otherwise we have nothing to show (see also Point 3. before Definition A.3). In view of Proposition A.7, we derive that

$$(\varphi v) \in W^{m,p}(U_1 \cap U_2), \text{ with } \|\varphi v\|_{W^{m,p}(U_1 \cap U_2)} \le C\Big(\|\varphi\|_{C_B^m(U_1 \cap U_2)}\Big)\|v\|_{W^{m,p}(U_1 \cap U_2)},$$

hence

[2]That is, $\mathbf{u}_k \rightharpoonup *\mathbf{u}$ in $\sigma(L^\infty(J;\mathcal{X}_2^*), L^1(J;\mathcal{X}_2))$. Note that $L^\infty(J;\mathcal{X}_2^*) \cong (L^1(J;\mathcal{X}_2))^*$ (see, e.g., Theorem 1, Section 1, Chapter IV in [8]).

$$\|\varphi v\|_{W^{m,p}(U_1\cap U_2)} \le C\Big(\|\phi\|_{C^m_{\mathrm{B}}(U_1)}\Big)\|u\|_{W^{m,p}(U_2)}.$$

Now, we consider $\{u_k\}_k \subset C_c^\infty(U_2)$, such that $u_k \to u$ in $W^{m,p}(U_2)$ and in an analogous manner we set

$$v_k := (\mathcal{R}(U_2, U_1\cap U_2))u_k, \ \forall k.$$

Evidently,

$$(\varphi v_k) \in C^m(U_1\cap U_2).$$

In particular, we can easily deduce that

$$(\varphi v_k) \in C^m_c(U_1\cap U_2).$$

Applying (A.3), we derive that

$$\begin{aligned}\|\varphi v_k - \varphi v\|_{W^{m,p}(U_1\cap U_2)} = \|\varphi(v_k - v)\|_{W^{m,p}(U_1\cap U_2)} \le C\|v_k - v\|_{W^{m,p}(U_1\cap U_2)} \le \\ \le C\|u_k - u\|_{W^{m,p}(U_2)} \to 0\end{aligned}$$

and the desired result follows from the definition of $W_0^{m,p}$-spaces. □

Theorem 3.1. *If U is bounded, then for every $u_0 \in H^1_0(U)$ and every bounded interval J_0 there exists a solution of (2.14), such that*

$$\|\mathbf{u}\|_{L^\infty(J_0;H^1(U))} + \|\mathbf{u}'\|_{L^\infty(J_0;H^{-1}(U))} \le C\Big(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\Big) \tag{3.1}$$

and also

$$E(\mathbf{u}) \le E(u_0) \text{ everywhere in } \overline{J_0}. \tag{3.2}$$

Moreover, if u_0 and ζ are real-valued, then the above solution satisfies $\mathbf{u}(t) = \bar{\mathbf{u}}(-t)$, for every $t \in \overline{J_0}$ with $|t| \le \operatorname{dist}(0, \partial J_0)$.

Proof. Here, we use the notation

$$\tilde{C} := C\Big(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta| - \rho\|_{L^2(U)}\Big)$$

and

$$\tilde{C}_{J_0} := C\Big(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\Big).$$

Now, based on

1. the fact that $U \in \{U_{\mathrm{P}}\}$ (see Subsection A.8.3) and also $H^1_0(U;\mathbb{R}) \hookrightarrow \hookrightarrow L^2(U;\mathbb{R})$ (see Proposition A.6),
2. the Fredholm theory and
3. the fact that the field $\mathbb{C}$ can be regarded as a vector space over the field $\mathbb{R}$,

we deduce that the complete set of eigenfunctions for the operator $-\Delta_w$ restricted to $H^1_0(U;\mathbb{R})$, is an orthogonal basis of both $H^1_0(U;\mathbb{C})$ and $L^2(U;\mathbb{C})$. Let $\{w_k\}_{k=1}^\infty \subset H^1_0(U;\mathbb{R})$ be the aforementioned basis, appropriately normalized so that $\{w_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(U;\mathbb{C})$. We then employ the standard Faedo-Galerkin method.

Step 1α

For every $m \in \mathbb{N}$, we define $d_m \in C^\infty(J_{0_m}; \mathbb{C}^m)$, with $d_m(t) := (d_m^k(t))_{k=1}^m$, to be the unique maximal solution of the initial-value problem

$$\begin{cases} d_m{}'(t) = F_m(d_m(t)), \ \forall t \in J_{0_m}^* \\ d_m(0) = ((w_k, u_0))_{k=1}^m \ \ (= (\langle u_0, w_k \rangle)_{k=1}^m, \text{in view of Lemma 3.1}), \end{cases}$$

where $F_m \in C^\infty(\mathbb{R}^{2m}; \mathbb{C}^m)$ with

$$F_m^k(z) := i\mathcal{N}\left[\sum_{l=1}^m z_l w_l, w_k\right], \text{ for every } z := (z_l)_{l=1}^m \in \mathbb{C}^m, \text{ for every } k = 1, \ldots, m.$$

We note that the smoothness of F_m follows by directly applying (N times, for arbitrary $N \in \mathbb{N}$) the common Leibniz integral rule. Now, we define $\mathbf{u}_m \in C^\infty(J_{0_m}; H_0^1(U; \mathbb{C}))$, with

$$\mathbf{u}_m(t) := \sum_{k=1}^m \overline{d_m^k}(t) w_k.$$

In view of *Lemma 3.1*, it is easy to verify that

$$\langle i\mathbf{u}_m{}', w_k \rangle + \mathcal{N}[u_m, w_k] = 0 \text{ everywhere in } J_{0_m}, \text{ for every } k = 1, \ldots, m. \tag{3.3}$$

Step 1β

By using the Bessel–Parseval identity, we get that

$$\mathbf{u}_m(0) \to u_0 \text{ in } L^2(U) \text{ and } \|\mathbf{u}_m(0)\|_{L^2(U)} \leq \|u_0\|_{L^2(U)}. \tag{3.4}$$

Furthermore, we can argue as in [12, Step 3. Theorem 2, Section 6.5], to deduce

$$\nabla_w \mathbf{u}_m(0) \to \nabla_w u_0 \text{ in } L^2(U) \text{ and } \|\nabla_w(\mathbf{u}_m(0))\|_{L^2(U)} \leq \|\nabla_w u_0\|_{L^2(U)}. \tag{3.5}$$

Now, these limiting relations have two immediate consequences: First, the bounds in (3.4) and (3.5) imply that $\|\mathbf{u}_m(0)\|_{H^1(U)} \leq \|u_0\|_{H^1(U)}$, hence, in view of (2.21), we derive

$$E(\mathbf{u}_m(0)) \leq \tilde{C}. \tag{3.6}$$

Second, the convergences in (3.4) and (3.5) imply that $\mathbf{u}_m(0) \to u_0$ in $H^1(U)$. Moreover, from (2.18), (2.20), and the convergence in (3.4), we get $G(\mathbf{u}_m(0)) \to G(u_0)$. Combining the last two convergences, we conclude to

$$E(\mathbf{u}_m(0)) \to E(u_0). \tag{3.7}$$

Step 2α

We multiply the variational equation in (3.3) by $d_m^k{}'(t)$, sum over $k = 1, \ldots, m$, and taking the real parts of both sides, to get

$$\frac{d}{dt} E(\mathbf{u}_m) = 0, \text{ that is } E(\mathbf{u}_m) = E(\mathbf{u}_m(0)) \text{ everywhere in } J_{0_m}. \tag{3.8}$$

Hence, from (3.6) we have that

$$E(\mathbf{u}_m) \leq \tilde{C} \text{ everywhere in } J_{0_m}, \text{ uniformly for every } m \in \mathbb{N}. \tag{3.9}$$

Therefore, from the fact that G is positive-valued, we deduce that $\|\nabla_w(\mathbf{u}_m+\zeta)\|_{L^2(U)} \le \tilde{C}$, which implies

$$\|\nabla_w \mathbf{u}_m\|_{L^2(U)} \le \tilde{C}. \tag{3.10}$$

In order to derive a bound for the L^2-norm which is independent of $|U|$, we follow a different route, instead of applying the Poincaré inequality. We note that

$$G(\mathbf{u}_m) \le \tilde{C}, \ \forall m \in \mathbb{N}, \tag{3.11}$$

which follows from (3.9). Moreover, in view of (2.17), we fix some $C_1 > 2\tau + 2$ and we have that

$$x^{2(\tau+1)} \le C_1 V(x), \ \text{for every } x \ge C_2, \ \text{for some } C_2 > \rho^{\frac{1}{2}}. \tag{3.12}$$

Setting

$$\Omega(t) := \{x \in U \mid |\mathbf{u}_m(t) + \zeta| \ge \max\{C_2, 1\}\} \subseteq U, \ \forall t \in \mathbb{R},$$

we get, from (3.12) and (3.11) (and the fact that $\lambda > 0$), that

$$\int_{\Omega(t)} |\mathbf{u}_m + \zeta|^s dx \le \tilde{C}, \ \forall m \in \mathbb{N}, \ \forall s \in (-\infty, 2(\tau+1)]. \tag{3.13}$$

Then, we multiply the variational Eq. (3.3) by $d_m^k(t)$, sum over $k = 1, ..., m$, and keeping imaginary parts on both sides, and thus, in view of *Lemma 3.1*, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_m\|^2_{L^2(U)} - \mathrm{Im}(\nabla_w \zeta, \nabla \mathbf{u}_m) - \mathrm{Im}\big((|\mathbf{u}_m+\zeta|^{2\tau} - \rho^{2\tau})(\mathbf{u}_m+\zeta), \mathbf{u}_m\big) = 0.$$

For the middle term, we apply Hölder's inequality ($p_1 = p_2 = 2$) and use the bound (3.10), while for the third term we expand in view of (2.13), to deduce that

$$\|\mathbf{u}_m\|^2_{L^2(U)} \le \tilde{C}\left(|t| + \left|\int_0^t \left(\int_U |\mathbf{u}_m|^{2\tau+1} dx\right) ds\right|\right), \ \forall t \in \mathbb{R}, \ \forall m \in \mathbb{N}. \tag{3.14}$$

In order to estimate the spatial integral, we write

$$\begin{aligned}\int_U |\mathbf{u}_m|^{2\tau+1} dx &= \int_{\Omega(t)^c \cap U} |\mathbf{u}_m|^2 |u_m|^{2\tau-1} dx + \int_{\Omega(t)} |\mathbf{u}_m|^{2\tau+1} dx \\ &\le \int_{\{x\in U \mid |\mathbf{u}_m| \le \max\{C_2,1\} + \|\zeta\|_{L^\infty(U)}\}} |\mathbf{u}_m|^2 |\mathbf{u}_m|^{2\tau-1} dx \\ &\quad + C\int_{\Omega(t)} |\zeta|^{2\tau+1} + |\mathbf{u}_m + \zeta|^{2\tau+1} dx \\ &\overset{(3.13)}{\le} C(\|\zeta\|_{L^\infty(U)})\|\mathbf{u}_m\|^2_{L^2(U)} + \tilde{C} \le \tilde{C}(1 + \|\mathbf{u}_m\|^2_{L^2(U)}).\end{aligned} \tag{3.15}$$

Let J_0 be arbitrary. From (3.14) and (3.15), we derive that

$$\|\mathbf{u}_m\|^2_{L^2(U)} \le \tilde{C}_{J_0}\left(1 + \left|\int_0^t \|\mathbf{u}_m\|^2_{L^2(U)} ds\right|\right), \ \forall t \in \overline{J_0}, \ \forall m \in \mathbb{N}.$$

Hence, by the Grönwall's inequality,

$$\|\mathbf{u}_m\|_{L^\infty(J_0;L^2(U))} \leq \tilde{C}_{J_0}, \text{ for every } m \in \mathbb{N}. \tag{3.16}$$

From (3.10) and (3.16) we conclude to

$$\|\mathbf{u}_m\|_{L^\infty(J_0;H^1(U))} \leq \tilde{C}_{J_0}, \ \forall m \in \mathbb{N}. \tag{3.17}$$

Step 2β

We fix an arbitrary $v \in H_0^1(U)$ with $\|v\|_{H^1(U)} \leq 1$ and write $v = \mathcal{P}v \oplus (\mathcal{I} - \mathcal{P})v$, where $\mathcal{P}$ is the projection on $\text{span}\{w_k\}_{k=1}^m$. Since $u_m' \in \text{span}\{w_k\}_{k=1}^m$ and $\mathcal{N}[h,g]$ is linear on g, from the variational equation in (3.3) we get that

$$\langle i\mathbf{u}_m', v\rangle = -\mathcal{N}[\mathbf{u}_m, \mathcal{P}v].$$

Applying (2.15) and (3.17) we derive

$$|\langle i\mathbf{u}_m', v\rangle| \leq \tilde{C}_{J_0}.$$

Therefore

$$\|\mathbf{u}_m'\|_{L^\infty(\mathbb{R};H^{-1}(U))} = \|i\mathbf{u}_m'\|_{L^\infty(\mathbb{R};H^{-1}(U))} \leq \tilde{C}_{J_0}, \ \forall m \in \mathbb{N}. \tag{3.18}$$

Step 3α

We fix an arbitrary bounded J_0. From (3.17), (3.18), point i) of Theorem 1.3.14 in [5][3] and point 1 of *Lemma 3.2*, there exist a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^\infty \subseteq \{\mathbf{u}_m\}_{m=1}^\infty$ and a function

$$\mathbf{u} = \mathbf{u}_{J_0} \in L^\infty\big(J_0; H_0^1(U)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U)\big),$$

such that

$$\mathbf{u}_{m_l} \rightharpoonup \mathbf{u} \text{ in } H_0^1(U) \text{ everywhere in } \overline{J_0} \text{ and also } \|\mathbf{u}\|_{L^\infty(J_0;H^1(U))} \leq \tilde{C}. \tag{3.19}$$

Step 3β

$H^{-1}(U)$ is separable since $H_0^1(U)$ is separable, hence by the Dunford–Pettis theorem (see, e.g., Theorem 1, Section 3, Chapter III in [8]) we have $L^\infty(J_0; H^{-1}(U)) \cong (L^1(J_0; H_0^1(U)))^*$ (see, e.g., Theorem 1, Section 1, Chapter IV in [8]). In virtue of the above, from (3.18), the Banach–Alaoglu–Bourbaki theorem (see, e.g., Theorem 3.16 in [3]) and Point 2 of *Lemma 3.2*, there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^\infty$ (not relabelled) and a function

$$\mathbf{h} \in L^\infty\big(J_0; H^{-1}(U)\big),$$

such that

$$\mathbf{u}_{m_l}' \overset{*}{\rightharpoonup} \mathbf{h} \text{ in } L^\infty\big(J_0; H^{-1}(U)\big) \text{with } \|\mathbf{h}\|_{L^\infty(J_0;H^{-1}(U))} \leq \tilde{C}. \tag{3.20}$$

Let $\psi \in C_c^\infty(J_0^\circ)$ and $v \in H_0^1(U)$ be arbitrary. From

1. the linearity of the functional,
2. the convergence in (3.20),

[3]We note that in [5], the normed space $(H_{0_{\mathbb{R}}}^1(U), \|\cdot\|_{H^1(U)})$ is considered instead of $H_0^1(U)$. However, it becomes clear from its proof that the aforementioned result is also valid in our case.

3. *Lemma* 1.1, Chapter III in [32],
4. the definition of the weak derivative,
5. *Lemma* 3.1,
6. the dominated convergence theorem and
7. the convergence in (3.19),

we obtain

$$\begin{aligned}\int_{J_0}\langle \mathbf{h}, v\rangle\psi dt &\overset{1.}{=} \int_{J_0}\langle \mathbf{h}, \psi v\rangle dt \overset{2.}{=} \lim_{l\to\infty}\int_{J_0}\langle \mathbf{u}_{m_l}{}', \psi v\rangle dt \overset{1.}{=} \lim_{l\to\infty}\int_{J_0}\langle \mathbf{u}_{m_l}{}', v\rangle\psi dt\\ &\overset{3.}{=} \lim_{l\to\infty}\int_{J_0}\langle \mathbf{u}_{m_l}, v\rangle'\psi dt \overset{4.}{=} -\lim_{l\to\infty}\int_{J_0}\langle \mathbf{u}_{m_l}, v\rangle\psi' dt \overset{5.}{=} -\lim_{l\to\infty}\int_{J_0}\overline{(\mathbf{u}_{m_l}, v)}\psi' dt\\ &\overset{6.}{=} -\int_{J_0}\lim_{l\to\infty}\overline{(\mathbf{u}_{m_l}, v)}\psi' dt \overset{7.}{=} -\int_{J_0}\overline{(\mathbf{u}, v)}\psi' dt \overset{5.}{=} -\int_{J_0}\langle \mathbf{u}, v\rangle\psi' dt\\ &\overset{4.}{=} \int_{J_0}\langle \mathbf{u}, v\rangle'\psi dt \overset{3.}{=} \int_{J_0}\langle \mathbf{u}', v\rangle\psi dt,\end{aligned}$$

hence $h \equiv u'$, since ψ and v are arbitrary.

Step 4

Since $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U) \hookrightarrow H^{-1}(U)$, from (3.17), (3.17) and the Aubin–Lions–Simon *Lemma* [23, Theorem 8.62], there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$ (still not relabelled) and $\mathbf{y} \in C(\overline{J_0}; L^2(U))$, such that

$$\mathbf{u}_{m_l} \to \mathbf{y} \text{ in } C\big(\overline{J_0}; L^2(U)\big). \tag{3.21}$$

From the convergence in (3.19), we deduce that $\mathbf{y} \equiv \mathbf{u}$. This fact has two direct consequences: First, $\mathbf{u}$ satisfies the initial condition, i.e.,

$$\mathbf{u}(0) \equiv u_0,$$

(as it follows from (3.21) for $t = 0$ combined with $\mathbf{u}_m(0) \to u_0$ in $L^2(U)$ from Step 1β). Second, from (2.5), (2.7), (2.9), (2.18), (2.20), as well as (3.17), the bound in (3.19) and (3.21), we get

$$g(\mathbf{u}_{m_l}) \to g(\mathbf{u}) \text{ in } C\big(\overline{J_0}; Y_\alpha(U)\big), \tag{3.22}$$

and also

$$G(\mathbf{u}_{m_l}) \to G(\mathbf{u}) \text{ uniformly in } \overline{J_0}. \tag{3.23}$$

Step 5

We will show that u satisfies the variational equation in (2.14). Let now $\psi \in C_c^\infty(J_0^\circ)$ and fix $N \in \mathbb{N}$. We choose m_l such that $N \le m_l$ and $v \in \operatorname{span}\{w_k\}_{k=1}^N$. By the linearity of the inner product and (3.3), we get that

$$\int_{J_0}\langle i\mathbf{u}_{m_l}{}', \psi v\rangle - \langle \Delta_w(\mathbf{u}_{m_l} + \zeta), \psi v\rangle + \langle g(\mathbf{u}_{m_l}), \psi v\rangle dt = 0.$$

From the convergence in (3.20) we get

$$\int_{J_0} \langle i\mathbf{u}_{m_l}{}', \psi v\rangle dt \to \int_{J_0} \langle i\mathbf{u}', \psi v\rangle dt,$$

while from the convergence in (3.19) we have

$$\langle \Delta_w v, \mathbf{u}_{m_l}\rangle \to \langle \Delta_w v, \mathbf{u}\rangle, \text{ everywhere in } \overline{J_0},$$

since the functional $\langle \Delta_w v, \cdot\rangle : H_0^1(U) \to \mathbb{C}$ is linear and bounded, thus

$$\overline{\langle \Delta_w v, \mathbf{u}_{m_l}\rangle} \to \overline{\langle \Delta_w v, \mathbf{u}\rangle} \text{ everywhere in } \overline{J_0}.$$

Thus, we deduce that

$$\langle \Delta_w \mathbf{u}_{m_l}, v\rangle = \overline{\langle \Delta_w v, \mathbf{u}_{m_l}\rangle} \to \overline{\langle \Delta_w v, \mathbf{u}\rangle} = \langle \Delta_w \mathbf{u}, v\rangle \text{ everywhere in } \overline{J_0},$$

and so

$$\langle \Delta_w(\mathbf{u}_{m_l} + \zeta), \psi v\rangle \to \langle \Delta_w(\mathbf{u} + \zeta), \psi v\rangle \text{ everywhere in } \overline{J_0}.$$

Applying next the dominated convergence theorem, we get

$$\int_{J_0} \langle \Delta_w(\mathbf{u}_{m_l} + \zeta), \psi v\rangle dt \to \int_{J_0} \langle \Delta_w(\mathbf{u} + \zeta), \psi v\rangle dt.$$

From Hölder's inequality ($p_1 = p_2 = 2$) and (3.22), we also deduce

$$\int_{J_0} \langle g(\mathbf{u}_{m_l}), \psi v\rangle dt \to \int_{J_0} \langle g(\mathbf{u}), \psi v\rangle dt.$$

Since ψ is arbitrary, $\mathbf{u}$ satisfies the variational equation for every $v \in \text{span}\{w_k\}_{k=1}^N$. We then get the desired result from a density argument, since N is arbitrary.

Step 6

We proceed to the proof of (3.2). Let $\epsilon > 0$ be arbitrary. From (3.7) and the equation in (3.8), we deduce that there exists $m_0 = m_0(\epsilon)$, such that

$$E(\mathbf{u}_m) \leq E(u_0) + \epsilon, \quad \text{everywhere in } \mathbb{R}, \text{ for every } m \geq m_0. \tag{3.24}$$

Moreover, from the convergence in (3.19) along with the fact the operator $\nabla_w : W^{1,p}(U) \to L^p(U)$ is linear and continuous for every $p \in [1, \infty]$ and every U, as well as the equivalence of continuity and weak continuity of linear functionals [3, Theorem 3.10], we deduce that $\nabla_w \mathbf{u}_{m_l} \rightharpoonup \nabla_w \mathbf{u}$ in $L^2(U)$, everywhere in $\overline{J_0}$. Hence, from the weak lower semi-continuity of the L^2-norm, we get

$$\|\nabla_w(\mathbf{u}_J + \zeta)\|_{L^2(U)} \leq \liminf_{l\to\infty} \|\nabla_w(\mathbf{u}_{m_l} + \zeta)\|_{L^2(U)}, \text{ everywhere in } \overline{J_0}. \tag{3.25}$$

Combining (3.23) and (3.25) we deduce that

$$E(\mathbf{u}) \leq \liminf_{l\to\infty} E(\mathbf{u}_{m_l}) \text{ everywhere in } \overline{J_0}. \tag{3.26}$$

From (3.26) and (3.24), we have

$$E(\mathbf{u}) \leq E(u_0) + \epsilon \text{ everywhere in } \overline{J_0}$$

which proves the claimed (3.2), since ϵ is arbitrary.

Step 7

Finally,

1. if ζ is real-valued, then $\overline{F_m}(z) = F_m(\bar{z})$, for every $z \in \mathbb{C}^m$ and
2. if u_0 is real-valued, then $d_m(0) \in \mathbb{R}^m$.

Hence, under these two assumptions, we apply *Lemma* 3.3 to get that $d_m(t) = \overline{d_m}(-t)$ and so $\overline{\mathbf{u}_m}(t) = \mathbf{u}_m(-t)$, for every $t \in \mathbb{R}$ and every $m \in \mathbb{N}$, which is equivalent to $\mathbf{u}_m(t) = \overline{\mathbf{u}_m}(-t)$, for every $t \in \mathbb{R}$ and every $m \in \mathbb{N}$. Now, the (conjugate) symmetry $\mathbf{u}(t) = \bar{\mathbf{u}}(-t)$, for every $t \in \overline{J_0}$ with $|t| \leq \operatorname{dist}(0, \partial J_0)$, follows from the aforementioned symmetry, along with the convergence in (3.19) or (3.21).

Theorem 3.2. Theorem 3.1 *is also valid for every unbounded U.*

Proof. In Step 1 we construct an approximation sequence for the initial data, and in Step 2 we consider an approximation sequence of problems considered in an expanding sequence of bounded sets that eventually cover the whole unbounded set. In Step 3, we take the limit of the aforementioned approximation sequence of solutions, and then, in Step 4, we verify that this limit is indeed a solution of the variational equation; the key for the proof of this step is the application of Proposition A.3. In the last Step 5, we verify the initial condition, the energy estimate, and the symmetry of the solution. As in the proof of Theorem 3.1, we write

$$\tilde{C}_{J_0} := C\Big(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\Big).$$

Step 1α

We fix an arbitrary $x_0 \in U$ and we set

$$B_k := B(x_0, k), \quad \forall k \in \mathbb{N}.$$

In view of Proposition A.8, we consider a sequence $\{\phi_k\}_k \subset C_c^\infty(\mathbb{R}^n; [0, 1])$, such that

1. $\operatorname{supp}(\phi_k) \subseteq \overline{B_{k+1}}$, for every k,
2. $\phi_k \equiv 1$ in $\overline{B_k}$, for every k, and
3. $\|\nabla \phi_k\|_{L^\infty(\mathbb{R}^n)} \leq C$, uniformly for every k.

We then set

$$\varphi_k := (\mathcal{R}(\mathbb{R}^n, B_{k+2}))\phi_k, \quad \forall k.$$

Evidently, $\varphi_k \in C_c^\infty(B_{k+2}; [0, 1])$ for every k, with

1. $\operatorname{supp}(\varphi_k) \subseteq \overline{B_{k+1}}$, for every k,
2. $\varphi_k \equiv 1$ in $\overline{B_k}$, for every k, and
3. $\|\nabla \varphi_k\|_{L^\infty(B_{k+2})} \leq C$, uniformly for every k.

Moreover, we set

$$U_k := B_k \cap U,\ \ v_k := (\mathcal{R}(B_{k+2}, U_{k+2}))\varphi_k,\ \ v_{0_k} := (\mathcal{R}(U, U_{k+2}))u_0 \ \text{ and } \\ u_{0_k} := v_k v_{0_k},\ \text{ for every } k.$$

In view of *Lemma* 3.4, we have that

$$u_{0_k} \in H_0^1(U_{k+2}),\ \text{ with } \|u_{0_k}\|_{H^1(U_{k+2})} \le C\|u_0\|_{H^1(U)},\ \text{ uniformly for every } k. \tag{3.27}$$

Step 1β

We set

$$u_{00_k} := (\mathcal{E}_0(U_{k+2}, U))u_{0_k},\ \ \forall k.$$

In virtue of Proposition A.2 along with (3.27), we deduce that

$$u_{00_k} \in H_0^1(U),\ \text{ with } \|u_{00_k}\|_{H^1(U)} = \|u_{0_k}\|_{H^1(U_{k+2})} \le C\|u_0\|_{H^1(U)}, \\ \text{uniformly for every } k. \tag{3.28}$$

Now, we claim that

$$u_{00_k} \to u_0 \ \text{ in } H^1(U). \tag{3.29}$$

Indeed, from

1. $(\mathcal{R}(U, U_k))u_{00_k} \equiv (\mathcal{R}(U, U_k))u_0$, for every k,
2. $(\mathcal{R}(U, U_k^{\mathsf{c}} \cap U_{k+1}))u_{00_k} \equiv ((\mathcal{R}(B_{k+2}, U_k^{\mathsf{c}} \cap U_{k+1}))\varphi_k)((\mathcal{R}(U, U_k^{\mathsf{c}} \cap U_{k+1}))u_0)$, for every k,
3. $(\mathcal{R}(U, U_{k+1}^{\mathsf{c}} \cap U))u_{00_k} \equiv 0$, for every k and
4. (A.3),

we have that

$$\begin{aligned} \|u_{00_k} - u_0\|_{H^1(U)} &\overset{1.}{=} \|\big(\mathcal{R}\big(U, U_k^{\mathsf{c}} \cap U\big)\big)(u_{00_k} - u_0)\|_{H^1\left(U_k^{\mathsf{c}} \cap U\right)} \\ &\overset{2.}{\underset{3.}{=}} \|\big(\big(\big(\mathcal{R}\big(B_{k+2}, U_k^{\mathsf{c}} \cap U_{k+1}\big)\big)\varphi_k\big) - 1\big)\big(\big(\mathcal{R}\big(U, U_k^{\mathsf{c}} \cap U_{k+1}\big)\big)u_0\big)\|_{H^1\left(U_k^{\mathsf{c}} \cap U_{k+1}\right)} \\ &\quad + \|\big(\mathcal{R}\big(U, U_{k+1}^{\mathsf{c}} \cap U\big)\big)u_0\|_{H^1\left(U_{k+1}^{\mathsf{c}} \cap U\right)} \\ &\overset{4.}{\le} C\|\big(\mathcal{R}\big(U, U_k^{\mathsf{c}} \cap U_{k+1}\big)\big)u_0\|_{H^1\left(U_k^{\mathsf{c}} \cap U_{k+1}\right)} + \|\big(\mathcal{R}\big(U, U_{k+1}^{\mathsf{c}} \cap U\big)\big)u_0\|_{H^1\left(U_{k+1}^{\mathsf{c}} \cap U\right)} \\ &\le C\|\big(\mathcal{R}\big(U, U_{k+1}^{\mathsf{c}} \cap U\big)\big)u_0\|_{H^1\left(U_{k+1}^{\mathsf{c}} \cap U\right)} \to 0, \text{as } k \to \infty. \end{aligned}$$

Step 1γ

We will show that

$$E(u_{00_k}) \to E(u_0). \tag{3.30}$$

Indeed, from (2.18) and (2.20), along with (3.29), we deduce the limit $G(u_{00_k}) \to G(u_0)$. Additionally, directly from (3.29), we get that $\|\nabla_w(u_{00_k} + \zeta)\|_{L^2(U)} \to \|\nabla_w(u_0 + \zeta)\|_{L^2(U)}$, and thus, (3.30) follows.

Step 2α

Let J_0 be arbitrary and bounded. For every k, we consider (2.14) in U_{k+2} instead of U, and u_{0_k} as the initial condition instead of u_0. Let

$$\mathbf{u}_k \in L^\infty\big(J_0; H_0^1(U_{k+2})\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U_{k+2})\big)$$

be the corresponding solution that Theorem 3.1 provides. For every k we have that

$$\begin{aligned} &\|\mathbf{u}_k\|_{L^\infty\left(J_0;H^1(U_{k+2})\right)} + \|\mathbf{u}_k'\|_{L^\infty\left(J_0;H^{-1}(U_{k+2})\right)} \le \\ &\le C\Big(\|u_{0_k}\|_{H^1(U_{k+2})}, \|\zeta_k\|_{X^1(U_{k+2})}, \|(|\zeta|-\rho)_k\|_{L^2(U_{k+2})}, |J_0|\Big), \end{aligned} \tag{3.31}$$

where

$$\zeta_k := (\mathcal{R}(U, U_{k+2}))\zeta \text{ and } (|\zeta|-\rho)_k := (\mathcal{R}(U, U_{k+2}))(|\zeta|-\rho),$$

and

$$E(\mathbf{u}_k) \le E(u_{0_k}) \le E(u_{00_k}) \text{ everywhere in } \overline{J_0}. \tag{3.32}$$

We also have that $\mathbf{u}_k(t) = \overline{\mathbf{u}_k}(-t)$, for every $t \in \overline{J_0}$ with $|t| \le \operatorname{dist}(0, \partial J_0)$, if u_0 (hence u_{0_k}) and ζ (ζ_k) are real-valued. From the bound in (3.27), along with the increasing property of C and the fact that the bound in (3.31) is independent of U, the estimate (3.31) can be written in the form

$$\|\mathbf{u}_k\|_{L^\infty\left(J_0;H^1(U_{k+2})\right)} + \|\mathbf{u}_k'\|_{L^\infty\left(J_0;H^{-1}(U_{k+2})\right)} \le \tilde{C}_{J_0}, \text{ uniformly for every } k. \tag{3.33}$$

Step 2β

In view of *Lemma* 3.4 and (3.33), we have that

$$(\nu_k \mathbf{u}_k) \in H_0^1(U_{k+2}), \text{ with } \|\nu_k \mathbf{u}_k\|_{H^1(U_{k+2})} \le C\|\mathbf{u}_k\|_{H^1(U_{k+2})} \le \tilde{C}_{J_0}, \text{ uniformly for every } k,$$

where ν_k is as in Step 1α. Hence, in view of Proposition A.2, we define

$$\begin{aligned} &\mathbf{v}_k \in L^\infty\big(J_0; H_0^1(U)\big) \text{ as } \mathbf{v}_k := (\mathcal{E}_0(U_{k+2}, U))(\nu_k \mathbf{u}_k), \text{ for every } k, \\ &\quad \text{with } \|\mathbf{v}_k\|_{L^\infty(J_0;H^1(U))} \le \tilde{C}_{J_0}, \text{ uniformly for every } k. \end{aligned} \tag{3.34}$$

Moreover, in view of *Lemma* 3.4 we have that

$$\begin{aligned} &(\nu_k((\mathcal{R}(U, U_{k+2}))v)) \in H_0^1(U_{k+2}), \text{ with} \\ &\|\nu_k((\mathcal{R}(U, U_{k+2}))v)\|_{H^1(U_{k+2})} \le C\|v\|_{H^1(U)}, \text{ for every } v \in H_0^1(U), \\ &\text{uniformly for every } k. \end{aligned}$$

Hence, employing (3.33), for every k we define

$$\begin{aligned} &\mathbf{f}_k \in L^\infty\big(J_0; H^{-1}(U)\big) \text{ by } \langle \mathbf{f}_k, v\rangle := \langle \mathbf{u}_k{}', \nu_k((\mathcal{R}(U, U_{k+2}))v)\rangle, \\ &\text{for every } v \in H_0^1(U), \text{ for every } k, \text{ with } \|\mathbf{f}_k\|_{L^\infty(H^{-1}(U))} \le \tilde{C}_{J_0}, \\ &\text{uniformly for every } k. \end{aligned} \tag{3.35}$$

We now claim that

$$\mathbf{v}_k \in L^\infty\big(J_0; H_0^1(U)\big) \cap L^\infty\big(J_0; H^{-1}(U)\big), \text{ with } \mathbf{v}_k' \equiv \mathbf{f}_k, \text{ for every } k. \tag{3.36}$$

Indeed, let $v \in H_0^1(U)$ be arbitrary. Employing

1. *Lemma* 1.1, Chapter III in [32],

2. *Lemma* 3.1, and
3. the fact that v_k is real-valued for every k,

we derive

$$\begin{aligned}\langle \mathbf{f}_k, v\rangle &= \langle {\mathbf{u}_k}', v_k((\mathcal{R}(U, U_{k+2}))v)\rangle \overset{1.}{=} \langle \mathbf{u}_k, v_k((\mathcal{R}(U, U_{k+2}))v)\rangle' \\ &\overset{2.}{=} \overline{(\mathbf{u}_k, v_k((\mathcal{R}(U, U_{k+2}))v))'} \overset{3.}{=} \overline{\left(\int_{U_{k+2}} \mathbf{u}_k \ \ v_k((\mathcal{R}(U, U_{k+2}))\bar{v})dx\right)'} \\ &= \overline{\left(\int_U ((\mathcal{E}_0(U_{k+2}, U))(v_k\mathbf{u}_k)) \ \ \bar{v}dx\right)'} = \overline{((\mathcal{E}_0(U_{k+2}, U))(v_k\mathbf{u}_k), v)'} \\ &\overset{2.}{=} \langle(\mathcal{E}_0(U_{k+2}, U))(v_k\mathbf{u}_k), v\rangle' = \langle \mathbf{v}_k, v\rangle'.\end{aligned}$$

Therefore, from the arbitrariness of v, along with [32, *Lemma* 1.1, Chapter III], we get (3.36).

Step 2γ

For every open and bounded $V \subset U$, there exists $k_V \in \mathbb{N}$, such that $V \subseteq U_{k+2}$ for every $k \geq k_V$. Now, for every fixed such V, we define[4]

$$\begin{gathered}\mathbf{v}_{V,k} := ((\mathcal{R}(U_{k+2}, V))\mathbf{u}_k) \in L^\infty\big(J_0; H^1(V)\big), \text{ for every } k \geq k_V, \\ \text{with } \|\mathbf{v}_{V,k}\|_{L^\infty(J_0;H^1(V))} \leq \tilde{C}_{J_0}, \text{ uniformly for every such } k.\end{gathered} \tag{3.37}$$

The bound above follows directly from the bound in (3.33). Moreover, in view of Definition A.2 and the bound in (3.33), we claim that

$$\begin{gathered}\mathbf{v}_{V,k} \in L^\infty\big(J_0; H^1(V)\big) \cap L^\infty\big(J_0; H^{-1}(V)\big), \text{ with } \mathbf{v}'_{V,k} \equiv (\mathcal{R}(U_{k+2}, V))\big(\mathbf{u}'_k\big), \\ \text{for every } k \geq k_V, \text{ thus } \|\mathbf{v}'_{V,k}\|_{L^\infty(J_0;H^{-1}(V))}, \text{ uniformly for every such } k.\end{gathered} \tag{3.38}$$

Indeed, let $v \in H_0^1(V)$ be arbitrary. From

1. Definition A.2,
2. *Lemma* 1.1, Chapter III in [32] and
3. *Lemma* 3.1,

we derive, for every $k \geq k_V$, that

$$\begin{aligned}\langle(\mathcal{R}(U_{k+2}, V))({\mathbf{u}_k}'), v\rangle &\overset{1.}{=} \langle {\mathbf{u}_k}', (\mathcal{E}_0(V, U_{k+2}))v\rangle \overset{2.}{=} \langle \mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))v\rangle' \\ &\overset{3.}{=} \overline{(\mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))v)'} = \overline{\left(\int_{U_{k+2}} \mathbf{u}_k \ \ ((\mathcal{E}_0(V, U_{k+2}))\bar{v})dx\right)'} \\ &= \overline{\left(\int_V ((\mathcal{R}(U_{k+2}, V))\mathbf{u}_k) \ \ \bar{v}dx\right)'} = \overline{((\mathcal{R}(U_{k+2}, V))\mathbf{u}_k, v)'} \\ &\overset{3.}{=} \langle(\mathcal{R}(U_{k+2}, V))\mathbf{u}_k, v\rangle' = \langle \mathbf{v}_{V,k}, v\rangle'.\end{aligned}$$

[4] We highlight that we don't claim that $\mathbf{v}_{V,k} \in L^\infty(J_0; H_0^1(V))$ for every $k \geq k_V$.

Thus, (3.38) follows from the arbitrariness of v along with [32, *Lemma* 1.1, Chapter III]. Additionally, we have

$$\Delta_w \mathbf{v}_{V,k} \equiv ((\mathcal{R}(U_{k+2}, V)) \circ \Delta_w)\mathbf{u}_k, \ \ \forall k \geq k_V. \tag{3.39}$$

To prove (3.39), we consider an arbitrary $v \in H_0^1(V)$, and from

1. Definition A.2,
2. (A.2),
3. (A.1), and
4. the definition in (3.37),

we get, for every $k \geq k_V$, that

$$\begin{aligned}
\langle(((\mathcal{R}(U_{k+2}, V)) \circ \Delta_w)\mathbf{u}_k, v\rangle &\overset{1.}{=} \langle \Delta_w \mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))v\rangle \\
&= -\int_{U_{k+2}} \nabla_w \overline{\mathbf{u}_k} \cdot ((\nabla_w \circ (\mathcal{E}_0(V, U_{k+2})))v)dx \\
&\overset{2.}{=} -\int_{U_{k+2}} \nabla_w \overline{\mathbf{u}_k} \cdot (((\mathcal{E}_0(V, U_{k+2})) \circ \nabla_w)v)dx \\
&= -\int_V (((\mathcal{R}(U_{k+2}, V)) \circ \nabla_w)\overline{\mathbf{u}_k}) \cdot \nabla_w v dx \\
&\overset{3.}{=} -\int_V ((\nabla_w \circ (\mathcal{R}(U_{k+2}, V)))\overline{\mathbf{u}_k}) \cdot \nabla_w v dx \\
&\overset{4.}{=} -\int_V \nabla_w \overline{\mathbf{v}_{V,k}} \cdot \nabla_w v dx = \langle \Delta_w \mathbf{v}_{V,k}, v\rangle.
\end{aligned}$$

Then, the claimed (3.39) follows from the arbitrariness of v. Finally, in view of (2.22) and the definition in (3.37), $g(\mathbf{v}_{V,k})$ is well defined for every $k \geq k_V$. Hence, we directly get

$$g(\mathbf{v}_{V,k}) \equiv ((\mathcal{R}(U_{k+2}, V)) \circ g)\mathbf{u}_k, \ \ \forall k \geq k_V. \tag{3.40}$$

Step 3α

In virtue of the bounds in (3.34) and (3.35) (along with (3.36)), we argue exactly as in Step 3 of the proof of Theorem 3.1, in order to derive that there exist $\{\mathbf{u}_{k_l}\}_{l=1}^\infty \subset \{\mathbf{u}_k\}_{k=1}^\infty$ and a function

$$\mathbf{u} = \mathbf{u}_{J_0} \in L^\infty\big(J_0; H_0^1(U)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U)\big),$$

such that

$$\begin{gathered}
\mathbf{v}_{k_l} = \big(\mathcal{E}_0(U_{k_l+2}, U)\big)(v_{k_l}\mathbf{u}_{k_l}) \rightharpoonup \mathbf{u} \text{ in } H_0^1(U) \text{ everywhere in } \overline{J_0} \\
\text{with } \|\mathbf{u}\|_{L^\infty(J_0; H^1(U))} \leq \tilde{C}_{J_0},
\end{gathered} \tag{3.41}$$

as well as

$$\mathbf{v}_{k_l}{}' \overset{*}{\rightharpoonup} \mathbf{u}' \text{ in } L^\infty\big(J_0; H^{-1}(U)\big) \text{ and also } \|\mathbf{u}'\|_{L^\infty(J_0; H^{-1}(U))} \leq \tilde{C}_{J_0}. \tag{3.42}$$

Step 3β

Let $V \subset U$ be a fixed, arbitrary, open, and bounded set. In virtue of the bounds in (3.37) and (3.38), again we work exactly as in Step 3 of the proof of Theorem 3.1, but with one exception. That is, we employ a slightly modified version of [5, point i), Theorem 1.3.14], being valid if one would replace $H^1(V)$ by $H_0^1(V)$, as we do herein. Hence, we may get a subsequence of $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$, (not relabelled), and we assume that $k_l \geq k_V$, for every $l \in \mathbb{N}$, where k_V is as in Step 2γ. We also get a function

$$\mathbf{u}_V = \mathbf{u}_{V,J_0} \in L^\infty\big(J_0; H^1(V)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(V)\big),$$

such that

$$\mathbf{v}_{V,k_l} = (\mathcal{R}(U_{k_l+2}, V))\mathbf{u}_{k_l} \rightharpoonup \mathbf{u}_V \text{ in } H^1(V) \text{ everywhere in } \overline{J_0}, \tag{3.43}$$

$$\mathbf{v}_{V,k_l}{}' \rightharpoonup *\mathbf{u}_V' \text{ in } L^\infty\big(J_0; H^{-1}(V)\big). \tag{3.44}$$

Step 3γ

We claim that

$$(\mathcal{R}(U,V))\mathbf{u} \equiv \mathbf{u}_V \text{ and } (\mathcal{R}(U,V))\mathbf{u}' \equiv \mathbf{u}_V', \\ \text{for every open and bounded } V \subset U. \tag{3.45}$$

For the proof, we note first that for every V as above, there exists $l_V \in \mathbb{N}$, such that $V \subseteq U_{k_l}$ for every $l \geq l_V$. Now, to justify the first equivalence in (3.45) we consider an arbitrary $\phi \in C_c^\infty(V)$. Then using

1. the convergence in (3.41),
2. $V \subseteq U_{k_l+2}$ for every $l \in \mathbb{N}$ by the definition of the sequence $\{u_{k_l}\}_{l=1}^{\infty}$,
3. $(\mathcal{R}(U_{k_l+2}, V))v_{k_l} \equiv 1$ for every $l \geq l_V$, since $(\mathcal{R}(U_{k_l+2}, U_{k_l}))v_{k_l} \equiv 1$ by the definition of v_{k_l} for every $l \in \mathbb{N}$, as well as $V \subseteq U_{k_l}$ for every $l \geq l_V$, and
4. (3.43),

we may deduce that

$$\begin{aligned}
\int_V ((\mathcal{R}(U,V))\mathbf{u})\ \phi dx &= \int_U \mathbf{u}\ ((\mathcal{E}_0(V,U))\phi) dx \\
&\overset{1.}{=} \lim_{l\to\infty} \int_U \mathbf{v}_{k_l}\ ((\mathcal{E}_0(V,U))\phi) dx = \lim_{l\to\infty} \int_V \big((\mathcal{R}(U,V))\mathbf{v}_{k_l}\big)\ \phi dx \\
&= \lim_{l\to\infty} \int_V \Big(\big((\mathcal{R}(U,V)) \circ \big(\mathcal{E}_0(U_{k_l+2}, U)\big)\big)(v_{k_l}\mathbf{u}_{k_l})\Big)\ \phi dx \\
&= \lim_{\substack{l\to\infty \\ l \geq l_V}} \int_V \Big(\big((\mathcal{R}(U,V)) \circ \big(\mathcal{E}_0(U_{k_l+2}, U)\big)\big)(v_{k_l}\mathbf{u}_{k_l})\Big)\ \phi dx \\
&\overset{2.}{=} \lim_{\substack{l\to\infty \\ l \geq l_V}} \int_V \big((\mathcal{R}(U_{k_l+2}, V))(v_{k_l}\mathbf{u}_{k_l})\big)\ \phi dx \\
&\overset{3.}{=} \lim_{\substack{l\to\infty \\ l \geq l_V}} \int_V ((\mathcal{R}(U_{k_l+2}, V))\mathbf{u}_{k_l})\ \phi dx = \lim_{\substack{l\to\infty \\ l \geq l_V}} \int_V \mathbf{v}_{V,k_l}\ \phi dx \\
&\overset{4.}{=} \int_V \mathbf{u}_V\ \phi dx,
\end{aligned}$$

everywhere in $\overline{J_0}$, and the result follows from the arbitrariness of ϕ. For the second equivalence of (3.45), let $\psi \in C_c^\infty(J_0^\circ)$ and $v \in H_0^1(V)$ be arbitrary. From

1. Definition A.2,
2. the linearity of the functional,

3. the convergence in (3.42),
4. *Lemma* 1.1, Chapter III in [32],
5. the definition of the weak derivative,
6. *Lemma* 3.1,
7. $V \subseteq U_{k_l+2}$ for every $l \in \mathbb{N}$ by the definition of the sequence $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$,
8. $(\mathcal{R}(U_{k_l+2}, V))v_{k_l} \equiv 1$ for every $l \geq l_V$, and
9. (3.44),

we have

$$
\begin{aligned}
&\int_{J_0} \langle (\mathcal{R}(U,V))\mathbf{u}', v\rangle \psi\, dt \overset{1.}{=} \int_{J_0} \langle \mathbf{u}', (\mathcal{E}_0(V,U))v\rangle \psi\, dt \\
&\overset{2.}{=} \int_{J_0} \langle \mathbf{u}', \psi((\mathcal{E}_0(V,U))v)\rangle dt \\
&\overset{3.}{=} \lim_{l\to\infty} \int_{J_0} \langle {\mathbf{v}_{k_l}}', \psi((\mathcal{E}_0(V,U))v)\rangle dt \\
&\overset{2.}{=} \lim_{l\to\infty} \int_{J_0} \langle {\mathbf{v}_{k_l}}', (\mathcal{E}_0(V,U))v\rangle \psi\, dt \\
&= \lim_{l\to\infty} \int_{J_0} \langle {\mathbf{u}_{k_l}}', v_{k_l}\big((\mathcal{R}(U,U_{k_l+2})) \circ (\mathcal{E}_0(V,U))\big)v\rangle \psi\, dt \\
&\overset{4.}{=} \lim_{l\to\infty} \int_{J_0} \langle \mathbf{u}_{k_l}, v_{k_l}\big((\mathcal{R}(U,U_{k_l+2})) \circ (\mathcal{E}_0(V,U))\big)v\rangle' \psi\, dt \\
&\overset{5.}{=} -\lim_{l\to\infty} \int_{J_0} \langle \mathbf{u}_{k_l}, v_{k_l}\big((\mathcal{R}(U,U_{k_l+2})) \circ (\mathcal{E}_0(V,U))\big)v\rangle \psi'\, dt \\
&\overset{6.}{=} -\lim_{l\to\infty} \int_{J_0} \overline{(\mathbf{u}_{k_l}, v_{k_l}\big((\mathcal{R}(U,U_{k_l+2})) \circ (\mathcal{E}_0(V,U))\big)v)} \psi'\, dt \\
&= -\lim_{l\to\infty} \int_{J_0} \overline{\left(\int_{U_{k_l+2}} \mathbf{u}_{k_l}\ \ v_{k_l}\big(((\mathcal{R}(U,U_{k_l+2})) \circ (\mathcal{E}_0(V,U)))\bar{v}\big)dx\right)} \psi'\, dt \\
&\overset{7.}{=} -\lim_{l\to\infty} \int_{J_0} \overline{\left(\int_V \big((\mathcal{R}(U_{k_l+2},V))(v_{k_l}\mathbf{u}_{k_l})\big)\bar{v}\,dx\right)} \psi'\, dt \\
&= -\lim_{l\geq l_V{}^{l\to\infty}} \textstyle\int_{J_0} \overline{\left(\int_V \big((\mathcal{R}(U_{k_l+2},V))(v_{k_l}\mathbf{u}_{k_l})\big)\bar{v}\,dx\right)} \psi'\, dt \\
&\overset{8.}{=} -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \overline{\left(\int_V ((\mathcal{R}(U_{k_l+2},V))\mathbf{u}_{\mathbf{k_l}})\bar{v}\,dx\right)} \psi'\, dt \\
&= -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \overline{(\mathbf{v}_{V,k_l}, v)} \psi'\, dt \overset{6.}{=} -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle \psi'\, dt \\
&= -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \overline{(\mathbf{v}_{V,k_l}, v)} \psi'\, dt \overset{6.}{=} -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle \psi'\, dt \\
&= -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \overline{(\mathbf{v}_{V,k_l}, v)} \psi'\, dt \overset{6.}{=} -\lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle \psi'\, dt \\
&\overset{5.}{=} \lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle' \psi\, dt \overset{4.}{=} \lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle {\mathbf{v}_{V,k_l}}', v\rangle \psi\, dt \\
&\overset{2.}{=} \lim_{l\geq l_V{}^{l\to\infty}} \int_{J_0} \langle {\mathbf{v}_{V,k_l}}', \psi v\rangle dt \overset{9.}{=} \int_{J_0} \langle {\mathbf{u}_V}', \psi v\rangle dt \overset{2.}{=} \int_{J_0} \langle {\mathbf{u}_V}', v\rangle \psi\, dt.
\end{aligned}
$$

Thus, the claimed equivalence follows, since ψ and v are arbitrary. We also claim that

$$((\mathcal{R}(U,V))\circ\Delta_w)\mathbf{u}\equiv\Delta_w\mathbf{u}_V,\ \text{for every open and bounded } V\subset U. \tag{3.46}$$

Indeed, let $v\in H_0^1(V)$ be arbitrary. From

1. Definition A.2,
2. (A.2),
3. (A.1), and
4. the first equivalence in (3.45),

we get

$$\begin{aligned}\langle(((\mathcal{R}(U,V))\circ\Delta_w)\mathbf{u},v\rangle &\overset{1.}{=} \langle\Delta_w\mathbf{u},(\mathcal{E}_0(V,U))v\rangle=-\int_U\nabla_w\bar{\mathbf{u}}\cdot((\nabla_w\circ(\mathcal{E}_0(V,U)))v)dx\\ &\overset{2.}{=} -\int_U\nabla_w\bar{\mathbf{u}}\cdot(((\mathcal{E}_0(V,U))\circ\nabla_w)v)dx\\ &= -\int_V(((\mathcal{R}(U,V))\circ\nabla_w)\bar{\mathbf{u}})\cdot\nabla_w v dx\\ &\overset{3.}{=} -\int_V((\nabla_w\circ(\mathcal{R}(U,V)))\bar{\mathbf{u}})\cdot\nabla_w v dx\\ &\overset{4.}{=} -\int_V\nabla_w\overline{\mathbf{u}_V}\cdot\nabla_w v dx=\langle\Delta\mathbf{u}_V,v\rangle.\end{aligned}$$

Then, the claimed equivalence (3.46) follows from the arbitrariness of v. Finally, we have

$$\big((\mathcal{R}(U,V))\circ g\big)\mathbf{u}\equiv g(\mathbf{u}_V),\ \text{for every open and bounded } V\subset U. \tag{3.47}$$

For the equivalence (3.47), we only need to notice that in view of the first equivalence in (3.45), along with (2.22), $g(\mathbf{u}_V)$ is well defined.

Step 4α

Since every $\mathbf{u}_{k_l}$ satisfies the variational equation in U_{k_l+2}, we have that

$$\langle i\mathbf{u}_{k_l}{}'-\Delta_w\big(\mathbf{u}_{k_l}+\zeta_{k_l}\big)+g(\mathbf{u}_{k_l}),v_{k_l}\rangle=0,\ \forall v_{k_l}\in H_0^1(U_{k_l+2}),\ \forall l\in\mathbb{N}.$$

Hence, for every open and bounded $V\subset U$ we have

$$\langle(\mathcal{R}(U_{k_l+2},V))\Big(i\mathbf{u}_{k_l}{}'-\Delta_w\big(\mathbf{u}_{k_l}+\zeta_{k_l}\big)+g(\mathbf{u}_{k_l})\Big),v\rangle=0,\ \forall v\in H_0^1(V),\ \forall l\in\mathbb{N}.$$

In virtue of the equivalence in (3.38), as well as the equivalences (3.39) and (3.40) (along with the definition of the sequence $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$), the above equation becomes

$$\langle i\mathbf{v}_{V,k_l}{}'-\Delta_w(\mathbf{v}_{V,k_l}+\zeta_V)+g(\mathbf{v}_{V,k_l}),v\rangle=0,\ \forall v\in H_0^1(V),\ \forall l\in\mathbb{N}, \tag{3.48}$$

where $\zeta_V:=(\mathcal{R}(U,V))\zeta$.

Step 4β

Directly from (3.44) we have

$$\int_{J_0}\langle i\mathbf{v}_{V,k_l}{}',\psi v\rangle dt\to\int_{J_0}\langle i\mathbf{u}_V{}',\psi v\rangle dt,\ \forall\psi\in C_c^\infty\big(J_0^\circ\big),\ \forall v\in H_0^1(V). \tag{3.49}$$

Moreover, in view of (3.43), we argue exactly as in Step 5 of the proof of Theorem 3.1 to obtain

$$\int_{J_0}\langle\Delta_w(\mathbf{v}_{V,k_l}+\zeta_V),\psi v\rangle dt\to\int_{J_0}\langle\Delta_w(\mathbf{u}_V+\zeta_V),\psi v\rangle dt, \tag{3.50}$$

for every ψ and v as above. Additionally, in virtue of the bound (3.34) for k_l instead of k, along with the scaling invariant compact embeddings (see Proposition A.5)

$$H_0^1(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))\big(L^2(U)\big) \text{ and } H_0^1(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))\big(L^4(U)\big) \ (n = \ 1,2,3),$$

we deduce that there exists a subsequence of $\{\mathbf{v}_{k_l}\}_{l=1}^{\infty}$ (not relabelled) and a function $\mathbf{z} \in \mathcal{X}(J_0; L^4(V))$, such that

$$(\mathcal{R}(U,V))\big(\mathbf{v}_{k_l}(t)\big) \to \mathbf{z}(t) \text{ in } L^2(V) \text{ and } (\mathcal{R}(U,V))\big(\mathbf{v}_{k_l}(t)\big) \to \mathbf{z}(t) \text{ in } L^4(V), \tag{3.51}$$

for every $t \in J_0$. Since

$$\mathbf{v}_{V,k_l} \equiv (\mathcal{R}(U,V))\mathbf{v}_{k_l}, \ \forall l \geq l_V, \tag{3.52}$$

where l_V is as in Step 3γ, we deduce, from (3.43), that

$$\mathbf{z} \equiv \mathbf{u}_V. \tag{3.53}$$

In virtue of (3.51), (3.52), (3.53), along with (2.24), (2.25), and (2.26), we derive that

$$g(\mathbf{v}_{V,k_l}) \to g(\mathbf{u}_V) \text{ in } \begin{cases} L^2(V), & \text{if } n = \ 1,2 \\ L^{\frac{4}{3}}(V) + L^2(V), & \text{if } n = 3, \end{cases} \text{ everywhere in } J_0.$$

Hence, the dominated convergence theorem implies the limit

$$\int_{J_0} \langle g(\mathbf{v}_{V,k_l}), \psi v\rangle dt \to \int_{J_0} \langle g(\mathbf{u}_V), \psi v\rangle dt, \ \forall \psi \in C_c^{\infty}\big(J_0^{\circ}\big), \ \forall v \in H_0^1(V). \tag{3.54}$$

Gathering (3.49), (3.50), and (3.54), we get from (3.48) that

$$\langle i\mathbf{u}_V{}' - \Delta_w(\mathbf{u}_V + \zeta_V) + g(\mathbf{u}_V), v\rangle = 0, \ \forall v \in H_0^1(V). \tag{3.55}$$

Step 4γ

In virtue of the second equivalence in (3.45), as well as the equivalences (3.46) and (3.47), we get from (3.55) that

$$(\mathcal{R}(U,V))\big(i\mathbf{u}' - \Delta_w(\mathbf{u} + \zeta) + g(\mathbf{u})\big) \overset{H^{\bar{1}}(V)}{\equiv} 0.$$

Since $V \subset U$ is arbitrary open and bounded, we deduce from Proposition A.3 that u satisfies the variational equation in U.

Step 5α

As far as the initial condition is concerned, we first note that

$$\mathbf{v}_k(0) \to u_0 \text{ in } H^1(U). \tag{3.56}$$

Indeed, we have

$$\mathbf{v}_k(0) = (\mathcal{E}_0(U_{k+2}, U))(\nu_k u_{0_k}), \ \forall k.$$

Thus, we get (3.56) by working exactly as in Step 1β. Therefore, by combining (3.56) with the convergence in (3.41) for $t = 0$, we deduce that $\mathbf{u}(0) \equiv u_0$.

Step 5β

We will show that

$$E(\mathbf{u}) \leq E(u_0) \text{ everywhere in } \overline{J_0}.$$

Indeed, we have from Theorem 3.1 that

$$E(\mathbf{u}_k) \leq E(u_{0_k}) \text{ everywhere in } \overline{J_0}.$$

Hence, from (3.32) and the fact that E is positive, we deduce

$$E(\mathbf{v}_{V,k}) \leq E(u_{00_k}) \text{ everywhere in } \overline{J_0}, \text{ for every open } V \subseteq U_{k+2}, \text{ for every } k.$$

Let now $\epsilon > 0$ be arbitrary. In virtue of (3.30), we have that there exists $k_0 = k_0(\epsilon)$, such that

$$E(\mathbf{v}_{V,k}) \leq E(u_0) + \epsilon \text{ everywhere in } \overline{J_0}, \text{ for every open } V \subseteq U_{k+2}, \text{ for every } k \geq k_0. \tag{3.57}$$

From (3.51), (3.52), (3.53), along with (2.27) and (2.28), we derive that

$$G(\mathbf{v}_{V,k_l}) \to G(\mathbf{u}_V), \text{ for every open and bounded } V \subset U. \tag{3.58}$$

Moreover, from (3.43) along with the (sequentially) weak lower semi-continuity of the H^1-norm, we deduce that

$$\|\mathbf{u}_V\|_{H^1(V)} \leq \liminf_{l\to\infty} \|\mathbf{v}_{Vk_l}\|_{H^1(V)} \text{ everywhere in } \overline{J_0}.$$

In virtue of the first convergence in (3.51) (along with (3.52) and (3.53)), the above inequality reads as

$$\|\nabla_w \mathbf{u}_V\|_{L^2(V)} \leq \liminf_{l\to\infty} \|\nabla_w \mathbf{v}_{Vk_l}\|_{L^2(V)} \text{ everywhere in } \overline{J_0}, \text{ for every } V \text{ as above.} \tag{3.59}$$

In addition, it is straightforward to check from (3.43) that

$$\mathrm{Re}(\mathbf{v}_{V,k_l}, \zeta) \to \mathrm{Re}(\mathbf{v}_V, \zeta) \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U. \tag{3.60}$$

Let now k_l for $l \leq l_V$ instead of k in (3.57). From (3.58), (3.59), and (3.60), we get

$$E(\mathbf{u}_V) \leq E(u_0) + \epsilon \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U,$$

or else,

$$E(\mathbf{u}_V) \leq E(u_0) \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U, \tag{3.61}$$

since ϵ is arbitrary. In virtue of the first equivalence in (3.45), it only remains to consider in (3.61) an increasing sequence $\{V_k \subset U\}_k$ of open and bounded sets with $V_k \to U$, e.g., $V_k = U_k$ for every k and to let $k \to \infty$, in order to conclude with the claimed energy estimate.

Step 5γ

The (conjugate) symmetry around $t = 0$, follows directly from the convergence in (3.41) along with the fact that every $\mathbf{v}_{k_l}$ satisfies the same symmetry.

Let us note that the defocusing nature of the equation permitted a stronger version of a local existence result: Instead of the standard statement "for every initial condition, there exists a bounded interval ... ", we established in Theorem 3.1 and Theorem 3.2 that "for any initial condition and any bounded interval J_0 ... ". Hence the local-in time-solutions can be extended to global ones as remarked in the proof of the following result.

Theorem 3.3. *Let* **u** *be as in Theorem 3.1, or as in Theorem 3.2. If*

i. i. $n = 1$,
ii. $n = 2$ *and* $\tau = 1$, *or*
iii. $U = \mathbb{R}^n$ $(n = 1, 2, 3)$,

then **u** *is unique and global, for which the energy is conserved.*

Proof. Since local-in-time solutions exist for any given bounded interval J_0, their uniqueness would imply their global-in-time existence.

As far as the uniqueness is concerned, we refer to [19], where for the case i. the embedding $H_0^m(U) \hookrightarrow L^\infty(U)$ is employed. As for ii., either the Trudinger or the following version of the Gagliardo–Nirenberg interpolation inequality[5] is employed:

$$\|u\|_{L^{2\tau}(U)} \le C\tau^{\frac{1}{2}} \|\nabla_w u\|_{L^2(U)}^{1-\frac{1}{\tau}} \|u\|_{L(U)}^{\frac{1}{\tau}}, \ \ \forall u \in H_0^1(U), \ \ \forall \tau \in [1, \infty), \ \ n = 2.$$

Finally, for the case iii., the result follows from the Strichartz (dispersive) estimates.

Moreover, one can utilize the above uniqueness result along with the backwards-in-time existence of the solution, in order to eventually establish conservation of energy for this solution. This crucial property can be proved as in [20, Proposition 8]. □

4. Regularity of solutions

In this section, we study the regularity of the solutions of Section 3. In particular, we consider the problem (2.14) only for the cases where τ is as in (1.7). We recall that a solution of such a problem possesses certain fine properties, such as uniqueness, global existence, and conservation of energy. We will establish that if the initial datum is infinitely smooth, then so is the solution.

Before we proceed to the statement and proof of the main results, we provide some preliminary ones. First, we derive an estimate with the use of the following Gagliardo–Nirenberg interpolation inequality

$$\|\nabla^j u\|_{L^{\frac{2m}{j}}(\mathbb{R}^n)} \le C \|\nabla^m u\|_{L^2(\mathbb{R}^n)}^{\frac{j}{m}} \|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{m}}, \ \ \forall\, j = 0, 1, \ldots, m, \ \ \forall u \in C_c^\infty(\mathbb{R}^n), \tag{4.1}$$

which allows us to handle certain types of non-linearities such as the ones assumed herein.

Proposition 4.1. *Let* $m \in \mathbb{N}$ *and* $f \in C^m([0, \infty); \mathbb{R})$. *Then, for every* $u \in C_c^\infty(\mathbb{R}^n)$,

$$\sum_{k=1}^m \|\nabla^k \left(f\left(|u|^2\right) u\right)\|_{L^2(\mathbb{R}^n)} \le C \left(\sum_{k=1}^m \|\nabla^k u\|_{L^2(\mathbb{R}^n)}\right) \left(\sum_{k=0}^m \|f^{(k)}\|_{L^\infty\left(\left(0, \|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)} \|u\|^{2k}_{L^\infty(\mathbb{R}^n)}\right). \tag{4.2}$$

[5]For an elegant proof of the form of the constant in this inequality we refer to *Lemma* 2 in [26] and the references therein.

Assuming further that $f \neq \text{const}$ *with* $f(0) = 0$, *the above estimate becomes*

$$\sum_{k=1}^{m} \|\nabla_w^k \Big(f\big(|u|^2\big) u \Big)\|_{L^2(U)} \leq C\left(\sum_{k=1}^{m} \|\nabla_w^k u\|_{L^2(U)} \right) \left(\sum_{k=1}^{m} \|f^{(k)}\|_{L^\infty\left(\left(0, \|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)} \|u\|^{2k}_{L^\infty(U)} \right),$$

for every $u \in C_c^\infty(U)$, *along with the obvious generalization for* $f^{(k)}(0) = 0$, *with* $k = 1, \ldots, m-1$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| \leq m$ be arbitrary. From the Leibniz rule and the multivariate Faá di Bruno formula (see [7, 9]), we have that[6]

$$\begin{aligned} D^\alpha\Big(f\big(|u|^2\big)u\Big) &= f\big(|u|^2\big)D^\alpha u + \sum_{\substack{|\alpha_1| + |\alpha_2| = |\alpha|, \\ |\alpha_1| \geq 1}} D^{\alpha_1}\Big(f\big(|u|^2\big)\Big)D^{\alpha_2}u = \\ &= f\big(|u|^2\big)D^\alpha u + \sum_{\substack{|\alpha_1| + |\alpha_2| = |\alpha|, \\ |\alpha_1| \geq 1}} \sum_{1 \leq |\beta| \leq |\alpha_1|} M_{\alpha_1, |\beta|}\big(|u|^2\big) D^{|\beta|} f\big(|u|^2\big) D^{\alpha_2} u := I_1 + I_2, \end{aligned}$$

where

$$M_{\alpha_1, |\beta|}\big(|u|^2\big) := \alpha_1! \sum_{s=1}^{|\alpha_1|} \sum_{p_s(\alpha_1, |\beta|)} \prod_{j=1}^{s} \frac{1}{\gamma_j!\big(\delta_j!\big)^{|\gamma_j|}} \big(D^{\delta_j}|u|^2\big)^{\gamma_j},$$

with $\gamma_j \in \mathbb{N}, \delta_j \in \mathbb{N}_0^n$,

$$p_s(\alpha_1, |\beta|) := \left\{ (\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_s) \mid 0 \prec \delta_1 \prec \ldots \prec \delta_s, \ \sum_{j=1}^{s} \gamma_j = |\beta|, \ \sum_{j=1}^{s} \gamma_j \delta_j = \alpha_1 \right\}$$

and $\mu \prec \nu$ for $\mu, \nu \in \mathbb{N}_0^n$ as in [7].

I_1 can be estimated easily. Indeed,

$$\|I_1\|_{L^2(\mathbb{R}^n)} \leq \|D^\alpha u\|_{L^2(\mathbb{R}^n)} \|f\|_{L^\infty\left(\left(0, \|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)}.$$

As far as I_2 is concerned, we have

$$\|I_2\|_{L^2(\mathbb{R}^n)} \leq C \sum_{\substack{|\alpha_1| + |\alpha_2| = |\alpha|, \\ |a_1| \geq 1}} \sum_{l=1}^{|\alpha_1|} \|f^{(l)}\|_{L^\infty\left(\left(0, \|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)} \sum_{s=1}^{|\alpha_1|} \sum_{p_s(\alpha_1, l)} I_{2'},$$

where

$$I_2' := \left\| \prod_{j=1}^{s} \big(D^{\delta_j}|u|^2\big)^{\gamma_j} D^{\alpha_2} u \right\|_{L^2(\mathbb{R}^n)} = \left\| \prod_{i_1=1}^{\gamma_1} \big(D^{\delta_1}|u|^2\big) \ldots \prod_{i_s=1}^{\gamma_s} \big(D^{\delta_s}|u|^2\big) D^{\alpha_2} u \right\|_{L^2(\mathbb{R}^n)}.$$

From Hölder's inequality for $p_{j, i_j} = \frac{|\alpha|}{|\delta_j|}$, for $i_j = 1, \ldots, \gamma_j, j = 1, \ldots, s$ and $p_{s+1} = \frac{|\alpha|}{|\alpha_2|}$, we get

$$I_2' \leq \prod_{i_1=1}^{\gamma_1} \|D^{\delta_1}|u|^2\|_{L^{\frac{2|\alpha|}{|\delta_1|}}(\mathbb{R}^n)} \ldots \prod_{i_s=1}^{\gamma_s} \|D^{\delta_s}|u|^2\|_{L^{\frac{2|\alpha|}{|\delta_s|}}(\mathbb{R}^n)} \|D^{\alpha_2}u\|_{L^{\frac{2|\alpha|}{|\alpha_2|}}(\mathbb{R}^n)}.$$

From the Leibniz rule, we have

[6] If $n = 1$, then $D^\beta = D^{|\beta|}$, for every multi-index β.

$$D^{\delta_j}|u|^2 = \sum_{|\delta_{1,j}|+|\delta_{2,j}|=|\delta_j|} D^{\delta_{1,j}}u \; D^{\delta_{2,j}}\bar{u},$$

thus, once again, from Hölder's inequality for $p_1 = \frac{|\delta_j|}{|\delta_{1,j}|}$ and $p_2 = \frac{|\delta_j|}{|\delta_{2,j}|}$, we get

$$\|D^{\delta_j}|u|^2\|_{L^{\frac{2|\alpha|}{|\delta_j|}}(\mathbb{R}^n)} \le \sum_{|\delta_{1,j}|+|\delta_{2,j}|=|\delta_j|} \|D^{\delta_{1,j}}u\|_{L^{\frac{2|\alpha|}{|\delta_{1,j}|}}(\mathbb{R}^n)} \|D^{\delta_{2,j}}u\|_{L^{\frac{2|\alpha|}{|\delta_{2,j}|}}(\mathbb{R}^n)}.$$

Hence, applying (4.1), we deduce that

$$\|D^{\delta_j}|u|^2\|_{L^{\frac{2|\alpha|}{|\delta_j|}}(\mathbb{R}^n)} \le C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)}^{\frac{|\delta_j|}{|\alpha|}} \|u\|_{L^\infty(\mathbb{R}^n)}^{2-\frac{|\delta_j|}{|\alpha|}}.$$

Again from (4.1), we get

$$\|D^{\alpha_2}u\|_{L^{\frac{2|\alpha|}{|\alpha_2|}}(\mathbb{R}^n)} \le C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)}^{\frac{|\alpha_2|}{|\alpha|}} \|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\alpha_2|}{|\alpha|}}.$$

Therefore,

$$I_2' \le C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)} \|u\|_{L^\infty(\mathbb{R}^n)}^{2l},$$

and so

$$\|I_2\|_{L^2(\mathbb{R}^n)} \le C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)} \sum_{1\le|\beta|\le|\alpha|} \sum_{l=1}^{|\beta|} \|f^{(l)}\|_{L^\infty\left(\left(0,\|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)} \|u\|_{L^\infty(\mathbb{R}^n)}^{2l}.$$

If $n < 2m$, we directly deduce that the above results hold for every $u \in H_0^m(U)$ and every arbitrary U, by employing the $\mathcal{E}_0(U,\mathbb{R}^n)$ operator and the scaling-invariant Sobolev embedding $H_0^m(U) \hookrightarrow L^\infty(U)$.

Now, in virtue of Theorem A.1, we extend Proposition 4.1 for functions in non-zero-trace Sobolev spaces.

Corollary 4.1. *Let U with $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L), m \in \mathbb{N}$ with $n < 2m, f \in C^m([0,\infty);\mathbb{R})$ and $u \in H^m(U)$. Then $(f(|u|^2)u) \in H^m(U)$, satisfying the inequality*

$$\sum_{k=1}^m \|\nabla_w^k\left(f(|u|^2)u\right)\|_{L^2(U)} \le C(K,L)\left(\sum_{k=0}^m \frac{1}{\varepsilon^{m-k}}\|\nabla_w^k u\|_{L^2(U)}\right) \times \left(\sum_{k=0}^m \|f^{(k)}\|_{L^\infty\left(\left(0,C(K)\|u\|^2_{L^\infty(U)}\right)\right)} \|u\|_{L^\infty(U)}^{2k}\right). \tag{4.3}$$

Proof. By considering the extended function, we see that (4.2) gets the form

$$\begin{aligned}\sum_{k=1}^m \|\nabla^k\left(f(|u|^2)u\right)\|_{L^2(U)} &\le \sum_{k=1}^m \left\|\nabla^k\left(f\left(|(\mathcal{E}(U,\mathbb{R}^n))u|^2\right)(\mathcal{E}(U,\mathbb{R}^n))u\right)\right\|_{L^2(\mathbb{R}^n)} \\ &\le C\left(\sum_{k=1}^m \|\left(\nabla^k \circ (\mathcal{E}(U,\mathbb{R}^n))\right)u\|_{L^2(\mathbb{R}^n)}\right) \\ &\times \left(\sum_{k=0}^m \|f^{(k)}\|_{L^\infty\left(\left(0,\|(\mathcal{E}(U,\mathbb{R}^n))u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)} \|(\mathcal{E}(U,\mathbb{R}^n))u\|_{L^\infty(\mathbb{R}^n)}^{2k}\right).\end{aligned}$$

From the bounds given in Theorem A.1 we obtain

$$\sum_{k=1}^{m}\|\big(\nabla^k\circ(\mathcal{E}(U,\mathbb{R}^n))\big)u\|_{L^2(\mathbb{R}^n)}\le C(K,L)\left(\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla^k u\|_{L^2(U)}\right).$$

Moreover, in view of Corollary A.2, we have that $u\in L^\infty(U)$. Therefore, again from the aforementioned bounds, we get

$$\|(\mathcal{E}(U,\mathbb{R}^n))u\|_{L^\infty(\mathbb{R}^n)}\le C(K)\|u\|_{L^\infty(U)}.$$

Thereby, the claimed result follows. □

For the next result, we notice that if $U\in\mathbb{R}$ with $\partial U\in\mathrm{Lip}^m(\varepsilon,K,L)$ for some $m\in\mathbb{N}$, then in fact $\partial U\in\mathrm{Lip}^1(\varepsilon,K,0)$, and vice versa.

Corollary 4.2. *Let $U\subset\mathbb{R}$ with $|U|<\infty$ as well as $\partial U\in\mathrm{Lip}^1(\varepsilon,K,0),m\in\mathbb{N}\setminus\{1\},f\in C^m([0,\infty);\mathbb{R}),u\in H^m(U)$ and $\zeta\in X^m(U)$. Then $(f(|u+\zeta|^2)(u+\zeta))\in H^m(U)$, satisfying the inequality*

$$\sum_{k=1}^{m}\|\nabla_w^k\Big(f\big(|u+\zeta|^2\big)(u+\zeta)\Big)\|_{L^2(U)}\le C\Big(\frac{1}{\varepsilon^m}\max\Big\{1,|U|^{\frac12}\Big\},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\Big)\times\left(1+\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}\right).\tag{4.4}$$

If, in addition, $u\in H^m(U)\cap H_0^1(U)$, as well as

$$\big(\Delta_w^j u\big)\in H_0^1(U),\ \ \forall j=\ 0,\dots,\left\lfloor\frac{m}{2}\right\rfloor-1,$$

then we have

$$\sum_{k=1}^{m}\|\nabla_w^k\Big(f\big(|u+\zeta|^2\big)(u+\zeta)\Big)\|_{L^2(U)}\le C\Big(\frac{1}{\varepsilon^m}\max\Big\{1,|U|^{\frac12}\Big\},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\Big)\times\left(1+\sum_{\substack{2j+1\le m\\ j\in\mathbb{N},}}\|(\nabla_w\circ\Delta_w^j)u\|_{L^2(U)}+\sum_{\substack{2j\le m\\ j\in\mathbb{N},}}\|\Delta_w^j u\|_{L^2(U)}\right).\tag{4.5}$$

Proof. We have that $\zeta\in H^m(U)$, since $|U|<\infty$, hence $(u+\zeta)\in H^m(U)$. Employing (4.3), we get

$$\sum_{k=1}^{m}\|\nabla_w^k\Big(f\big(|u+\zeta|^2\big)(u+\zeta)\Big)\|_{L^2(U)}\le C(K)\left(\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k(u+\zeta)\|_{L^2(U)}\right)\times\left(\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,C(K)\|u+\zeta\|^2_{L^\infty(U)}\right)\right)}\|u+\zeta\|^{2k}_{L^\infty(U)}\right).$$

For the term inside the first parenthesis we have

$$
\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k(u+\zeta)\|_{L^2(U)}\\
&\le \sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k u\|_{L^2(U)}+\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k\zeta\|_{L^2(U)}\\
&\le \max\left\{1,\frac{1}{\varepsilon^m}\right\}\sum_{k=0}^{m}\|\nabla_w^k u\|_{L^2(U)}+\frac{1}{\varepsilon^m}\|\zeta\|_{L^2(U)}+\max\left\{1,\frac{1}{\varepsilon^m}\right\}\sum_{k=1}^{m}\|\nabla_w^k\zeta\|_{L^2(U)}\\
&\le C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}+\|u\|_{H^1(U)}\right)+\frac{1}{\varepsilon^m}|U|^{\frac{1}{2}}\|\zeta\|_{X^m(U)}\\
&+C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\|\zeta\|_{X^m(U)}\\
&\le C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}+\|u\|_{H^1(U)}\right)\\
&+\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\|\zeta\|_{X^m(U)}+C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\|\zeta\|_{X^m(U)}\\
&\le C\max\left\{1,\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\right\}\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}+\|u\|_{H^1(U)}+\|\zeta\|_{X^m(U)}\right)\\
&\le C\max\left\{1,\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\right\}\max\left\{1,\|u\|_{H^1(U)}+\|\zeta\|_{X^m(U)}\right\}\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}+1\right)\\
&=C\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}+1\right).
\end{aligned}
$$

As for the term inside the second parenthesis, we have that

$$\|u\|_{L^\infty(U)}\le C\left(\frac{1}{\varepsilon},K\right)\|u\|_{H^1(U)},$$

from the scaling dependent embedding $H^1(U)\hookrightarrow L^\infty(U)$ (see Corollary A.2), which implies

$$\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,C(K)\|u+\zeta\|^2_{L^\infty(U)}\right)\right)}\|u+\zeta\|^{2k}_{L^\infty(U)}\le C\left(\frac{1}{\varepsilon},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right).$$

Directly from (4.4) and the bound in Proposition A.9, we get (4.5). □

Lastly, the following version of the Brezis–Gallouët–Wainger inequality

$$\|u\|_{L^\infty(\mathbb{R}^2)}\le C(\|u\|_{H^1(\mathbb{R}^2)})\left(1+\left(\ln\left(1+\|\nabla^2 u\|_{L^2(\mathbb{R}^2)}\right)\right)^{\frac{1}{2}}\right),\quad \forall u\in C_c^\infty(\mathbb{R}^2),\tag{4.6}$$

which can be proved by a straightforward adaptation of [4, *Lemma* 2], is essential in order to establish the following useful auxiliary inequalities.

Lemma 4.1. *Let $U \subseteq \mathbb{R}^2$ with $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L)$ and $m \in \mathbb{N} \setminus \{1\}$. Then*

$$\|u\|_{L^\infty(U)} \le C\left(\frac{1}{\varepsilon}, K, L, \|u\|_{H^1(U)}\right)\left(1 + \left(\ln\left(1 + \sum_{k=2}^{m} \|\nabla_w^k u\|_{L^2(U)}\right)\right)^{\frac{1}{2}}\right), \quad \forall u \in H^m(U). \tag{4.7}$$

Proof. Let $u \in H^m(U)$ be arbitrary. Since $m \ge 2$, then $u \in L^\infty(U)$. Considering the extended function, (4.6) becomes

$$\|u\|_{L^\infty(U)} \le C\left(\frac{1}{\varepsilon}, K, L, \|u\|_{H^1(U)}\right)\left(1 + \left(\ln\left(1 + \|\nabla_w^2 u\|_{L^2(U)}\right)\right)^{\frac{1}{2}}\right),$$

whereby (4.7) follows. □

Corollary 4.3. *Let $U \subset \mathbb{R}^2$ with $|U| < \infty$ as well as $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L), m \in \mathbb{N} \setminus \{1\}, u \in H^m(U)$ and $\zeta \in X^m(U)$. Then $(|u + \zeta|^2 (u + \zeta)) \in H^m(U)$, satisfying*

$$\begin{aligned}\sum_{k=1}^{m} \|\nabla_w^k (|u+\zeta|^2 (u+\zeta))\|_{L^2(U)} &\le C\left(\frac{1}{\varepsilon^m} \max\left\{1, |U|^{\frac{1}{2}}\right\}, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\right) \\ &\times \left(1 + \sum_{k=2}^{m} \|\nabla_w^k u\|_{L^2(U)}\right)\left(1 + \ln\left(1 + \left(\sum_{k=2}^{m} \|\nabla_w^k u\|_{L^2(U)}\right)^2\right)\right).\end{aligned} \tag{4.8}$$

If, in addition, $\partial U \in \mathrm{Lip}^m(\varepsilon, K, L), u \in H^m(U) \cap H_0^1(U)$, as well as

$$(\Delta_w^j u) \in H_0^1(U), \quad \forall j = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1,$$

then we have

$$\begin{aligned}&\sum_{k=1}^{m} \|\nabla_w^k (|u+\zeta|^2 (u+\zeta))\|_{L^2(U)} \le C\left(\frac{1}{\varepsilon^m} \max\{1, |U|^{\frac{1}{2}}\}, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\right) \\ &\times \left(1 + \sum_{\substack{2j+1 \le m \\ j \in \mathbb{N},}} \|(\nabla_w \circ \Delta_w^j) u\|_{L^2(U)} + \sum_{\substack{2j \le m \\ j \in \mathbb{N},}} \|\Delta_w^j u\|_{L^2(U)}\right) \\ &\times \left(1 + \ln\left(1 + \sum_{\substack{2j+1 \le m \\ j \in \mathbb{N},}} \|(\nabla_w \circ \Delta_w^j) u\|_{L^2(U)}^2 + \sum_{\substack{2j \le m \\ j \in \mathbb{N},}} \|\Delta_w^j u\|_{L^2(U)}^2\right)\right).\end{aligned} \tag{4.9}$$

Proof. We have that $\zeta \in H^m(U)$, since $|U| < \infty$. Hence, $(u + \zeta) \in H^m(U)$. Employing (4.3), we get

$$\sum_{k=1}^{m} \|\nabla_w^k (|u+\zeta|^2 (u+\zeta))\|_{L^2(U)} \le C(K, L)\left(\sum_{k=0}^{m} \frac{1}{\varepsilon^{m-k}} \|\nabla_w^k (u+\zeta)\|_{L^2(U)}\right)\|u+\zeta\|_{L^\infty(U)}^2.$$

In order to estimate the term inside the parenthesis, we work exactly as in Corollary 4.2 and we deduce that

$$\sum_{k=1}^{m}\|\nabla_w^k\big(|u+\zeta|^2(u+\zeta)\big)\|_{L^2(U)} \le C\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right) \times\left(1+\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}\right)\|u+\zeta\|^2_{L^\infty(U)}.$$

For the last term, we employ (4.7) to get

$$\begin{aligned}\|u+\zeta\|^2_{L^\infty(U)} &\le C\Big(\|u\|^2_{L^\infty(U)}+\|\zeta\|^2_{L^\infty(U)}\Big) \le C\Big(\|\zeta\|_{X^m(U)}\Big)\Big(1+\|u\|^2_{L^\infty(U)}\Big)\\ &\le C\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(1+\ln\left(1+\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}\right)\right)\\ &\le C\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(1+\ln\left(1+\left(\sum_{k=2}^{m}\|\nabla_w^k u\|_{L^2(U)}\right)^2\right)\right).\end{aligned}$$

Now, directly from (4.8) and the bound in Proposition A.9, we get (4.9). □

We are ready to proceed to the statement and the proof of the main results of this section.

Theorem 4.1. *Let $n=1,2$, U be bounded, τ be as in (1.7), $u_0\in H_0^1(U)$ and $\boldsymbol{u}$ be the (unique and global) solution of (2.14) that* Theorem 3.1 *provides. If*

1. $\partial U\in\bigcap_{m=1}^{\infty}\mathrm{Lip}^m(\varepsilon,K,L_m)$,
2. $\zeta\in\bigcap_{m=1}^{\infty}X^m(U)$ *and*
3. $u_0\in\bigcap_{m=2}^{\infty}H^m(U)\cap H_0^1(U)$, *with* $(\Delta^j u_0)\in H_0^1(U)$ *for every* $j\in\mathbb{N}_0$,

then $\mathbf{u}\in L^\infty_{\mathrm{loc}}(\mathbb{R};\cap_{m=2}^{\infty}H^m(U)\cap H_0^1(U))\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\cap_{m=0}^{\infty}H^m(U))$, *satisfying*

$$\|\mathbf{u}\|_{L^\infty(J_0;H^m(U))}+\|\mathbf{u}'\|_{L^\infty(J_0;H^{m-2}(U))} \le C\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L_m,\|u_0\|_{H^m(U)},\|\zeta\|_{X^{m+2}(U)},\||\zeta|-\rho\|_{L^2(U)},|J_0|\right), \tag{4.10}$$

for every $m\in\mathbb{N}\setminus\{1\}$ and every J_0.

Proof. It suffices to show (4.10). Let $m\in\mathbb{N}\setminus\{1\}$ and J_0 be arbitrary. We set

$$\tilde{C}:=C\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L_m,\|u_0\|_{H^m(U)},\|\zeta\|_{X^{m+2}(U)},\||\zeta|-\rho\|_{L^2(U)},|J_0|\right).$$

Step 1

Let $\{\mathbf{u}_k\}_{k=1}^{\infty}$ be the Faedo-Galerkin approximations, as in the proof of Theorem 3.1. We recall that for every w_l there exists $\lambda_l>0$, such that $-\Delta_w w_l=\lambda_l w_l$ in $H^{-1}(U)$. In virtue of Theorem A.3, $-\Delta_w w_l=\lambda_l w_l$ everywhere in U (and not just almost everywhere). Therefore, $(-1)^j\Delta_w^j w_l=\lambda_l^j w_l$ everywhere in U, for every $j\in\mathbb{N}$, that is $\Delta_w^j(\mathbf{u}_k(0))\in\mathrm{span}\{w_l\}_{l=1}^{k}$, for every $j\in\mathbb{N}_0$, and so

$$\{\mathbf{u}_k\}_{k=1}^{\infty} \subset C^{\infty}\left(\mathbb{R}, \bigcap_{m=2}^{\infty} H^m(U) \cap H_0^1(U)\right),$$

as well as

$$\left(\Delta^j(\mathbf{u}_k(0)), \mathbf{u}_k(0)\right) = \left(\Delta^j(\mathbf{u}_k(0)), u_0\right), \ \forall j \in \mathbb{N}_0. \tag{4.11}$$

Moreover, we have

$$\left(\Delta^i(\mathbf{u}_k(0)), \Delta^j(\mathbf{u}_k(0))\right) = \left((\mathbf{u}_k(0)), \Delta^{i+j}(\mathbf{u}_k(0))\right), \ \forall i,j \ \in \mathbb{N}_0, \tag{4.12}$$

as it follows directly from the common integration by parts formula. Now, we claim that

$$\sum_{\substack{j \in \mathbb{N}, \\ 2j+1 \le m}} \|(\nabla \circ \Delta^j)(\mathbf{u}_k(0))\|_{L^2(U)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \le m}} \|\Delta^j(\mathbf{u}_k(0))\|_{L^2(U)} \le C\left(\frac{1}{\varepsilon}, K, L_m\right)\|u_0\|_{H^m(U)}.$$

In view of Proposition A.9, it suffices to show that

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{N}, \\ 2j+1 \le m}} \|(\nabla \circ \Delta^j)(\mathbf{u}_k(0))\|_{L^2(U)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \le m}} \|\Delta^j(\mathbf{u}_k(0))\|_{L^2(U)} \\
&\le C\left(\sum_{\substack{j \in \mathbb{N}_0, \\ 2j+1 \le m}} \|(\nabla \circ \Delta^j)u_0\|_{L^2(U)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \le m}} \|\Delta^j u_0\|_{L^2(U)}\right).
\end{aligned}$$

Indeed, from

1. (4.11),
2. (4.12),
3. the common integration by parts formula,

we obtain, for every $j \in \mathbb{N}$, that

$$\begin{aligned}
\|\Delta^j(\mathbf{u}_k(0))\|_{L^2(U)}^2 &= \left(\Delta^j(\mathbf{u}_k(0)), \Delta^j(\mathbf{u}_k(0))\right) \stackrel{1.}{=} \left(\mathbf{u}_k(0), \Delta^{2j}(\mathbf{u}_k(0))\right) \\
&\stackrel{2.}{=} \left(u_0, \Delta^{2j}(\mathbf{u}_k(0))\right) \stackrel{1.}{=} \left(\Delta^j u_0, \Delta^j(\mathbf{u}_k(0))\right) \\
&\le \frac{1}{2}\|\Delta^j(\mathbf{u}_k(0))\|_{L^2(U)}^2 + \frac{1}{2}\|\Delta^j u_0\|_{L^2(U)}^2,
\end{aligned}$$

as well as

$$\begin{aligned}
\|(\nabla \circ \Delta^j)(\mathbf{u}_k(0))\|_{L^2(U)}^2 &= \left((\nabla \circ \Delta^j)(\mathbf{u}_k(0)), (\nabla \circ \Delta^j)(\mathbf{u}_k(0))\right) \\
&\stackrel{3.}{=} -\left(\Delta^j(u_k(0)), \Delta^{j+1}(\mathbf{u}_k(0))\right) \stackrel{1.}{=} -\left(\mathbf{u}_k(0), \Delta^{2j+1}(\mathbf{u}_k(0))\right) \\
&\stackrel{2.}{=} -\left(u_0, \Delta^{2j+1}(\mathbf{u}_k(0))\right) \underset{3.}{\stackrel{1.}{=}} \left((\nabla \circ \Delta^j)u_0, (\nabla \circ \Delta^j)(\mathbf{u}_k(0))\right) \\
&\le \frac{1}{2}\|(\nabla \circ \Delta^j)(\mathbf{u}_k(0))\|_{L^2(U)}^2 + \frac{1}{2}\|(\nabla \circ \Delta^j)u_0\|_{L^2(U)}^2.
\end{aligned}$$

Step 2

We multiply the variational Eq. (3.3) by

$$\begin{cases} d_k^l(t)\lambda_l^{2j}, & \text{for every } j \in \mathbb{N} \text{ such that } 2j \leq m, \\ -d_k^l(t)\lambda_l^{2j+1}, & \text{for every } j \in \mathbb{N} \text{ such that } 2j+1 \leq m, \end{cases}$$

sum on $l = 1, ..., k$, integrate by parts keeping imaginary parts of both sides to find

$$\frac{1}{2}\frac{d}{dt}\|\Delta^j \mathbf{u}_k\|^2_{L^2(U)} - \operatorname{Im}\big(\Delta^{j+1}\zeta, \Delta^j \mathbf{u}_k\big) - \operatorname{Im}\big(\Delta^j\big(\big(|\mathbf{u}_k + \zeta|^{2\tau}\big)(\mathbf{u}_k + \zeta)\big), \Delta^j \mathbf{u}_k\big) = 0,$$

for every $j \in \mathbb{N}$ with $2j \leq m$, and

$$\frac{1}{2}\frac{d}{dt}\|(\nabla \circ \Delta^j)\mathbf{u}_k\|^2_{L^2(U)} - \operatorname{Im}\big((\nabla \circ \Delta^{j+1})\zeta, (\nabla \circ \Delta^j)\mathbf{u}_k\big) - \\ -\operatorname{Im}\big((\nabla \circ \Delta^j)\big(\big(|\mathbf{u}_k + \zeta|^{2\tau}\big)(\mathbf{u}_k + \zeta)\big), (\nabla \circ \Delta^j)\mathbf{u}_k\big) = 0,$$

for every $j \in \mathbb{N}$ with $2j + 1 \leq m$. We sum the above equations for every j, integrate with respect to t, employ Young's and Hölder's inequalities, as well as (4.5) and (4.9), along with the estimate for the H^1-norm of each u_k given in the proof of Theorem 3.1. With such a procedure, we derive the estimates

$$A \leq \tilde{C}\left(1 + \left|\int_0^t A ds\right|\right), \quad \text{for every } t \in J_0, \text{ if } n = 1,$$

and also

$$A \leq \tilde{C}\left(1 + \left|\int_0^t A(1 + \ln(1 + A))ds\right|\right), \quad \text{for every } t \in J_0, \text{ if } n = 2,$$

where

$$A := \sum_{\substack{j \in \mathbb{N}, \\ 2j+1 \leq m}} \|(\nabla \circ \Delta^j)\mathbf{u}_k\|^2_{L^2(U)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \leq m}} \|\Delta^j \mathbf{u}_k\|^2_{L^2(U)}.$$

Consequently, $A \leq \tilde{C}$ everywhere in J_0, which, if combined with the estimate for the H^1-norm of each $\mathbf{u}_k$ given in the proof of Theorem 3.1, gives us the bound

$$\|\mathbf{u}_k\|_{L^\infty(J_0;H^m(U))} \leq \tilde{C}, \quad \forall k \in \mathbb{N}.$$

This is due to the fact that every $\mathbf{u}_k$ satisfies the necessary compatibility conditions for the validity of Proposition A.9. Now, working in an analogous manner as in Step 3β of the proof of Theorem 3.1, we deduce that $\mathbf{u} \in L^\infty(J_0; H^m(U))$ with

$$\|\mathbf{u}\|_{L^\infty(J_0;H^m(U))} \leq \tilde{C}.$$

Moreover, directly from the differential equation, we deduce that $\mathbf{u}' \in L^\infty(J_0; H^{m-2}(U))$ with

$$\|\mathbf{u}'\|_{L^\infty(J_0;H^{m-2}(U))} \leq \tilde{C}.$$

Employing the same arguments as in the proof of Theorem 4.1, after the differentiation of the approximating equations with respect to the temporal variable[7], we can show by induction, its following generalization (the proof is omitted for the sake of brevity).

Corollary 4.4. *Let $n = 1, 2$, U be bounded, τ be as in (1.7), $u_0 \in H_0^1(U)$ and* **u** *be the (unique and global) solution of (2.14) that Theorem 3.1 provides. If*

1. 1. $\partial U \in \cap_{m=1}^{\infty} \mathrm{Lip}^m(\varepsilon, K, L_m)$,
2. 2. $\zeta \in \cap_{m=1}^{\infty} X^m(U)$ *and*
3. 3. $u_0 \in \cap_{m=2}^{\infty} H^m(U) \cap H_0^1(U)$, *with* $(\Delta^j u_0) \in H_0^1(U)$ *for every* $j \in \mathbb{N}_0$,

then $\mathbf{u} \in \cap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}(\mathbb{R}; \cap_{m=2}^{\infty} H^m(U))$, *with*

$$\|\mathbf{u}^{(j)}\|_{L^\infty(J_0;H^m(U))} \le C\left(\frac{1}{\varepsilon^m}\max\left\{1, |U|^{\frac{1}{2}}\right\}, K, L_m, \|u_0\|_{H^m(U)}, \|\zeta\|_{X^{m+2}(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\right), \tag{4.13}$$

for every $j \in \mathbb{N}_0$, every $m \in \mathbb{N} \setminus \{1\}$ and every J_0.

We conclude, by showing the corresponding regularity result for the case where $U = \mathbb{R}^n, n = 1, 2$.

Theorem 4.2. *Let $n = 1, 2$, τ be as in (1.7), $u_0 \in H^1(\mathbb{R}^n)$ and* **u** *be the (unique and global) solution of (2.14) that Theorem 3.2 provides. If*

1. $\zeta \in \cap_{m=1}^{\infty} X^m(\mathbb{R}^n)$ *and*
2. $u_0 \in \cap_{m=2}^{\infty} H^m(\mathbb{R}^n)$,

then $\mathbf{u} \in \cap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}(\mathbb{R}; \cap_{m=1}^{\infty} H^m(\mathbb{R}^n))$, *with*

$$\|\mathbf{u}^{(j)}\|_{L^\infty\left(J_0;H^m(\mathbb{R}^n)\right)} \le C\left(\|u_0\|_{H^m(\mathbb{R}^n)}, \|\zeta\|_{X^{m+2}(\mathbb{R}^n)}, \||\zeta| - \rho\|_{L^2(\mathbb{R}^n)}, |J_0|\right), \tag{4.14}$$

for every $j \in \mathbb{N}_0$, every $m \in \mathbb{N} \setminus \{1\}$ and every J_0.

Proof. We set again

$$\tilde{C} := C\left(\|u_0\|_{H^m(\mathbb{R}^n)}, \|\zeta\|_{X^{m+2}(\mathbb{R}^n)}, \||\zeta| - \rho\|_{L^2(\mathbb{R}^n)}, |J_0|\right).$$

Let $\{\mathbf{u}_k\}_{k=1}^{\infty}$ be the sequence of solutions, as in the proof of Theorem 3.2. Since

$$B_1 \in \bigcap_{m=1}^{\infty} \mathrm{Lip}^m(\varepsilon, K, L_m),$$

then, in view of Proposition A.4, we deduce that

$$U_k \equiv B_k \in \bigcap_{m=1}^{\infty} \mathrm{Lip}^m(k\ \varepsilon, K, L_m), \ \ \forall k \in \mathbb{N}.$$

[7]As we have already noticed in Step 1α of the proof of Theorem 3.1, the Faedo-Galerkin approximations are infinitely smooth with respect to t.

Hence, $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \cap_{k=0}^{\infty} W_{\text{loc}}^{k,\infty}(\mathbb{R}; \cap_{m=1}^{\infty} H^m(U_{k+2}) \cap H_0^1(U_{k+2}))$, with

$$\begin{aligned} &\|\mathbf{u}_k^{(j)}\|_{L^\infty(J_0;H^m(U_{k+2}))} \\ &\le C\left(\frac{1}{((k+2)\varepsilon)^m}\max\left\{1, |U_{k+2}|^{\frac{1}{2}}\right\}, K, L_m, \|u_0\|_{H^m(\mathbb{R}^n)}, \|\zeta\|_{X^{m+2}(\mathbb{R}^n)}, \||\zeta| - \rho\|_{L^2(\mathbb{R}^n)}, |J_0|\right), \end{aligned}$$

for every $j \in \mathbb{N}_0$, every $m \in \mathbb{N} \setminus \{1\}$ and every J_0. Since

$$\frac{1}{((k+2)\varepsilon)^m}\max\left\{1, |U_{k+2}|^{\frac{1}{2}}\right\} = \frac{|U_{k+2}|^{\frac{1}{2}}}{((k+2)\varepsilon)^m} \le C(k+2)^{\frac{n}{2}-m} \le C \text{ uniformly for every } k \in \mathrm{N},$$

we have that

$$\|\mathbf{u}_k^{(j)}\|_{L^\infty(J_0;H^m(U_{k+2}))} \le \tilde{C},$$

for every j, m and J_0 as above. The same bounds are true for the respective norms of $\mathbf{v}_k^{(j)}$. Now, working as in Step 3β of the proof of Theorem 3.1, we deduce that $\mathbf{u}^{(j)} \in L^\infty(J_0; H^m(\mathbb{R}^n))$ with

$$\|\mathbf{u}^{(j)}\|_{L^\infty\left(J_0;H^m(\mathbb{R}^n)\right)} \le \tilde{C}.$$

Remark 4.1. *The usual regularity results for unbounded sets appearing in the literature (see, e.g., Chapter 10 in [3]) also concern sets with bounded boundaries, such as exterior domains, and not only the whole Euclidean space. Such results can be obtained for the classical version of our problem, i.e., for $\zeta, \rho \equiv 0$, by using the techniques presented herein. However, it is not possible to consider $\varepsilon_k = (k+2)\varepsilon \to \infty$ in Theorem 4.2 for the case of a bounded boundary.*

Remark 4.2. *We can also deal with the regular problem in the half-line, by simply considering the odd or the even extension for both u_0 and ζ, depending on the behaviour of these functions at the boundary. This approach is analogous to the use of the sine or cosine Fourier transform for solving problems in the half-line. See also [14], where the Fokas transform method is employed, as well as [11], where the Laplace transform method and the Bourgain $X^{s,b}$ method are combined.*

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Appendix A

In this appendix, we present certain useful, known and new, definitions, and results.

A.1 Restriction and extension operators on open subsets of $\mathbb{R}^n$

Definition A.1. For every $U_1 \subseteq U_2$, we write

$$\mathcal{R}(U_2, U_1) : \mathcal{X}(U_2) \to \mathcal{X}(U_1)$$

for the following (linear) restriction operator

$$[(\mathcal{R}(U_2, U_1))u](x) := u(x), \ \forall x \in U_1, \ \forall u \in \mathcal{X}(U_2)$$

and also

$$\mathcal{E}_0(U_1, U_2) : \mathcal{X}(U_1) \to \mathcal{X}(U_2)$$

for the (linear) extension operator

$$[(\mathcal{E}_0(U_1,U_2))u](x) := \begin{cases} u(x), & \text{if } x \in U_1 \\ 0, & \text{if } x \in U_2 \setminus U_1, \end{cases} \quad \forall u \in \mathcal{X}(U_1).$$

We further define the set of restricted functions from $\mathcal{X}(U_2)$ to $\mathcal{X}(U_1)$

$$(\mathcal{R}(U_2,U_1))(\mathcal{X}(U_2)) := \{(\mathcal{R}(U_2,U_1))u \mid u \in \mathcal{X}(U_2)\},$$

and the set of extended functions (by zero) from $\mathcal{X}(U_1)$ to $\mathcal{X}(U_2)$

$$(\mathcal{E}_0(U_1,U_2))(\mathcal{X}(U_1)) := \{(\mathcal{E}_0(U_1,U_2))u \mid u \in \mathcal{X}(U_1)\}.$$

Proposition A.1. *Let $m \in \mathbb{N}_0, p \in [1,\infty]$ and $U_1 \subseteq U_2$ be arbitrary. Then $\mathcal{R}(U_2,U_1)$ maps isometrically $W^{m,p}(U_2)$ into (but not onto) $W^{m,p}(U_1)$, and*

$$(D_w^\alpha \circ (\mathcal{R}(U_2,U_1)))u = ((\mathcal{R}(U_2,U_1)) \circ D_w^\alpha)u, \text{a.e.inU}_1, \text{for every } \alpha \in \mathbb{N}_0^n \text{ with } 0 \le |\alpha| \le m, \tag{A.1}$$

for every $u \in W^{m,p}(U_2)$. Hence, $W^{m,p}(U_2) \hookrightarrow (\mathcal{R}(U_2,U_1))(W^{m,p}(U_2))$, if we consider the space on the right-hand side as a normed space equipped with its natural norm.

Proposition A.2. *Let $m \in \mathbb{N}_0, p \in [1,\infty]$ and $U_1 \subseteq U_2$ be arbitrary. Then $\mathcal{E}_0(U_1,U_2)$ maps isometrically $W_0^{m,p}(U_1)$ into (not onto) $W_0^{m,p}(U_2)$, and*

$$(D_w^\alpha \circ (\mathcal{E}_0(U_1,U_2)))u = ((\mathcal{E}_0(U_1,U_2)) \circ D_w^\alpha)u, \text{a.e.inU}_2, \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } 0 \le |\alpha| \le \text{m}, \tag{A.2}$$

for every $u \in W_0^{m,p}(U_1)$. Hence, $W_0^{m,p}(U_1) \hookrightarrow (\mathcal{E}_0(U_1,U_2))(W_0^{m,p}(U_1))$, if we consider the space on the right-hand side as a normed space equipped with its natural norm.

Definition A.2. For every $m \in \mathbb{N}_0, p \in [1,\infty]$ and $U_1 \subseteq U_2$, we define

$$\mathcal{R}(U_2,U_1) : W^{-m,p}(U_2) \to W^{-m,p}(U_1)$$

by

$$\langle (\mathcal{R}(U_2,U_1))f, u\rangle := \langle f, (\mathcal{E}_0(U_1,U_2))u\rangle, \ \forall u \in H_0^1(U_2), \ \forall f \in W^{-m,p}(U_2).$$

Evidently,

$$\|(\mathcal{R}(U_2,U_1))f\|_{W^{-m,p}(U_1)} \le \|f\|_{W^{-m,p}(U_1)}, \ \forall f \in W^{-m,p}(U_2),$$

hence, $W^{-m,p}(U_2) \hookrightarrow (\mathcal{R}(U_2,U_1))(W^{-m,p}(U_2))$, if we consider the space on the right-hand side as a normed space equipped with its natural norm.

Proposition A.3. *Let $m \in \mathbb{N}_0, p \in [1,\infty)$, U and $f_1, f_2 \in W^{-m,p}(U)$. If*

$$(\mathcal{R}(U,V))f_1 \equiv (\mathcal{R}(U,V))f_2, \text{ for every open } V \subset\subset U \text{ with } \partial V \text{ being Lipschitz continuous,}$$

then $f_1 \equiv f_2$.

A.2 Uniformly m-Lipschitz boundaries

In this subsection, we recall and generalize some basic results relevant to open sets of $\mathbb{R}^n$ with Lipschitz boundaries. We will generalize the known definition [23, Definition 13.11]), recalling the following:

1. $y = \Phi(x) \in \mathbb{R}^n$ are local coordinates (in this case, $x \in \mathbb{R}^n$ are the background coordinates) when Φ is a rigid motion, i.e., an affine transformation of the form $\Phi(x) = x_0 + ax$, where $x_0 \in \mathbb{R}^n$ and $a \in \mathbb{R}^{n\times n}$ being orthogonal,
2. $f(\cup_{i\in\mathcal{I}} U_i) = \cup_{i\in\mathcal{I}} f(U_i)$ and $f(\cap_{i\in\mathcal{I}} U_i) = \cap_{i\in\mathcal{I}} f(U_i)$, for every bijective f,
3. every function $f : \emptyset \to \mathbb{R}$ is just a real constant and
4. x' stands for the $(n-1)$-dimensional vector obtained by removing the n-th component of a given n-dimensional vector x, i.e., $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

The generalization of the aforementioned definition is as follows.

Definition A.3. Let $m \in \mathbb{N}, \varepsilon \in (0,\infty], K \in \mathbb{N}, L \in [0,\infty)$ and U be an open set. We say that ∂U is uniformly m-Lipschitz with constants ε, K, L and we write $\partial U \in \mathrm{Lip}^m(\varepsilon, K, L)$ if there exists a locally finite countable open cover $\{U_k\}_k$ of ∂U, such that

1. if $x \in \partial U$, then $B(x,\varepsilon) \subseteq U_k$ for some $k \in \mathbb{N}$,
2. every collection of $K+1$ of $\mathrm{U_k}$'s has empty intersection and
3. for every k there exist local coordinates $y_k = \Phi_k(x)$ and a function $\gamma_k : \mathbb{R}^{n-1} \to \mathbb{R}$, such that
4. $\nabla^{j-1}\gamma_k$ is (globally) Lipschitz continuous, for every $j = 1, ..., m$ and every k, with

$$\max_{j=1,\dots,m}\{\mathrm{Lip}(\nabla^{j-1}\gamma_k)\} \le L, \text{ uniformly for every } k,$$

and

i. $\Phi_k(U_k \cap U)(= \Phi_k(U_k) \cap \Phi_k(U)) = \Phi_k(U_k) \cap \{y_k \in \mathbb{R}^n \mid y_{n_k} > \gamma_k(y_k')\}$.

The following result is crucial for Section 4.

Proposition A.4. *If U is such that $\partial U \in \mathrm{Lip}^m(\varepsilon, K, L)$, as well as if Φ is a transformation of the form $\Phi(x) := x_0 + \lambda x$, where $x_0 \in \mathbb{R}^n$ and $\lambda > 1$, then $\partial(\Phi(U)) \in \mathrm{Lip}^m(\lambda\varepsilon, K, L)$ also.*

For the uniformly 1-Lipschitz boundaries we also have the following well-known result (see, e.g., Theorem 13.17 in [23]), concerning the Stein total extension operator (see Paragraph 5.17 in [1] for the definition of these operators).

Theorem A.1. *Let U with $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L)$. Then there exists a linear extension operator*

$$\mathcal{E}(U,\mathbb{R}^n) : W^{m,p}(U) \to W^{m,p}(\mathbb{R}^n), \ \forall m \in \mathbb{N}_0, \ \forall p \in [1,\infty],$$

such that, for every $m \in \mathbb{N}_0$, every $p \in [1,\infty]$ and every $u \in W^{m,p}(U)$, we have

$$\|(\mathcal{E}(U,\mathbb{R}^n))u\|_{L^p(\mathbb{R}^n)} \le C(K)\|u\|_{L^p(U)} \text{ and}$$

$$\|(\nabla_w^k \circ (\mathcal{E}(U,\mathbb{R}^n)))u\|_{L^p(\mathbb{R}^n)} \le C(K,L)\sum_{j=0}^{k}\frac{1}{\varepsilon^{k-j}}\|\nabla_w^j u\|_{L^p(U)}, \text{ for every } k = 1,\dots,m, \text{ if } m \ne 0.$$

Hence, we can write that $W^{m,p}(U) \hookrightarrow (\mathcal{E}(U,\mathbb{R}^n))(W^{m,p}(U))$, if we consider a notation similar to that of Theorem A.1. The space on the right-hand side is a normed space equipped with its natural norm.

A.3 The continuous Sobolev embeddings

In this subsection, we comment on the classical Sobolev embeddings in terms of the restriction and extension operators discussed in Subsection A.1. Recalling the standard Sobolev embedding Theorems [3, Corollary 9.13], we present two consequences.

Corollary A.1. *Let $m \in \mathbb{N}, p \in [1,\infty)$ For every open $V \subseteq U$ we have that (see Definition A.1)*

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in \left[p, \frac{np}{n-mp}\right], \text{ if } n > mp,$$
$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in [p,\infty), \text{ if } n = mp,$$
$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^\infty(U)), \text{ if } n < mp.$$

In particular, for the case $n < mp$ we have

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor,\gamma}(\bar{U})\right) \cap (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor-1,1}(\bar{U})\right),$$
$$\text{for } \begin{cases} \gamma = m - \frac{n}{p} - \left\lfloor m - \frac{n}{p} \right\rfloor, & \text{if } \left(m - \frac{n}{p}\right) \in /\mathbb{N} \\ \forall \gamma \in (0,1), & \text{if } \left(m - \frac{n}{p}\right) \in \mathbb{N}, \end{cases}$$

where the right-hand space is considered as a normed space equipped with its natural norm.

All of the above embeddings are scaling invariant, that is, the constants of the respective inequalities are uniform, i.e., independent of U. The embeddings are also independent of the choice of V.

Corollary A.2. *Let $m \in \mathbb{N}, p \in [1,\infty)$ and U with $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L)$. For every open $V \subseteq U$ we have that*

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in \left[p, \frac{np}{n-mp}\right], \text{ if } n > mp,$$
$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in [p,\infty), \text{ if } n = mp,$$
$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^\infty(U)), \text{ if } n < mp.$$

In particular, for the case $n < mp$ we have

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor,\gamma}(\bar{U})\right) \cap (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor-1,1}(\bar{U})\right),$$
$$\text{for } \begin{cases} \gamma = m - \frac{n}{p} - \left\lfloor m - \frac{n}{p} \right\rfloor, & \text{if } \left(m - \frac{n}{p}\right) \in /\mathbb{N} \\ \forall \gamma \in (0,1), & \text{if } \left(m - \frac{n}{p}\right) \in \mathbb{N}. \end{cases}$$

All of the above embeddings are scaling dependent, that is the constants of the respective inequalities depend (increasingly) on $\frac{1}{\varepsilon}$, K and L, yet they are independent of the choice of V.

A.4 The compact Rellich–Kondrachov embeddings

Here, we provide useful versions of the well-known Rellich–Kondrachov compactness theorem in terms of the restriction operators of Subsection A.1. For convenience, we consider only the case $m = 1$, since it is the only one used for the proofs of our main results.

Proposition A.5. Let $m \in \mathbb{N}, p \in [1,\infty)$. For every open $V \subseteq U$ we have that

$$W_0^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in \left[1, \frac{np}{n-p}\right), \text{ if } n > p \text{ and } |V| < \infty,$$
$$W_0^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in [1,\infty), \text{ if } n = p \text{ and } |V| < \infty,$$
$$W_0^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))\left(C(\bar{U})\right), \text{ if } n < p \text{ and } V \text{ is bounded.}$$

In any case, $W_0^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^p(U))$ for every bounded $V \subseteq U$.

All the above embeddings are scaling invariant, that is the constants of the respective inequalities are uniform, i.e., independent of U. The embeddings are also independent of the choice of V.

Proposition A.6. Let $p \in [1,\infty)$ and U *be an open set* with $\partial U \in \mathrm{Lip}^1(\varepsilon, K, L)$. For every open $V \subseteq U$ we have that

$$W^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in \left[1, \frac{np}{n-p}\right), \text{ if } n > p \text{ and } |V| < \infty,$$
$$W^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q \in [1,\infty), \text{ if } n = p \text{ and } |V| < \infty,$$
$$W^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))\big(C(\bar{U})\big), \text{ if } n < p \text{ and } V \text{ is bounded.}$$

In any case, $W^{1,p}(U) \hookrightarrow \hookrightarrow (\mathcal{R}(U,V))(L^p(U))$ for every bounded $V \subseteq U$.

All of the above embeddings are scaling dependent, that is the constants of the respective inequalities depend (increasingly) on $\frac{1}{\varepsilon}$, K and L, but they are independent of the choice of V.

A.5 The Leibniz formula

Here, we state a useful generalization of the Leibniz rule for the product of a smooth function with a function which belongs to a Sobolev space [12, Theorem 1, Section 5.2]. Let us recall that for every $m \in \mathbb{N}_0$ and every U, $C^m_{\mathrm{B}}(U)$ stands for the Banach space

$$C^m_{\mathrm{B}}(U) := \left\{u \in C^m(U) \mid D^\alpha u \text{ is bounded everywhere in } U, \text{ for every } 0 \le v|\alpha| \le m\right\},$$

equipped with its natural norm (see, e.g., paragraph 1.27 in [1]).

Proposition A.7. *Let* $m \in \mathbb{N}_0, p \in [1,\infty]$. *If* $\phi \in \cap_{m=0}^{\infty} C^m_{\mathrm{B}}(U)$ *and* $u \in W^{m,p}(U)$, *then we have that*

1. $(\phi u) \in W^{m,p}(U)$ also, with

$$\|\phi u\|_{W^{m,p}(U)} \le C(\|\phi\|_{C^m_{\mathrm{B}}(U)})\|u\|_{W^{m,p}(U)} \tag{A.3}$$

and

2.
$$D^\alpha_w(\phi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^\beta \phi)(D^{\alpha-\beta}_w u) \text{ a.e. in U, for every } \alpha \in \mathbb{N}^n_0 \text{ with } 0 \le |\alpha| \le \text{m}. \tag{A.4}$$

Its proof is quite similar to the proof of the original version cited above, noticing the fact that $(\phi\psi) \in C^\infty_c(U)$ for every $\psi \in C^\infty_c(U)$.

A.6 Cut-off functions

Setting

$$U^\delta := U \cup \bigcup_{x \in \partial U} B(x,\delta),$$

we can have the following basic, yet crucial, result.

Proposition A.8. *Let* U *be an open set and* $\delta > 0$. *Then there exists* $\phi \in C^\infty_c(\mathbb{R}^n; [0,1])$ *such that*

1. $\mathrm{supp}(\phi) \subseteq \overline{U^\delta}$,
2. $\phi \equiv 1$ *in* $\bar{U}$ *and*
3. $\|\nabla^k \phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{C_k}{\delta^k}$, *for every* $k \in \mathbb{N}_0$ $(C_0 = 1)$.

A.7 Second-order, symmetric, uniformly elliptic operators

Definition A.4. For $a = (a_{ij})_{i,j=1}^{n} \in L^{\infty}(U)$ satisfying

$$a = \overline{a^{\mathrm{T}}}, \text{i.e.}, a_{ij} = \overline{a_{ji}}, \text{ a.e. in } U \tag{A.5}$$

and

$$\xi \cdot a\bar{\xi} \overset{(A.5)}{=} \mathrm{Re}(\xi \cdot a\bar{\xi}) \geq \theta|\xi|^2, \text{ a.e. in U, for every } \xi \in \mathbb{C}^n, \text{ for some } \theta > 0, \tag{A.6}$$

we write

$$\mathcal{L}_w = \mathcal{L}_w(a,\theta) : \{u \in L^p(U) \text{ for some } p \in [1,\infty] \mid \nabla_w u \in L^2(U)\} \to H^{-1}(U)$$

for the linear and bounded operator

$$\langle \mathcal{L}_w u, v\rangle := \int_U \nabla_w v \cdot a \nabla_w \bar{u} dx = \int_U \sum_{i,j=1}^{n} a_{ij} (\partial_w^i \bar{u})(\partial_w^j v) dx,$$
$$\text{for every } u \in \{u \in L^p(U), \text{ for some } p \in [1,\infty] \mid \nabla_w u \in L^2(U)\}, \text{ for every } v \in H_0^1(U).$$

Moreover, we set

$$\mathcal{L} : \{u \in L^1_{\mathrm{loc}}(U) \mid \nabla_w u \in L^2(U)\}^2 \to \mathbb{R}$$

for the double-entry form

$$\mathcal{L}[u,v] := \mathrm{Re}\left(\int_U \nabla_w v \cdot a \nabla_w \bar{u} dx\right) = \mathrm{Re}\left(\int_U \sum_{i,j=1}^{n} a_{ij} (\partial_w^i \bar{u})(\partial_w^j v) dx\right),$$
$$\text{for every } u,v \in \{u \in L^1_{\mathrm{loc}}(U) \mid \nabla_w u \in L^2(U)\}.$$

Additionally, if $a \in W^{1,\infty}(U)$ we define

$$L_w = L_w(a,\theta) : \{u \in L^1_{\mathrm{loc}}(U) \mid \nabla_w^j u \in L^2(U), \text{ for } j = 1,2\} \to L^2(U)$$

for the linear operator

$$L_w u := -\mathrm{div}_w(a^{\mathrm{T}} \nabla_w u) = \sum_{i,j=1}^{n} \partial_w^j \left(a_{ji} (\partial_w^i u)\right),$$
$$\text{for every } u \in \left\{u \in L^1_{\mathrm{loc}}(U) \mid \nabla_w^j u \in L^2(U), \text{ for } j = 1,2\right\}.$$

A.8 Elliptic regularity theory for uniformly m-Lipschitz boundaries

A.8.1 Interior regularity

Theorem A.2. *Let $m \in \mathbb{N} \setminus \{1\}$, U be an open set and $(u,f) \in H^1(U) \times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, then $u \in H^m(U_\delta)$ for every $\delta > 0$, with*

$$\sum_{j=2}^{m} \|\nabla_w^j u\|_{L^2(U_\delta)} \leq C\left(\frac{1}{\delta - \delta'}, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)}\right), \ \forall\ 0 < \delta' < \delta.$$

A.8.2 Up to the boundary regularity

Theorem A.3. *Let $m \in \mathbb{N} \setminus \{1\}$, U with $\partial U \in \mathrm{Lip}^m(\varepsilon, K, L)$ and $(u,f) \in H_0^1(U) \times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, then $u \in H^m(U) \cap H_0^1(U)$, with*

$$\sum_{j=2}^{m} \|\nabla_w^j u\|_{L^2(U)} \le C\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)}\right).$$

A.8.3 A priori estimates

We write

$$\{U_{\mathrm{P}}\} := \left\{U \text{ satisfies the assumptions for the validity of the Poincare inequality for } \mathrm{H}_0^1(\mathrm{U})\right\}.$$

Theorem A.4. *Let $m \in \mathbb{N}, U_{\mathrm{P}}$ be as above with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^m(\varepsilon, K, L)$ and $\mathcal{L}_w(a, \theta)$ with $a \in W^{m-1,\infty}(U_{\mathrm{P}})$. Then,*

1. *$\mathcal{L}_w$ induces an isomorphism from $H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$ onto $H^{m-2}(U_{\mathrm{P}})$ and*
2. *for $m \neq 1$ and every $u \in H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$ we have*

$$\sum_{j=2}^{m} \|\nabla_w^j u\|_{L^2(U_{\mathrm{P}})} \le C\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U_{\mathrm{P}})}\right)\left(\|\nabla_w u\|_{L^2(U_{\mathrm{P}})} + \|L_w u\|_{H^{m-2}(U_{\mathrm{P}})}\right).$$

Proposition A.9. *Let $m \in \mathbb{N} \setminus \{1\}, U_{\mathrm{P}}$ be as above with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^m(\varepsilon, K, L), \mathcal{L}_w(a, \theta)$ with $a \in W^{m-1,\infty}(U_{\mathrm{P}})$ and $u \in H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$. If*

$$\left(L_w^j u\right) \in H_0^1(U_{\mathrm{P}}), \ \forall j = 0, \dots, \left\lfloor \frac{m}{2} \right\rfloor - 1 \ \text{(compatibility conditions)},$$

then we have

$$\sum_{j=2}^{m} \|\nabla_w^j u\|_{L^2(U_{\mathrm{P}})} \le C\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U_{\mathrm{P}})}\right) \times \left(\sum_{2j+1 \le m^{j \in \mathbb{N}_0},} \|\left(\nabla_w \circ L_w^j\right) u\|_{L^2(U_{\mathrm{P}})} + \sum_{2j \le m^{j \in \mathbb{N}},} \|L_w^j u\|_{L^2(U_{\mathrm{P}})}\right).$$