A quantitative approach on the solvability of evolution problems in open sets of certain geometries

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September 11, 2021

Abstract

We study the engagement of the boundary data in solving certain evolution problems - that incorporate the standard second order, symmetric, uniformly elliptic operators - in arbitrary open sets. We extend the range of the open sets considered, by proposing a unified approach for both bounded and unbounded sets. The core behind this approach is a revision of (i) the extension operator theory and (ii) the elliptic regularity theory. We illustrate the abstract results for the case of the non vanishing, non linear Schrödinger equation (NLSE) with the defocusing pure power non linearity.

Keywords: evolution problems, arbitrary open sets, boundary analysis, extension operator theory, elliptic operator, elliptic regularity theory, non linear Schrödinger equation, non-linear, nonlinear, NLSE, NLS, defocusing, non vanishing, non-vanishing, regular.

MSC: 35B30, 37L05, 35K58, 35K10, 35J15, 35D30, 35B65, 35Q55.

1 Introduction

In this work we present a unified rigorous scheme for solving evolution problems that incorporate the standard second order, symmetric, uniformly elliptic operators, considered in arbitrary open sets. This scheme depends on and utilizes the boundary data of the problems. First, we need to point out how the boundary data get involved in the solutions. We emphasize that the engagement of the boundary data of certain open sets concerns (i) the use of the extension operator and (ii) the elliptic regularity theory. It turns out that the kind of dependence on the boundary data is the same in those two cases. Second, we need to generalize the above two concepts in a useful way, in order to broaden the gamut of the open sets to be considered. In this regard, we propose the notions of *frontal extension* and *almost regular boundary*. Eventually, the realm of admissible open sets ranges from bounded ones with corners, to certain unbounded ones like the whole euclidean space $\mathbb{R}^n$, its upper half $\mathbb{R}^n_+$ and similar unbounded open sets, as well as exterior open sets.

We recall that extension operators (see, e.g., [24, 1, 16] for the Stein extension operator, while [12, 18, 3, 4] for the Jones extension operator, etc.) constitute the tool that allows the validity of the Sobolev embeddings for open sets other than $\mathbb{R}^n$. Here, we only need Lipschitz boundaries and we then choose to work with the Stein extension operator. In particular, we consider the Stein "frontal extension operator" (see Theorem 2.1.2), that permits us to extend the functions beyond a particular connected component of the boundary of their domain. This particular extension plays a key role in our scheme, as it is essential for the existence of regular solutions of evolution problems considered in exterior open sets (see the corresponding paragraph of §3.2.2).

The notion of almost regular boundaries (see Definition 2.2.4) is a straightforward generalization of that of classical regular ones, where all the fine properties of the latter are preserved, while

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points of certain singularity are additionally allowed. In particular, almost regular boundaries may include corners[1], yet the elliptic regularity theory (see, e.g., [6]) remains valid in open sets with such boundaries (see §2.2.2 and §2.2.3). This fact is the decisive issue behind the existence of regular solutions considered in bounded sets with corners (see §3.1.2), as well as in $\mathbb{R}^n_+$ or in similar unbounded open sets (see the corresponding paragraph of §3.2.2). We emphasize the key features of such an almost smooth boundary geometry: the set of its singularities is a *closed null set of the boundary manifold*, i.e. a closed set of zero $(n-1)$-dimensional Hausdorff measure, and the derivatives - in the open set where they exist - of its local chart functions are *bounded*; these features assure that such a geometry is still a Lipschitz one. We note that regarding the regularity of elliptic PDEs, certain cases of sets of boundaries with corners have already been studied (see, e.g., [11, 14], etc). A trivial example is the regularity up to the boundary of the eigenfunctions of a second order, symmetric, uniformly elliptic operator defined in an n-dimensional rectangle endowed with the zero boundary conditions, since these functions are nothing but the n-product of the respective regular one-dimensional eigenfunctions considered in edges. In any case, here we present a unified, quantitative approach on this subject, which provides us with useful information of the dependence on the boundary data that can be utilized by our scheme.

The scheme proposed in §3 incorporates certain bounds that a solution in a bounded open set satisfies, which are independent (see Theorem 3.1.2) in the case of weak solutions, or dependent - in a manageable manner (see Theorem 3.1.3 and Proposition 3.1.1) - in the case of regular solutions, on the boundary data. Our results are based on *quantitative bounds* (see, e.g., [23] for a similar concept), that render sufficient information of the dependence on the boundary data, and not on “rough” estimates, as the ones usually appearing in the literature. These quantitative bounds are then used for an expansion process which ultimately provides us with a solution (i) in abstract unbounded geometries if the solution is weak (see Theorem 3.2.1), and (ii) in appropriate unbounded geometries if the solution is regular (see Theorem 3.2.3, Theorem 3.2.4 and Theorem 3.2.5). Regarding the regular solutions in unbounded open sets U, we have, among others, the following options and corresponding approaches:

- If $U = \mathbb{R}^n$, we employ the bounds in geometries of smooth boundaries (e.g., balls with respect to euclidean norm) to bound the background; this has been done in [7].
- If $U = \mathbb{R}^n_+$, we employ the bounds in almost smooth geometries.
- In the case where U is an exterior open set, we may (depending on the nature of the bounded fixed boundary) employ the bounds in almost smooth geometries, but we mainly use the frontal extension operator.

These unbounded open sets are of great importance in applications of evolution equations and their special common characteristic is that

they can be obtained by a sequence of homotheties of almost regular boundaries. (P)

We recall that an homothety (of ratio $\lambda \in \mathbb{R}$) is a special case of an affine function from $\mathbb{R}^n$ to itself; in particular it is the sum of a translation plus a dilation (of ratio λ). A property of non local nature such as (P), along with the manageable behavior of an almost regular boundary under homotheties (see Proposition 2.2.2), is the essence behind the applicability of our scheme in search of regular solutions in unbounded open sets.

As far as the exposition is concerned, we choose the non vanishing, defocusing, pure power NLSE problem (see the introduction of §3) as our model case for an illustration of the above scheme. We do so, in order to extend previous work, done by two of the present authors [9, 8, 7], to the above presented geometric framework. Moreover, the NLSE is of great importance in applications as corresponding initial boundary value problems are often encountered in Quantum Mechanics, Non Linear Optics, the Theory of Superconductivity, Water Waves, Bose-Einstein Condensates, and a plethora of other fields of Physics and Engineering. In particular, the defocusing, pure power NLSE

$$i\frac{\partial v}{\partial t} - \Delta v + |v|^{2\tau} v = 0, \ \tau \in \mathbb{N},$$

[1]The notion “almost regular boundaries” has also been used in the bibliography (see, e.g., [17]) as an alternative characterization of “boundaries/manifolds with corners” (see, e.g., [15]).

is one of the most important non linear integrable PDEs, a principal universal mathematical model for wave propagation in non linear dispersive media. It has been a topic of intensive and extensive research activity from both the physical and mathematical viewpoints. The physical interest is partly due to the complex phenomenology associated to the existence of localized waveforms supported on the top of a stable continuous background. In one-dimensional frameworks, one of the most famous such waveforms is the "dark soliton": when $\tau = 1$, i.e., for the cubic one-dimensional defocusing NLSE, the simplest expression of the dark soliton solution has the form $v(t,x) = e^{it}\tanh\left(2^{-\frac{1}{2}}x\right)$. In two and three (spatial) dimensions, the corresponding solutions have distinctly non trivial generalizations, described by the so called "vortices" (in two-dimensional setups), and "vortex rings" (in three-dimensional setups); see [13] and references therein.

We emphasize that in the non vanishing NLSE model that we consider, the bounds of §3 are employed in order to control the - non bounded - background, in our search for regular solutions in unbounded open sets of the aforementioned appropriate geometries. Thus, the toolbox developed in the present paper is essential to the regularity of solutions in certain unbounded setups, which makes the proposed scheme noteworthy.

2 On boundary analysis

First, we review some known concepts, on the processes that involve the boundary of open sets.

2.1 Extension open set

It is well known that the embeddings that hold true in $\mathbb{R}^n$ continue to be valid in an extension open set, and this is the essence behind the search for such sets.

2.1.1 Lipschitz boundary

Definition 2.1.1 (Lipschitz boundary)**.** *We say that ∂U is Lipschitz, and we write*

$$\partial U \in \mathrm{Lip},$$

iff $\forall x_0 \in \partial U$, $\exists$

1. *local coordinates* $y_{x_0} = \Phi_{x_0}(x)$,
2. *a Lipschitz function* $\gamma_{x_0} \colon \mathbb{R}^{n-1} \to \mathbb{R}$, *and*
3. *a constant* $\rho_{x_0} > 0$,

such that

$$\Phi_{x_0}(B(x_0,\rho_{x_0}) \cap U) = \Phi_{x_0}(B(x_0,\rho_{x_0})) \cap \Phi_{x_0}(U) = \\ = \Phi_{x_0}(B(x_0,\rho_{x_0})) \cap \left\{ y_{x_0} \in \mathbb{R}^n \,\middle|\, y_{x_0 n} > \gamma_{x_0}({y_{x_0}}') \right\}.$$

Remark 2.1.1. *It is straightforward to show that, under the conditions of Definition 2.1.1, we have that*

$$\Phi_{x_0}(B(x_0,\rho_{x_0}) \cap \partial U) = \Phi_{x_0}(B(x_0,\rho_{x_0})) \cap \left\{ y_{x_0} \in \mathbb{R}^n \,\middle|\, y_{x_0 n} = \gamma_{x_0}({y_{x_0}}') \right\},$$

which implies that ∂U is a $(n-1)$-dimensional manifold. In particular, ∂U is smoother than a topological manifold (e.g., no cusps appear).

A way to "quantify" the conditions appearing in Definition 2.1.1, is described below.

Definition 2.1.2 (uniformly Lipschitz boundary)**.** *Let*

$$(\varepsilon, K, L) \in (0,\infty) \times \mathbb{N} \times [0,\infty).$$

We say that ∂U is uniformly Lipschitz of constants ε, K and L, and we write

$$\partial U \in \mathrm{Lip}(\varepsilon, K, L),$$

iff $\exists$ locally finite countable open cover $\{U_k\}_k$ of ∂U, such that

1. $\forall x \in \partial U$, $\exists k_x$ *such that* $B(x,\varepsilon) \subseteq U_{k_x}$,
2. $\bigcap_{j=1}^{K+1} U_{k_j} = \varnothing$, *where* $U_{k_j} \neq U_{k_i}$, *and*
3. $\forall k$, $\exists$ *local coordinates* $y_k = \Phi_k(x)$ *and a Lipschitz function* $\gamma_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$, *such that*
   - *i.* $\mathrm{Lip}(\gamma_k) \leq L$, *and*
   - *ii.* $\Phi_k(U_k \cap U) = \Phi_k(U_k) \cap \{y_k \in \mathbb{R}^n \,|\, y_{k_n} > \gamma(y_k{}')\}$.

Remark 2.1.2. *1. The conclusion of Remark 2.1.1 remains also valid for the case of uniformly Lipschitz boundaries.*

2. It is easy to deduce that, when ∂U is bounded, then ∂U is Lipschitz iff it is uniformly Lipschitz.

Moreover, the following result can be found in [7, Proposition A.4].

Proposition 2.1.1. *If U is such that $\partial U \in \mathrm{Lip}(\varepsilon, K, L)$, as well as Φ is a transformation of the form $\Phi(x) = c + \lambda x$, where $c \in \mathbb{R}^n$ and $\lambda > 1$, then $\partial(\Phi(U)) \in \mathrm{Lip}(\lambda\varepsilon, K, L)$.*

Remark 2.1.3. *Evidently, if U is such that $\partial U \in \mathrm{Lip}(\varepsilon_1, K_1, L_1)$, then $\partial U \in \mathrm{Lip}(\varepsilon_2, K_2, L_2)$, $\forall$ $\varepsilon_2 \leq \varepsilon_1$, $K_2 \geq K_1$, and $L_2 \geq L_1$.*

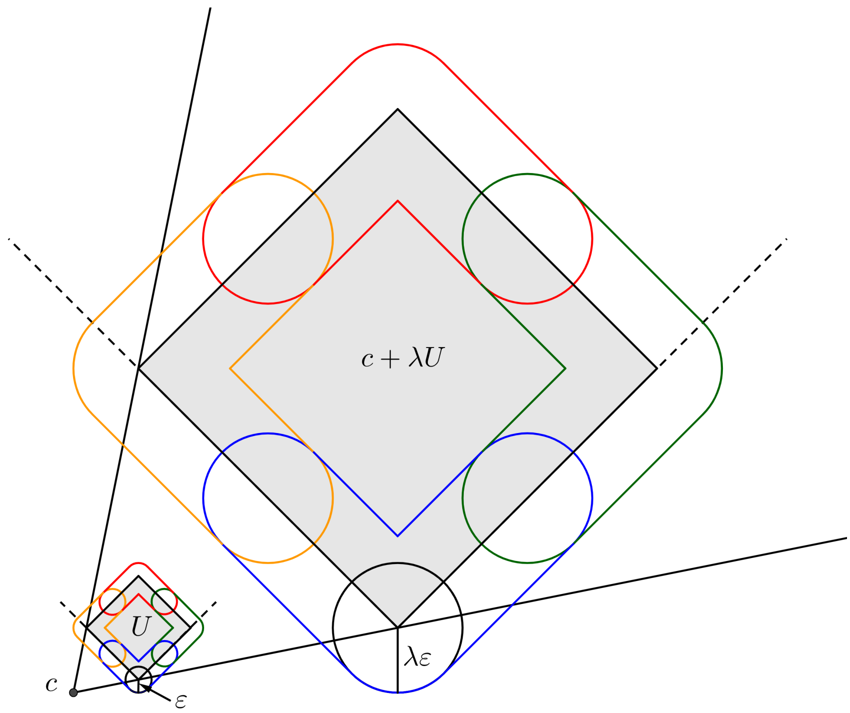

Figure 1: $\partial U \in \mathrm{Lip}(\varepsilon, 2, 1)$ & $\partial(c + \lambda U) \in \mathrm{Lip}(\lambda\varepsilon, 2, 1)$.

As we will point out later, it is sometimes useful to consider separately the connected components of the boundary of an open set.

Definition 2.1.3 (uniformly Lipschitz connected component of the boundary)**.** *Let*

$$(\varepsilon, K, L) \in (0,\infty) \times \mathbb{N} \times [0,\infty)\,.$$

We say that the connected component $\widetilde{\partial U}$ of ∂U is uniformly Lipschitz of constants ε, K and L, and we write

$$\widetilde{\partial U} \in \mathrm{Lip}(\varepsilon, K, L),$$

iff $\exists$ a locally finite countable open cover $\{U_k\}_k$ of $\widetilde{\partial U}$, such that

1. $\forall x \in \widetilde{\partial U}$, $\exists k_x$ *such that* $B(x,\varepsilon) \subseteq U_{k_x}$,

2. $\bigcap_{j=1}^{K+1} U_{k_j} = \varnothing$, *where* $U_{k_j} \neq U_{k_i}$, *and*

3. $\forall k$, $\exists$ *local coordinates* $y_k = \Phi_k(x)$ *and a Lipschitz function* $\gamma_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$, *such that*

    i. $\operatorname{Lip}(\gamma_k) \leq L$, *and*

    ii. $\Phi_k(U_k \cap U) = \Phi_k(U_k) \cap \left\{ y_k \in \mathbb{R}^n \,\middle|\, y_{k_n} > \gamma(y_k{}') \right\}$.

A straightforward adaptation of Proposition 2.1.1 gives us the following result.

Proposition 2.1.2. *Let U be bounded and connected, such that ∂U has at least two connected components where $\widetilde{\partial U}$ stands for the exterior one. If $\widetilde{\partial U} \in \operatorname{Lip}(\varepsilon, K, L)$, as well as Φ is a transformation of the form $\Phi(x) = c + \lambda x$, where $c \in \mathbb{R}^n$ and $\lambda > 1$, then $\partial\widetilde{(\Phi(U))} \in \operatorname{Lip}(\lambda\varepsilon, K, L)$, where $\partial\widetilde{(\Phi(U))}$ stands for the connected component of $\partial(\Phi(U))$ such that $\partial\widetilde{(\Phi(U))} = \Phi\big(\widetilde{\partial U}\big)$.*

2.1.2 Extension operator

Below follows a well known result (see, e.g., [16, Theorem 13.17]), concerning the Stein total extension operator (see [1, Paragraph 5.17] for the definition of such an operator), defined in Sobolev spaces for open sets of uniformly Lipschitz boundaries.

Theorem 2.1.1 (extension operator)**.** *Let U be such that $\partial U \in \operatorname{Lip}(\varepsilon, K, L)$. Then $\exists$ linear extension operator*

$$\mathcal{E} \colon W^{m,p}(U) \to W^{m,p}(\mathbb{R}^n), \ \forall m \in \mathbb{N}_0, \ \forall p \in [1, \infty],$$

such that, $\forall$ $m \in \mathbb{N}_0$, $p \in [1,\infty]$ & $u \in W^{m,p}(U)$, we have

$$\|\mathcal{E}u\|_{L^p(\mathbb{R}^n)} \leq C(K) \|u\|_{L^p(U)}, \ \textit{and}$$
$$\left\| \left(\nabla_w^k \circ \mathcal{E}\right) u \right\|_{L^p(\mathbb{R}^n)} \leq C(K,L) \sum_{j=0}^{k} \frac{1}{\varepsilon^{k-j}} \left\| \nabla_w^j u \right\|_{L^p(U)}, \ \forall k \in \{1, \dots, m\}, \ \textit{if } m \neq 0.$$

We can write that $W^{m,p}(U) \hookrightarrow \mathcal{E}(W^{m,p}(U))$, if we consider the space in right hand side as a normed space equipped with its natural norm.

The proof of the following straightforward adaptation of the above result, concerning the connected components of the boundary, is omitted.

Theorem 2.1.2 (frontal extension operator)**.** *Let the connected component $\widetilde{\partial U}$ of ∂U be such that $\widetilde{\partial U} \in \operatorname{Lip}(\varepsilon, K, L)$. Further, let $\widetilde{U}$ be the corresponding connected component of U^e that shares the same boundary with $\widetilde{\partial U}$. Then $\exists$ linear extension operator*

$$\mathcal{E} \colon W^{m,p}(U) \to W^{m,p}\big(U \cup \widetilde{\partial U} \cup \widetilde{U}\big), \ \forall m \in \mathbb{N}_0, \ \forall p \in [1, \infty],$$

such that, $\forall$ $m \in \mathbb{N}_0$, $p \in [1,\infty]$ & $u \in W^{m,p}(U)$, we have

$$\|\mathcal{E}u\|_{L^p(U_2)} \leq C(K) \|u\|_{L^p(U_1)}, \ \textit{and}$$
$$\left\| \left(\nabla_w^k \circ \mathcal{E}\right) u \right\|_{L^p(U_2)} \leq C(K,L) \sum_{j=0}^{k} \frac{1}{\varepsilon^{k-j}} \left\| \nabla_w^j u \right\|_{L^p(U_1)}, \ \forall k \in \{1, \dots, m\}, \ \textit{if } m \neq 0,$$

where $U_1 = U \cap \bigcup_{x \in \widetilde{\partial U}} B(x, \varepsilon)$, and $U_2 = \widetilde{U} \cup \bigcup_{x \in \widetilde{\partial U}} B(x, \varepsilon)$.

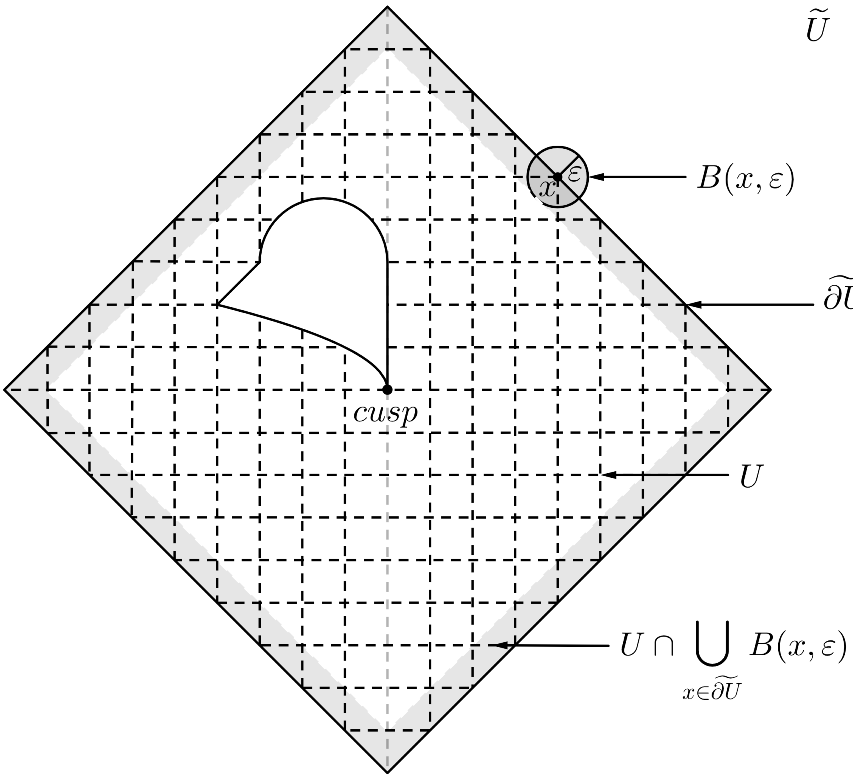

Figure 2: Only for the connected component $\widetilde{\partial U}$ of ∂U we have that $\widetilde{\partial U} \in \mathrm{Lip}(\varepsilon, K, L)$, since the other one has a cusp. Hence, U (dashed) is an extension set only across the front $\widetilde{\partial U}$ to the set $\widetilde{U}$. For the corresponding frontal extension operator, we only need data from the front $\widetilde{\partial U}$, as well as data from the function-to-be-extended in the ε-relative-neighbourhood of the front, that is $U \cap \bigcup_{x\in\widetilde{\partial U}} B(x,\varepsilon)$.

Remark 2.1.4. *It is direct to see that the estimates in Theorem 2.1.2 can take the form*

$$\|\mathcal{E}u\|_{L^p(U_3)} \le C(K)\|u\|_{L^p(U)}, \text{ and}$$
$$\left\|\left(\nabla_w^k \circ \mathcal{E}\right)u\right\|_{L^p(U_3)} \le C(K,L)\sum_{j=0}^{k}\frac{1}{\varepsilon^{k-j}}\left\|\nabla_w^j u\right\|_{L^p(U)}, \ \forall k \in \{1,\dots,m\}, \text{ if } m \neq 0,$$

where $U_3 = U \cup \widetilde{\partial U} \cup \widetilde{U}$, which justifies the corresponding embedding.

2.2 Elliptic regularity

Here, we focus on elliptic regularity theory, in order to extract certain useful results, which generalize already known ones for open sets with adequately smooth boundaries. The new result, here, is the boundary regularity for solutions in open sets with boundaries as in Definition 2.2.4.

Before we proceed, we recall the necessary notions concerning second order, symmetric, uniformly elliptic operators, and we state the interior regularity result, for which no smoothness of the boundary is required.

Definition 2.2.1. *For $A = (a_{ij})_{i,j=1}^n \in L^\infty(U)$ satisfying*

$$\mathrm{Re}\left(\xi \cdot A\bar{\xi}\right) \ge \theta|\xi|^2, \ \text{a.e. in } U, \ \forall \xi \in \mathbb{C}^n, \ \text{for some } \theta > 0 \ (\text{uniform ellipticity of } A)$$

and

$$A = \overline{A^{\mathrm{T}}}, \ \text{i.e., } a_{ij} = \overline{a_{ji}}, \ \text{a.e. in } U \ (\text{self-adjointness of } A),$$

we write

$$\mathcal{L}_w = \mathcal{L}_w(A,\theta) \colon \left\{u \in L^p(U) \text{ for some } p \in [1,\infty] \,\middle|\, \nabla_w u \in L^2(U)\right\} \to H^{-1}(U)$$

for the linear and bounded operator

$$\langle \mathcal{L}_w u, v\rangle = \int_U \nabla_w v \cdot A \nabla_w \bar{u} dx = \int_U \sum_{i,j=1}^{n} a_{ij} \left(\partial_w^i \bar{u}\right) \left(\partial_w^j v\right) dx,$$
$$\forall\ u \in \left\{u \in L^p(U) \,\middle|\, p \in [1,\infty]\ \ \&\ \nabla_w u \in L^2(U)\right\}\ \&\ v \in H_0^1(U).$$

Moreover, we set

$$\mathcal{L}\colon \left\{u \in L^1_{\mathrm{loc}}(U) \,\middle|\, \nabla_w u \in L^2(U)\right\}^2 \to \mathbb{R}$$

for the bi-linear form

$$\mathcal{L}[u,v] = \mathrm{Re}\left(\int_U \nabla_w v \cdot A \nabla_w \bar{u} dx\right) = \mathrm{Re}\left(\int_U \sum_{i,j=1}^{n} a_{ij} \left(\partial_w^i \bar{u}\right) \left(\partial_w^j v\right) dx\right),$$
$$\forall\, u, v\ \in \left\{u \in L^1_{\mathrm{loc}}(U) \,\middle|\, \nabla_w u \in L^2(U)\right\}.$$

Additionally, if $A \in W^{1,\infty}(U)$, we define the linear operator

$$L_w = L_w(A,\theta)\colon \left\{u \in L^1_{\mathrm{loc}}(U) \,\middle|\, \nabla_w^j u \in L^2(U),\ \textit{for } j \in \{1,2\}\right\} \to L^2(U)$$

by

$$L_w u = -\,\mathrm{div}_w\left(A^{\mathrm{T}} \nabla_w u\right) = \sum_{i,j=1}^{n} \partial_w^j\left(a_{ji}\left(\partial_w^i u\right)\right),\ \ \forall u \in \left\{u \in L^1_{\mathrm{loc}}(U) \,\middle|\, \nabla_w^j u \in L^2(U),\ \textit{for } j \in \{1,2\}\right\}.$$

Definition 2.2.2. *$\forall m \in \mathbb{N}_0$, we consider that the space $H^m(U) = W^{m,2}(U)$ is equipped with the inner product $(*,\star)_{H^m(U)} \to \mathbb{C}$ defined as*

$$(u,v)_{H^m(U)} = \sum_{0 \le |\alpha| \le m} \int_U \left(D_w^\alpha u\right)\left(D_w^\alpha \bar{v}\right) dx,\ \ \forall\, u, v\ \in H^m(U).$$

When $m = 0$, we simply write $(*,\star) = (*,\star)_{H^0(U)} = (*,\star)_{L^2(U)}$.

Definition 2.2.3. *We write*

$$\{U_P\} = \left\{U \textit{ satisfies the criterion for the validity of Poincaré's inequality for } H_0^1(U)\right\}.$$

We recall that Poincaré's inequality for the space $H_0^1(U)$ for some U (see, e.g., [16, Theorem 13.19], or [1, Theorem, Paragraph 6.30]) implies that there exists $C = C_U$ such that

$$\|u\|_{H^1(U)} \le C \|\nabla_w u\|_{L^2(U)},\ \ \forall u \in H_0^1(U).$$

Evidently, $C \ge 1$. $\forall U_P$, we write $C_{U_P} \ge 1$ for the "smallest" constant of the respective inequality, that is

$$C_{U_P} = \inf\left\{C \,\middle|\, \|u\|_{H^1(U)} \le C\|\nabla_w u\|_{L^2(U)},\ \ \forall u \in H_0^1(U)\right\} \ge 1.$$

Proposition 2.2.1. *Every $\mathcal{L}_w(A,\theta)$ induces an isomorphism from $H_0^1(U_P)$ onto $H^{-1}(U_P)$.*

Below follows the regularity away from the boundary, where the classical Nirenberg's difference quotients approach (see, e.g., [21]) along with a standard induction argument are employed, for which we refer to [7, Theorem A.2].

Theorem 2.2.1. *Let $m \in \mathbb{N} \smallsetminus \{1\}$ and $(u,f) \in H^1(U) \times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $A \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, then $u \in H^m(U_\delta)$ $\forall \delta > 0$, with*

$$\sum_{j=2}^{m} \left\|\nabla_w^j u\right\|_{L^2(U_\delta)} \le C\left(\frac{1}{\delta}, \frac{1}{\theta}, \|A\|_{W^{m-1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)}\right).$$

2.2.1 Almost C^m boundary

Below follows a certain sub case of Definition 2.1.2, which is not only useful for the elliptic regularity problem but also allows us to consider certain "bad", yet manageable, non smoothness points.

Definition 2.2.4 (uniformly almost C^m boundary)**.** *Let*

$$(\varepsilon, K, L, m) \in (0,\infty) \times \mathbb{N} \times [0,\infty) \times \mathbb{N}.$$

We say that ∂U is uniformly almost C^m, of constants ε, K and L, and we write

$$\partial U \sim C^m(\varepsilon, K, L),$$

iff $\exists$ a locally finite countable open cover $\{U_k\}_k$ of ∂U and $S \subsetneq \partial U$, such that

1. *$\forall x \in \partial U$, $\exists k_x$ such that $B(x,\varepsilon) \subseteq U_{k_x}$,*
2. *$\bigcap_{j=1}^{K+1} U_{k_j} = \varnothing$, where $U_{k_j} \neq U_{k_i}$, and*
3. *$\forall k$, $\exists$ local coordinates $y_k = \Phi_k(x)$ and a Lipschitz continuous function $\gamma_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$, such that*
   - *i.* $\mathrm{Lip}(\gamma_k) \leq L$,
   - *ii.* $\Phi_k(U_k \cap U) = \Phi_k(U_k) \cap \left\{ y_k \in \mathbb{R}^n \,\middle|\, y_{k_n} > \gamma(y_k') \right\}$,
   - *iii.* $S_k := \Phi_k(U_k \cap S) = \Phi_k(U_k) \cap \left\{ y_k \in \mathbb{R}^n \,\middle|\, y_{k_n} = \gamma(y_k') \;\&\; \exists j \in \{1,\dots,m\} \text{ s.t. } \nexists \nabla^j \gamma_k(y_k') \right\}$,
   - *iv.* *$\mathfrak{p}(S_k)$ is closed null set of $\mathbb{R}^{n-1}$, and*
   - *v.* $\gamma_k|_{(\mathfrak{p}(S_k))^c} \in C_b^m\big((\mathfrak{p}(S_k))^c\big)$ *with* $\|\gamma_k\|_{C_b^m((\mathfrak{p}(S_k))^c)} \leq L$.

If $S = \varnothing$ above, then we say that ∂U is uniformly C^m, of constants ε, K and L, and we write

$$\partial U \in C^m(\varepsilon, K, L).$$

Remark 2.2.1.

1. *Evidently, uniformly almost C^m boundaries are special cases of uniformly Lipschitz boundaries.*
2. *According to the theory of manifolds, the set S of Definition 2.2.4 is nothing but a null set of the manifold ∂U.*
3. *The analogous statement of Remark 2.1.3 is also true, that is if U is such that $\partial U \sim C^m(\varepsilon_1, K_1, L_1)$, then $\partial U \sim C^m(\varepsilon_2, K_2, L_2)$, $\forall\ \varepsilon_2 \leq \varepsilon_1$, $K_2 \geq K_1$, and $L_2 \geq L_1$.*
4. *If $U \subseteq \mathbb{R}$, i.e., $n = 1$, then the conditions $\partial U \sim C^m(\varepsilon, K, L)$, $\partial U \in C^m(\varepsilon, K, L)$ and $\partial U \in \mathrm{Lip}(\varepsilon, K, 0)$ are equivalent.*

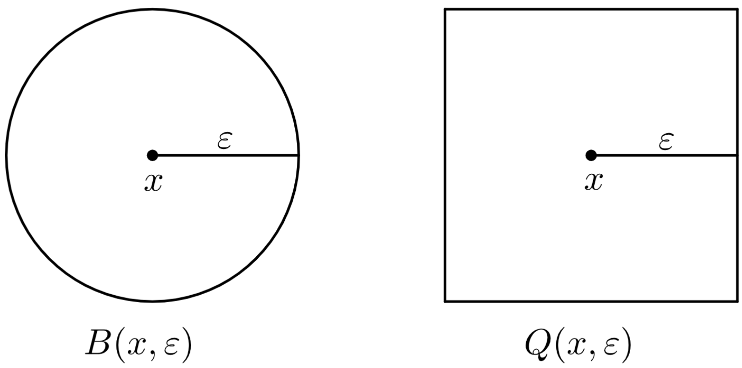

Figure 3: $\forall m \in \mathbb{N}$, a ball with respect to the euclidean norm, $B(x,\varepsilon)$, has uniformly C^m boundary, while a ball with respect to the supremum norm, $Q(x,\varepsilon)$, has almost uniformly C^m boundary. The set S of Definition 2.2.4 is $\varnothing$ in the first case, as well as the set of vertices (in $\mathbb{R}^2$) and (hyper-)edges (in $\mathbb{R}^n$ with $n \geq 3$) in the second.

By a straightforward adaptation of Proposition 2.1.1, we can have the following result.

Proposition 2.2.2. *If U is such that $\partial U \sim C^m(\varepsilon, K, L)$ for $m \in \mathbb{N}$, as well as Φ is a transformation of the form $\Phi(x) = c + \lambda x$, where $c \in \mathbb{R}^n$ and $\lambda > 1$, then $\partial(\Phi(U)) \sim C^m(\lambda\varepsilon, K, L)$.*

We can generalize Definition 2.2.4 by characterizing only parts of the boundary, as we did in Definition 2.1.3 in comparison to Definition 2.1.2. However, we will not need such a perspective in the present work.

Before we proceed to the demonstration of the usefulness of almost regular boundaries to the elliptic regularity theory, we provide an alternative approach on the concept of such boundaries. This alternative approach is compatible with the classic theory of manifolds with boundaries (see, e.g., [15] and [20]) and it does not require the use of the measure theoretic toolbox of differentiation.

Definition 2.2.5 (uniformly almost C^m boundary, alternative approach)**.** *Let*

$$(\varepsilon, K, L, m) \in (0, \infty) \times \mathbb{N} \times [0, \infty) \times \mathbb{N}.$$

We say that ∂U is uniformly almost C^m, of constants ε, K and L, and we write

$$\partial U \sim C^m(\varepsilon, K, L),$$

iff

1. *$\overline{U}$ is an n-dimensional Lipschitz manifold in $\mathbb{R}^n$ with boundary, and*
2. *$\exists$ an n-dimensional C^m-manifold, S, in $\mathbb{R}^n$ with boundary, such that*
   - *i.* $\operatorname{int} S = U$,
   - *ii.* $\partial U \setminus \operatorname{bd} S$ *is a null set of the ($(n-1)$-dimensional Lipschitz) manifold* $\operatorname{bd} \overline{U} = \partial U$,
   - *iii. $\exists$ a locally finite collection of boundary charts, $\{(S_k, f_k)\}_k$, of S, such that*
     - *a.* $\operatorname{bd} S \subseteq \bigcup\limits_k f_k(S_k)$,
     - *b.* $\forall x \in \operatorname{bd} S$, $\exists k_x$ *such that* $B(x, \varepsilon) \subseteq f_{k_x}(S_{k_x})$,
     - *c.* $\bigcap\limits_{j=1}^{K+1} f_{k_j}\big(S_{k_j}\big) = \varnothing$, *where* $f_{k_j}\big(S_{k_j}\big) \neq f_{k_i}(S_{k_i})$, *and*
     - *d.* $\forall k$, $\|f_k\|_{C_b^m(S_k)} \leq L$.

If $\partial U \setminus \operatorname{bd} S = \varnothing$ above, then we say that ∂U is uniformly C^m, of constants ε, K and L, and we write

$$\partial U \in C^m(\varepsilon, K, L).$$

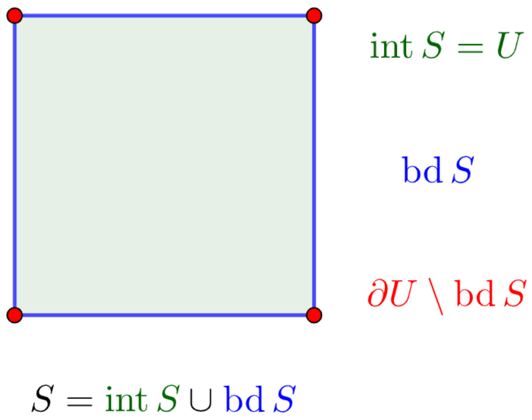

Figure 4: An example of the geometry behind the Definition 2.2.5. In order to characterize ∂U we introduce an auxiliary regular manifold, S, of maximum dimension with boundary. In contrast to Definition 2.2.4, this alternative approach does not involve Measure Theory.

Remark 2.2.2. *1. Indeed, Definition 2.2.5 does not require the differentiability of Lipschitz functions: the definition of a Lipschitz manifold is irrelevant to the Rademacher theorem.*

2. *We emphasize the need of utilizing Lipschitz manifolds instead of merely C-manifolds in Definition 2.2.5: a Lipschitz manifold is the least regular of the commonly used manifolds that allows us to consider null sets of it.*

3. *Concerning the "bad" null set $\partial U \setminus \operatorname{bd} S$ of ∂U in Definition 2.2.5 (as well as the corresponding "bad" set in Definition 2.2.4), its relative closedness in ∂U, hence also its closedness, are straightforward. Thus, this set is relatively nowhere dense in ∂U.*

The equivalence of Definition 2.2.4 and Definition 2.2.5 follows by the use of standard arguments.

2.2.2 Regularity up to the boundary

If we impose certain smoothness to ∂U, we can extend the interior elliptic regularity of Theorem 2.2.1 to the whole U.

Theorem 2.2.2. *Let $m \in \mathbb{N} \setminus \{1\}$, U be of $\partial U \sim C^m(\varepsilon, K, L)$ and $(u, f) \in H_0^1(U) \times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $A \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, then $u \in H^m(U) \cap H_0^1(U)$, with*

$$\sum_{j=2}^{m} \|\nabla_w^j u\|_{L^2(U)} \le C\Big(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|A\|_{W^{m-1,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)}\Big).$$

Proof. With Theorem 2.2.1 at hand, we only need to deduce elliptic regularity near the boundary. The proof for this result for $m = 1$, where the classical Nirenberg's difference quotients approach is employed after the straightening of the boundary, is quite similar to the corresponding one of the result in the case of uniformly C^m boundaries (see Definition 2.1.2), for which we refer to [7, Theorem A.3]. The only and evident adaptation that has to be made, concerns the closed null sets $\mathfrak{p}(S_k) \subsetneq \mathbb{R}^{n-1}$ in Definition 2.2.4. Those sets contain the "bad non smooth" points of the boundary, yet they do not contribute at all to the quantities represented by integrals. The higher regularity follows by induction. □

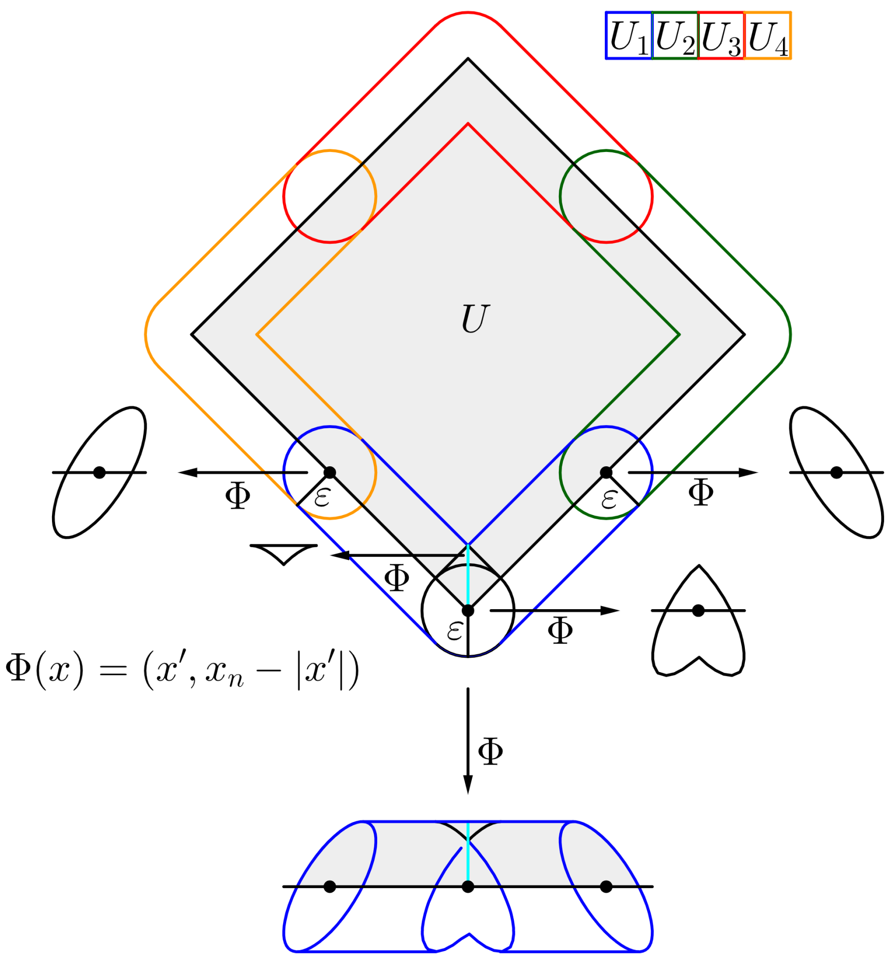

Figure 5: An example of the geometry of Theorem 2.2.2. Here, $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is (in local coordinates) the function that locally straightens-out the boundary ∂U. We see how it is applied to U_1 which belongs to an appropriate cover of ∂U, as in Definition 2.2.4. Notice that since Φ is a Lipschitz function, then it has the Luzin N property, i.e., it maps the cyan colored null subset of $U \cap U_1$ - the set of points where $\Phi|_{U\cap U_1}$ is not smooth - to a null set. As for the reason why this subset of $U \cap U_1$ is a null set, it follows directly from the fact that the set of vertices (in $\mathbb{R}^2$) and (hyper-)edges (in $\mathbb{R}^n$ with $n \geq 3$) of U is a null set of the manifold ∂U.

Remark 2.2.3. *In the standard reference books (see, e.g., [10, 6, 19]), the second order uniformly elliptic operator in $H^1(U)$ has the general form $\mathcal{L}_w^{\mathrm{g}} = \mathcal{L}_w + b \cdot \nabla_w u + cu$, for $b = (b_i)_{i=1}^n \in L^\infty(U)$ and $c \in L^\infty(U)$, which in general does not induce a symmetric bilinear form, nor an isomorphism, from $H_0^1(U_P)$ onto $H^{-1}(U_P)$ (see Definition 2.2.3). However, we note that the elliptic regularity results for a solution $u \in H^1(U)$ of $\mathcal{L}_w u = f$ in $H^{-1}(U)$ appearing so far in this section are trivially true also for $\mathcal{L}_w^{\mathrm{g}}$, since all we have to do is to consider $f^{\mathrm{g}} = f - \bar{b} \cdot \nabla_w \bar{u} - \bar{c}\bar{u}$ instead of f in the variational equation.*

2.2.3 A priori estimates

In this subsection, we are interested in the sets U_P (see Definition 2.2.3) with appropriate boundaries. In the light of Theorem 2.2.2, the following results are a straightforward adaptation of the corresponding ones in [7, Section A.8.3].

Theorem 2.2.3. *Let $m \in \mathbb{N}$, U_P with $\partial U \sim C^m(\varepsilon, K, L)$ and $\mathcal{L}_w(A, \theta)$ with $A \in W^{m-1,\infty}(U_P)$. Then,*

1. *$\mathcal{L}_w$ induces an isomorphism from $H^m(U_P) \cap H_0^1(U_P)$ onto $H^{m-2}(U_P)$ &*
2. *for $m \neq 1$ and every $u \in H^m(U_P) \cap H_0^1(U_P)$ we have*

$$\sum_{j=2}^{m} \left\| \nabla_w^j u \right\|_{L^2(U_P)} \leq C\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|A\|_{W^{m-1,\infty}(U_P)}\right) \left(\|\nabla_w u\|_{L^2(U_P)} + \|L_w u\|_{H^{m-2}(P)} \right).$$

Proposition 2.2.3. *Let $m \in \mathbb{N} \setminus \{1\}$, U_P with $\partial U \sim C^m(\varepsilon, K, L)$, $\mathcal{L}_w(A, \theta)$ with $A \in W^{m-1,\infty}(U_P)$ and $u \in H^m(U_P) \cap H_0^1(U_P)$. If*

$$\left(L_w^j u\right) \in H_0^1(U_P), \;\; \forall j \in \left\{0, \dots, \left\lfloor \frac{m}{2} \right\rfloor - 1\right\} \text{ (compatibility conditions)},$$

then we have

$$\sum_{j=2}^{m} \left\| \nabla_w^j u \right\|_{L^2(U_P)} \leq C\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|A\|_{W^{m-1,\infty}(U_P)}\right) \times$$
$$\times \left(\sum_{\substack{j \in \mathbb{N}_0, \\ 2j+1 \leq m}} \left\| \left(\nabla_w \circ L_w^j\right) u \right\|_{L^2(U_P)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \leq m}} \left\| L_w^j u \right\|_{L^2(U_P)} \right).$$

3 A scheme for solving evolution problems in arbitrary open sets

Here, we apply the previous theory in order to develop a scheme for the solvability and regularity of evolution problems that incorporate the elliptic operator of Definition 2.2.1, in arbitrary open sets. In order to keep the presentation as compact as possible, we consider a representative problem. The model case should be rather general; so we select it to be

1. *semi linear*,
2. of *first order with respect to time*, and
3. *non vanishing at the boundary* and *at infinity*.

In view of the above assumptions, we consider the following problem involving the NLSE with the defocusing pure power non linearity

$$\begin{cases} i\frac{\partial u}{\partial t} - \Delta(u+\zeta) + \left(|u+\zeta|^{2\tau} - \rho^{2\tau}\right)(u+\zeta) = 0, \text{ in } (J_0 \smallsetminus \{0\}) \times U, \\ u = u_0, \text{ in } \{t=0\} \times \overline{U}, \\ u = 0, \text{ in } J_0 \times \partial U, \text{ and } u \xrightarrow{|x| \nearrow \infty} 0, \text{ in } J_0 \times \overline{U}, \end{cases} \tag{3.1}$$

for $x \in U \subseteq \mathbb{R}^n$, where $n \in \{1,2,3\}$, for $t \in J_0 \subseteq \mathbb{R}$, where J_0 stands for an open interval containing $t_0 = 0$, with $u\colon J_0 \times \overline{U} \to \mathbb{C}$ and $\zeta\colon \overline{U} \to \mathbb{C}$ being sufficiently smooth, as well as $\rho > 0$ and

$$\begin{cases} \tau \in \mathbb{N}, & \text{if } n \in \{1,2\}, \\ \tau = 1, & \text{if } n = 3. \end{cases}$$

Just for the sake of convenience and brevity, we do not consider the elliptic operator of Definition 2.2.1 in its generality, i.e., $-\operatorname{div}\left(A^{\mathrm{T}}\nabla\right)$. We emphasize that ζ does not (necessarily) vanish at the boundary of U and at infinity. In this context, ζ is considered as an element of a Zhidkov space; recall that, $\forall m \in \mathbb{N}$, the Zhidkov space over U, is defined as the Banach space

$$X^m(U) = \left\{u \in L^\infty(U) \,\middle|\, \nabla_w^k u \in L^2(U), \text{ for } k \in \{1,\dots,m\}\right\},$$

endowed with its natural norm

$$\|u\|_{X^m(U)} = \|u\|_{L^\infty(U)} + \sum_{k=1}^m \left\|\nabla_w^k u\right\|_{L^2(U)}.$$

A characteristic example is the hyperbolic tangent, for which we have $\tanh \in \bigcap_{m=1}^{\infty} X^m(\mathbb{R})$. Evidently, $X^m(U) \hookrightarrow H^m(U)$ only if $|U| < \infty$.

We note that the problem (3.1) is actually a Gross-Pitaevskii one, which follows from the substitution of

$$v(t,x) = e^{i\rho^{2\tau}t}\left(u(t,x) + \zeta(x)\right),$$

to the classic defocusing NLSE problem with the pure power non linearity

$$\begin{cases} i\frac{\partial v}{\partial t} - \Delta v + |v|^{2\tau} v = 0, \text{ in } (J_0 \smallsetminus \{0\}) \times U, \\ v = v_0, \text{ in } \{t=0\} \times \overline{U}, \\ v \neq 0, \text{ in } J_0 \times \partial U, \text{ and } v \not\to 0 \text{ when } |x| \nearrow \infty, \text{ in } J_0 \times \overline{U}. \end{cases}$$

The problem (3.1) has already been studied, to some detail, in [9], [8] or [7], where certain open sets, bounded or not, are considered. Here, we generalize the results of the aforementioned papers by broadening the range of the considered open sets.

3.1 Bounded open sets

For the existence results in bounded open sets, the standard *Faedo-Galerkin scheme* can be utilized.

3.1.1 Weak solutions

The basic result here is already known from [9], [8] and [7], and the key tools needed are the following:

1. The equations

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla_w(\mathbf{u}_k+\zeta)\|^2_{L^2(U)} + \int_U \underbrace{\left(\frac{1}{2\tau+2}|\mathbf{u}_k+\zeta|^{2\tau+2} - \frac{1}{2}\rho^{2\tau}|\mathbf{u}_k+\zeta|^2 + \frac{\tau}{2\tau+2}\rho^{2\tau+2}\right)}_{\geq 0} dx\right) = 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_k\|^2_{L^2(U)} - \operatorname{Im}(\nabla_w\zeta, \nabla_w\mathbf{u}_k) - \operatorname{Im}\left(\left(|\mathbf{u}_k+\zeta|^{2\tau} - \rho^{2\tau}\right)(\mathbf{u}_k+\zeta), \mathbf{u}_k\right) = 0,$$

where $\{\mathbf{u}_k\}_{k=1}^\infty \subsetneq C^\infty\left(\mathbb{R}; H_0^1(U;\mathbb{C})\right)$ stands for the Faedo-Galerkin approximation sequence.

2. The version of the Gagliardo-Nirenberg interpolation inequality, see, e.g., [5, Theorem 1.3.7],

$$\|u\|_{L^{\alpha+2}(\mathbb{R}^n)} \leq C \|\nabla_w u\|_{L^2(\mathbb{R}^n)}^{\frac{n\alpha}{2(\alpha+2)}} \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n\alpha}{2(\alpha+2)}}, \ \forall u \in C_c^\infty(\mathbb{R}^n),$$

for

$$\alpha \in \begin{cases} (0,\infty), & \text{if } n \in \{1,2\}, \\ \left(0, \frac{4}{n-2}\right), & \text{otherwise,} \end{cases}$$

or else

$$\|u\|_{L^{\alpha+2}(U)} \leq C \|\nabla_w u\|_{L^2(U)}^{\frac{n\alpha}{2(\alpha+2)}} \|u\|_{L^2(U)}^{1-\frac{n\alpha}{2(\alpha+2)}}, \ \forall u \in H_0^1(U),$$

which is utilized for handling the non linear term.

Theorem 3.1.1. *If U is bounded and $\zeta \in X^1(U)$, then for every $u_0 \in H_0^1(U)$ and every bounded interval J_0 there exists a solution*

$$\mathbf{u} \in L^\infty\left(J_0; H_0^1(U)\right) \cap W^{1,\infty}\left(J_0; H^{-1}(U)\right)$$

of (3.1), such that

$$\|\mathbf{u}\|_{L^\infty(J_0;H^1(U))} + \|\mathbf{u}'\|_{L^\infty(J_0;H^{-1}(U))} \leq C\Big(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta|-\rho\|_{L^2(U)}, |J_0|\Big).$$

We note that there is no restriction in Theorem 3.1.1 concerning the choice of U, the boundary of which can have cusps, or the set can lie on each side of the boundary, etc. Moreover, we emphasize that the estimate in this result does not depend directly on U, e.g., on $|U|$.

For the sake of completeness, we also state the following already known (see [9]) uniqueness/globality result, where for the case $n=1$ the Sobolev embedding $H_0^m(U) \hookrightarrow L^\infty(U)$ is employed and for $n=2$, either the Trudinger (see, e.g., [5, Remark 1.3.6]), or the following version of the Gagliardo-Nirenberg interpolation inequality (see, e.g., [22, Lemma 2])

$$\|u\|_{L^{2s}(U)} \leq C s^{\frac{1}{2}} \|\nabla_w u\|_{L^2(U)}^{1-\frac{1}{s}} \|u\|_{L(U)}^{\frac{1}{s}}, \ \forall u \in H_0^1(U), \ \forall s \in [1,\infty), \ n=2,$$

is employed.

Theorem 3.1.2. *Let $\mathbf{u}$ be as in Theorem 3.1.1. If*

$$\begin{cases} n=1, \ \textit{or} \\ n=2 \ \textit{and} \ \tau=1, \end{cases}$$

then $\mathbf{u}$ is unique and global.

3.1.2 Regular solutions

In order to get regularity of weak solutions, we need to impose certain smoothness to the boundary of U. The key tools needed here are the following:

1. The equations

$$\frac{1}{2}\frac{d}{dt}\|\Delta_w^j \mathbf{u}_k\|_{L^2(U)}^2 - \operatorname{Im}\left(\Delta_w^{j+1}\zeta, \Delta_w^j \mathbf{u}_k\right) - \operatorname{Im}\Big(\Delta_w^j\Big(\left(|\mathbf{u}_k+\zeta|^{2\tau}\right)(\mathbf{u}_k+\zeta)\Big), \Delta_w^j \mathbf{u}_k\Big) = 0,$$

$\forall j \in \mathbb{N}$ with $2j \leq m$, and

$$\frac{1}{2}\frac{d}{dt}\left\|\left(\nabla_w \circ \Delta_w^j\right)\mathbf{u}_k\right\|_{L^2(U)}^2 - \operatorname{Im}\left(\left(\nabla_w \circ \Delta_w^{j+1}\right)\zeta, \left(\nabla_w \circ \Delta_w^j\right)\mathbf{u}_k\right) -$$
$$- \operatorname{Im}\Big(\left(\nabla_w \circ \Delta_w^j\right)\Big(\left(|\mathbf{u}_k+\zeta|^{2\tau}\right)(\mathbf{u}_k+\zeta)\Big), \left(\nabla_w \circ \Delta_w^j\right)\mathbf{u}_k\Big) = 0,$$

$\forall j \in \mathbb{N}$ with $2j+1 \leq m$, where $\{\mathbf{u}_k\}_{k=1}^\infty$ stands for the regular enough Faedo-Galerkin approximation sequence. Those equations are valid when the arbitrary U has almost C^m boundary ∂U.

2. For handling the non linear term, we utilize the following results (see [7]), where the version of the Gagliardo-Nirenberg interpolation inequality

$$\left\|\nabla^j u\right\|_{L^{\frac{2m}{j}}(\mathbb{R}^n)} \leq C\|\nabla^m u\|_{L^2(\mathbb{R}^n)}^{\frac{j}{m}}\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{m}}, \ \ \forall\, j \in \{0,1,\dots,m\}, \ \ \forall u \in C_c^\infty(\mathbb{R}^n),$$

the Sobolev embedding $H^m(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $n < 2m$, the version of the Brezis-Gallouët-Wainger inequality (see, e.g., [2, Lemma 2])

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C\Big(\|u\|_{H^1(\mathbb{R}^2)}\Big)\left(1 + \left(\ln\Big(1 + \left\|\nabla^2 u\right\|_{L^2(\mathbb{R}^2)}\Big)\right)^{\frac{1}{2}}\right), \ \ \forall u \in C_c^\infty(\mathbb{R}^2),$$

as well as Theorem 2.1.1 and Proposition 2.2.3, are employed.

Corollary 3.1.1 $(n = 1)$**.** *Let $U \subsetneq \mathbb{R}$ with $|U| < \infty$ as well as $\partial U \in \mathrm{Lip}(\varepsilon, K, 0)$, $m \in \mathbb{N}\setminus\{1\}$, $f \in C^m([0,\infty);\mathbb{R})$, $u \in H^m(U)$ and $\zeta \in X^m(U)$. Then $\Big(f\Big(|u+\zeta|^2\Big)(u+\zeta)\Big) \in H^m(U)$, satisfying the inequality*

$$\sum_{j=1}^m \left\|\nabla_w^j\Big(f\Big(|u+\zeta|^2\Big)(u+\zeta)\Big)\right\|_{L^2(U)} \leq C\bigg(\frac{1}{\varepsilon^m}\max\Big\{1, |U|^{\frac{1}{2}}\Big\}, K, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\bigg)\times$$
$$\times\left(1 + \sum_{j=2}^m \left\|\nabla_w^j u\right\|_{L^2(U)}\right).$$

If, in addition, $u \in H^m(U) \cap H_0^1(U)$, as well as

$$\left(\Delta_w^j u\right) \in H_0^1(U), \ \ \forall j \in \left\{0, \dots, \left\lfloor\frac{m}{2}\right\rfloor - 1\right\},$$

then we have

$$\sum_{j=1}^m \left\|\nabla_w^j\Big(f\Big(|u+\zeta|^2\Big)(u+\zeta)\Big)\right\|_{L^2(U)} \leq C\bigg(\frac{1}{\varepsilon^m}\max\Big\{1, |U|^{\frac{1}{2}}\Big\}, K, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\bigg)\times$$
$$\times\left(1 + \sum_{\substack{j\in\mathbb{N},\\ 2j+1\leq m}} \left\|\left(\nabla_w \circ \Delta_w^j\right)u\right\|_{L^2(U)} + \sum_{\substack{j\in\mathbb{N},\\ 2j\leq m}} \left\|\Delta_w^j u\right\|_{L^2(U)}\right).$$

Corollary 3.1.2 $(n = 2)$**.** *Let $U \subsetneq \mathbb{R}^2$ with $|U| < \infty$ as well as $\partial U \in \mathrm{Lip}(\varepsilon, K, L)$, $m \in \mathbb{N}\setminus\{1\}$, $u \in H^m(U)$ and $\zeta \in X^m(U)$. Then $\Big(|u+\zeta|^2(u+\zeta)\Big) \in H^m(U)$, satisfying*

$$\sum_{j=1}^m \left\|\nabla_w^j\Big(|u+\zeta|^2(u+\zeta)\Big)\right\|_{L^2(U)} \leq C\bigg(\frac{1}{\varepsilon^m}\max\Big\{1, |U|^{\frac{1}{2}}\Big\}, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\bigg)\times$$
$$\times\left(1 + \sum_{j=2}^m \left\|\nabla_w^j u\right\|_{L^2(U)}\right)\left(1 + \ln\left(1 + \left(\sum_{j=2}^m \left\|\nabla_w^j u\right\|_{L^2(U)}\right)^2\right)\right).$$

If, in addition, $\partial U \sim C^m(\varepsilon, K, L)$, $u \in H^m(U) \cap H_0^1(U)$, as well as

$$\left(\Delta_w^j u\right) \in H_0^1(U), \ \ \forall j \in \left\{0, \dots, \left\lfloor\frac{m}{2}\right\rfloor - 1\right\},$$

then we have

$$\sum_{j=1}^m \left\|\nabla_w^j\Big(|u+\zeta|^2(u+\zeta)\Big)\right\|_{L^2(U)} \leq C\bigg(\frac{1}{\varepsilon^m}\max\Big\{1, |U|^{\frac{1}{2}}\Big\}, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\bigg)\times$$
$$\times\left(1 + \sum_{\substack{j\in\mathbb{N},\\ 2j+1\leq m}} \left\|\left(\nabla_w \circ \Delta_w^j\right)u\right\|_{L^2(U)} + \sum_{\substack{j\in\mathbb{N},\\ 2j\leq m}} \left\|\Delta_w^j u\right\|_{L^2(U)}\right)\times$$
$$\times\left(1 + \ln\left(1 + \sum_{\substack{j\in\mathbb{N},\\ 2j+1\leq m}} \left\|\left(\nabla_w \circ \Delta_w^j\right)u\right\|_{L^2(U)}^2 + \sum_{\substack{j\in\mathbb{N},\\ 2j\leq m}} \left\|\Delta_w^j u\right\|_{L^2(U)}^2\right)\right).$$

We also state a straightforward adaptation of Corollary 3.1.2, which is crucial for the case of "exterior domains" below, where Theorem 2.1.2 is employed instead of Theorem 2.1.1.

Corollary 3.1.3 ($n = 2$ and connected U with ∂U having at least two connected components)**.** *Let $U \subsetneq \mathbb{R}^2$ be bounded and connected, such that ∂U has at least two connected components with one of which, $\widetilde{\partial U}$, is such that $\widetilde{\partial U} \in \mathrm{Lip}(\varepsilon_1, K, L)$, as well as $\left(\partial U \smallsetminus \widetilde{\partial U}\right) \in \mathrm{Lip}(\varepsilon_2, K, L)$. Moreover, let $m \in \mathbb{N} \smallsetminus \{1\}$, $u \in H^m(U)$ and $\zeta \in X^m(U)$. Then $\left(|u+\zeta|^2 (u+\zeta)\right) \in H^m(U)$, satisfying*

$$\sum_{j=1}^{m} \left\| \nabla_w^j \left(|u+\zeta|^2 (u+\zeta) \right) \right\|_{L^2(U)} \leq C\Big(I_1, I_2, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\Big) \times \\ \times \left(1 + \sum_{j=2}^{m} \left\|\nabla_w^j u\right\|_{L^2(U)}\right) \left(1 + \ln\left(1 + \left(\sum_{j=2}^{m} \left\|\nabla_w^j u\right\|_{L^2(U)}\right)^2\right)\right),$$

where

$$I_1 = \frac{1}{{\varepsilon_1}^m} \max\left\{1, \left|U \cap \bigcup_{x \in \widetilde{\partial U}} B(x, \varepsilon_1)\right|^{\frac{1}{2}}\right\} \;\&\; I_2 = \frac{1}{{\varepsilon_2}^m} \max\left\{1, \left|U \cap \bigcup_{x \in \partial U \smallsetminus \widetilde{\partial U}} B(x, \varepsilon_2)\right|^{\frac{1}{2}}\right\}.$$

If, in addition, $\partial U \sim C^m(\min\{\varepsilon_1, \varepsilon_2\}, K, L)$, $u \in H^m(U) \cap H_0^1(U)$, as well as

$$\left(\Delta_w^j u\right) \in H_0^1(U), \;\; \forall j \in \left\{0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1\right\},$$

then we have

$$\sum_{j=1}^{m} \left\| \nabla_w^j \left(|u+\zeta|^2 (u+\zeta) \right) \right\|_{L^2(U)} \leq C\Big(I_1, I_2, K, L, \|u\|_{H^1(U)}, \|\zeta\|_{X^m(U)}\Big) \times \\ \times \left(1 + \sum_{\substack{j \in \mathbb{N}, \\ 2j+1 \leq m}} \left\|\left(\nabla_w \circ \Delta_w^j\right) u\right\|_{L^2(U)} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \leq m}} \left\|\Delta_w^j u\right\|_{L^2(U)}\right) \times \\ \times \left(1 + \ln\left(1 + \sum_{\substack{j \in \mathbb{N}, \\ 2j+1 \leq m}} \left\|\left(\nabla_w \circ \Delta_w^j\right) u\right\|_{L^2(U)}^2 + \sum_{\substack{j \in \mathbb{N}, \\ 2j \leq m}} \left\|\Delta_w^j u\right\|_{L^2(U)}^2\right)\right).$$

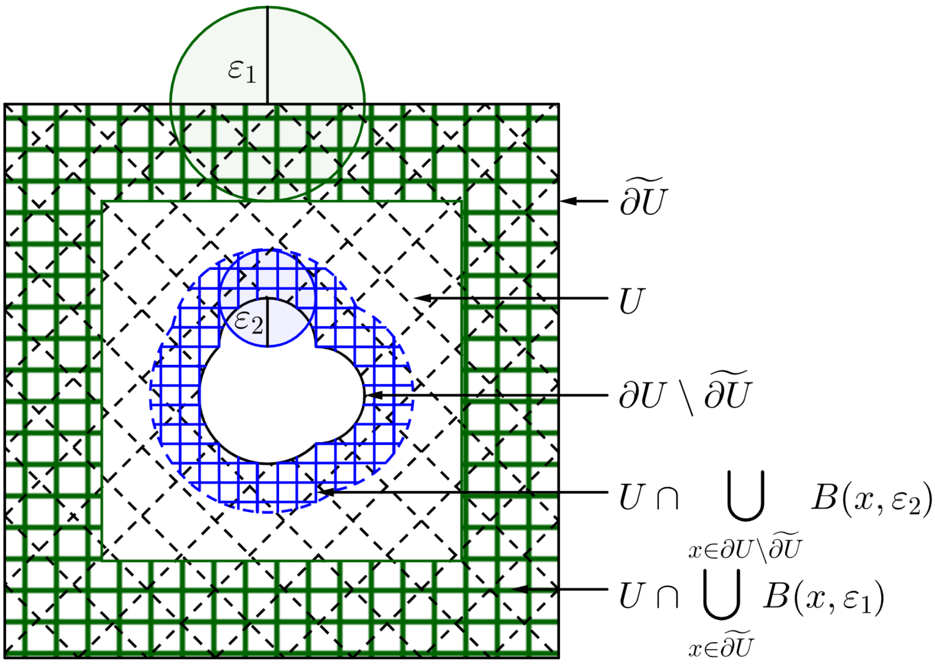

Figure 6: An example of the geometry of Corollary 3.1.3.

Hence, in view of Corollary 3.1.1 and Corollary 3.1.2 we have the following.

Theorem 3.1.3. *Let* $n \in \{1,2\}$*,* U *be bounded,*

$$\begin{cases} \tau \in \mathbb{N}, & \textit{if } n = 1, \\ \tau = 1, & \textit{if } n = 2, \end{cases}$$

$u_0 \in H_0^1(U)$ *and* $\mathbf{u}$ *be the (unique and global) solution of (3.1) that Theorem 3.1.1 provides. If*

1. $\partial U \sim \bigcap_{m=1}^{\infty} C^m(\varepsilon, K, L_m)$,

2. $\zeta \in \bigcap_{m=1}^{\infty} X^m(U)$ *and*

3. $u_0 \in \bigcap_{m=2}^{\infty} H^m(U) \cap H_0^1(U)$*, with* $\left(\Delta^j u_0\right) \in H_0^1(U)$ $\forall j \in \mathbb{N}_0$,

then $\mathbf{u} \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\Big(\mathbb{R}; H_0^1(U) \cap \bigcap_{m=2}^{\infty} H^m(U)\Big)$,

$$\|\mathbf{u}^{(j)}\|_{L^\infty(J_0;H^m(U))} \leq C\bigg(\frac{1}{\varepsilon^m} \max\Big\{1, |U|^{\frac{1}{2}}\Big\}, K, L_m, \|u_0\|_{H^m(U)}, \|\zeta\|_{X^{m+2}(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\bigg),$$

$\forall\ j \in \mathbb{N}_0$*,* $m \in \mathbb{N} \smallsetminus \{1\}$ *and* J_0*.*

In view of Corollary 3.1.3 instead, Theorem 3.1.3 takes the following form.

Proposition 3.1.1. *Let* $n = 2$*,* U *be bounded and connected, such that* ∂U *has at least two connected components where* $\widetilde{\partial U}$ *stands for the exterior one,* $\tau = 1$*,* $u_0 \in H_0^1(U)$ *and* $\mathbf{u}$ *be the (unique and global) solution of (3.1) that Theorem 3.1.1 provides. If*

1. $\widetilde{\partial U} \sim \bigcap_{m=1}^{\infty} C^m(\varepsilon_1, K, L_m)$ *&* $\big(\partial U \smallsetminus \widetilde{\partial U}\big) \sim \bigcap_{m=1}^{\infty} C^m(\varepsilon_2, K, L_m)$,

2. $\zeta \in \bigcap_{m=1}^{\infty} X^m(U)$ *and*

3. $u_0 \in \bigcap_{m=2}^{\infty} H^m(U) \cap H_0^1(U)$*, with* $\left(\Delta^j u_0\right) \in H_0^1(U)$ $\forall j \in \mathbb{N}_0$,

then $\mathbf{u} \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\Big(\mathbb{R}; H_0^1(U) \cap \bigcap_{m=2}^{\infty} H^m(U)\Big)$,

$$\|\mathbf{u}^{(j)}\|_{L^\infty(J_0;H^m(U))} \leq C\Big(I_1, I_2, K, L_m, \|u_0\|_{H^m(U)}, \|\zeta\|_{X^{m+2}(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\Big),$$

$\forall\ j \in \mathbb{N}_0$*,* $m \in \mathbb{N} \smallsetminus \{1\}$ *and* J_0*, for* I_1 *and* I_2 *as in Corollary 3.1.3.*

3.2 Unbounded open sets

For the existence results in unbounded open sets, the scheme that will be utilized has the following form:

$\boxed{\text{I}}$ *Certain estimates in bounded open sets.*
The solutions in bounded open sets satisfy a certain estimate; the characteristics of the domains are incorporated, in a specific manageable way, in this estimate.

$\boxed{\text{II}}$ *Approximating the unbounded open set from inside.*
We approximate the unbounded open set by an appropriate increasing sequence of bounded subsets.

$\boxed{\text{III}}$ *Constructing an approximation sequence of the solution.*
In every bounded approximation open set as above, we consider the corresponding approximation problem. The sequence of the solutions of those problems will eventually converge to the desired solution of the initial problem in the unbounded open set.

Before we proceed with the application of the above scheme to the problem (3.1), let us be more specific about the crucial steps $\boxed{\text{II}}$ and $\boxed{\text{III}}$ (we have already studied step $\boxed{\text{I}}$):

- From the analytical point of view, these steps are based on the following results, for which we refer to [7, Proposition A.8 and Lemma 3.4, respectively].

  Proposition 3.2.1 (cut off function)**.** *Let $\delta > 0$. Then there exists $\phi \in C_c^\infty(\mathbb{R}^n; [0,1])$ such that*

  1. $\operatorname{supp}(\phi) \subseteq \overline{U^\delta}$,
  2. $\phi|_{\overline{U}} = 1$ *and*
  3. $\|\nabla^m \phi\|_{L^\infty(\mathbb{R}^n)} \leq \dfrac{C_m}{\delta^m}$, $\forall m \in \mathbb{N}_0$ $(C_0 = 1)$.

  Proposition 3.2.2. *Let $m \in \mathbb{N}$, $p \in [1,\infty]$, $\phi \in C_c^\infty(U_1)$ and $u \in W_0^{m,p}(U_2)$. Then*

  $$(\phi|_{U_1 \cap U_2} u|_{U_1 \cap U_2}) \in W_0^{m,p}(U_1 \cap U_2), \text{ with } \|\phi u\|_{W^{m,p}(U_1 \cap U_2)} \leq C\Big(\|\phi\|_{C_b^m(U_1)}\Big)\|u\|_{W^{m,p}(U_2)}.$$

  Hence, with the above arsenal at hand, we can cut a given initial datum $u_0 \in H_0^1(U)$ off, for unbounded U, by utilizing a sequence of cut off functions

  $$\left\{\phi \in C_c^\infty(\mathbb{R}^n; [0,1]) \,\middle|\, \operatorname{supp}(\phi) \subseteq \overline{U_{k+1}} \text{ and } \phi|_{\overline{U_k}} = 1\right\}_{k=1}^\infty,$$

  in order to get a sequence $\left\{u_{0k} \in H_0^1(U \cap U_k)\right\}_{k=1}^\infty$, where

  1. $U_k \subset\subset U_{k+1} \subset\subset \mathbb{R}^n$ $\forall m \in \mathbb{N}$ &
  2. $U_k \nearrow \mathbb{R}^n$,

  such that

  $$u_{0k+1}|_{U \cap U_k} = u_0|_{U \cap U_k}$$

  and

  $$\widetilde{u_{0k}} \xrightarrow{H_0^1(U)} u_0, \text{ here } \widetilde{\cdot} \text{ stands for the extension-by-zero operator.}$$

  Moreover, if $\{U_k\}_{k=1}^\infty$ is such that

  3. $\operatorname{dist}(\partial U_k, \partial U_{k+1}) \geq C$ uniformly $\forall k \in \mathbb{N}$,

  then

  $$\|u_{0k}\|_{H^1(U \cap U_k)} \leq C \|u_0\|_{H^1(U)} \text{ uniformly } \forall k \in \mathbb{N},$$

  due to the increasing property of C. This way, we have constructed a sequence of new initial data for the corresponding sequence of Cauchy problems, which converges to the initial datum. Moreover, these initial data have uniformly bounded, therefore manageable (by the use of a standard compactness argument), norms.

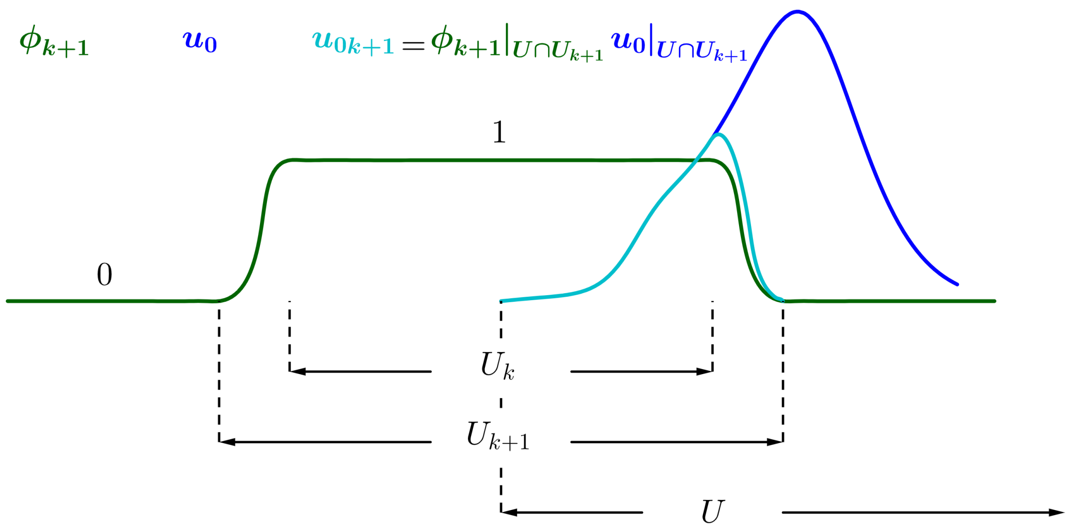

Figure 7: A sketch of how the sequence $\{u_{0k} \in H_0^1(U \cap U_k)\}_{k=1}^{\infty}$ is constructed. At the end of the day, the approximation sequence $\{\widetilde{\mathbf{u}_k}\}_{k=1}^{\infty} \subsetneq L^{\infty}(J_0; H_0^1(U))$ has the following properties: 1. every $\widetilde{\mathbf{u}_{k+1}}|_{U \cap U_{k+1}} = \mathbf{u}_{k+1}$ solves the NLSE and 2. $\widetilde{\mathbf{u}_{k+1}}|_{U \cap U_k}(0) = \mathbf{u}_{k+1}|_{U \cap U_k}(0) = u_0|_{U \cap U_k}$ $\forall k \in \mathbb{N}$.

- As for the geometrical point of view, which is crucial only for case of regular solutions as we have already seen, the above steps are based on Proposition 2.2.2. Hence, we can demand that the increasing sequence $\{U \cap U_k\}_{k=1}^{\infty}$ of the previous point carries the same geometrical properties, i.e., $\partial(U \cap U_k) \sim C^m(\lambda_k \varepsilon, K, L)$, $\forall k \in \mathbb{N}$, for appropriate $\lambda_k \nearrow \infty$. This enables us to manage the quantities appeared in estimates, concerning the properties of the boundaries, since in that case we get

$$C\left(\frac{1}{\lambda_k \varepsilon}, K, L\right) \leq C \text{ uniformly } \forall k \in \mathbb{N},$$

  due to the increasing property of C. Moreover, we will utilize the results of Section 2.1.2 as well as Proposition 2.1.1, and combine them with the fact that $\lambda_k \nearrow \infty$, in order to manage the non vanishing term $|U \cap U_k|^{\frac{1}{2}}$ that arises from the term $\|\zeta\|_{L^2(U \cap U_k)}$.

3.2.1 Weak solutions

Since the estimate in Theorem 3.1.1 is independent of the choice of the bounded U, then, employing the scheme described above, we get the following, already known from [9], [8] and [7], result.

Theorem 3.2.1. *Theorem 3.1.1 is also valid for every unbounded U, if, in addition, $\||\zeta| - \rho\|_{L^2(U)}$.*

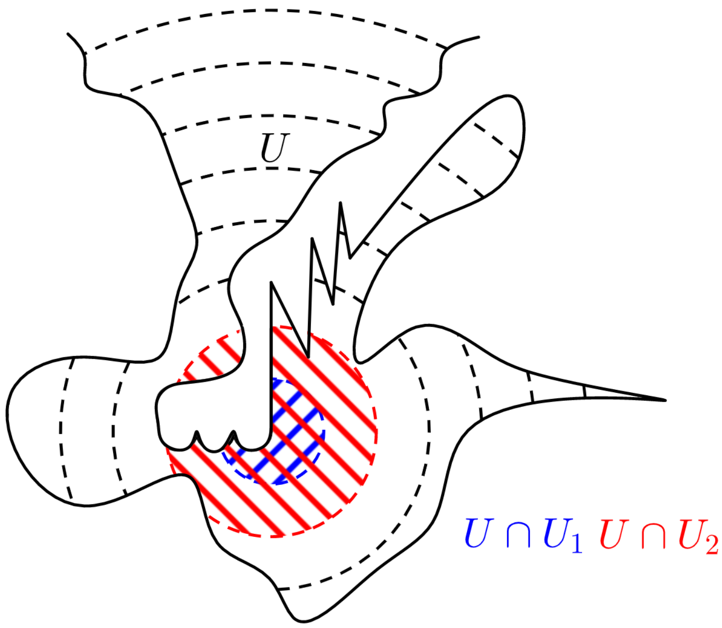

Figure 8: For the existence of weak solution, neither the boundary of the unbounded U nor the geometry of the increasing sequence $\{U \cap U_k\}_{k=1}^{\infty}$, are important.

We note that the additional condition implies that ζ decays on a constant background ρ as $|x| \nearrow \infty$.

As far as the uniqueness/globality is concerned, we state the analogous of Theorem 3.1.2, where, for the case of $\mathbb{R}^n$, the Strichartz (dispersive) estimates are employed, see, e.g., [5, Theorem 2.3.3].

Theorem 3.2.2. *Let* $\mathbf{u}$ *be as in Theorem 3.2.1. If*

$$\begin{cases} n = 1, \ \textit{or} \\ n = 2 \ \& \ \tau = 1, \ \textit{or} \\ U = \mathbb{R}^n, \ (n \in \{1,2,3\}), \end{cases}$$

then $\mathbf{u}$ *is unique and global.*

3.2.2 Regular solutions

As we have already stated, the property (P) characterizes the admissible unbounded open sets where regular solutions exists. This fact is about to become clear in the following analysis.

- **The case $U = \mathbb{R}^n$.**

If $U = \mathbb{R}^n$, then U can be covered from inside with an expanding sequence of open sets of simple geometry. In fact the boundaries of those sets can be considered of class $\bigcap_{m=1}^{\infty} C^m$, e.g.,

$$U = \bigcup_{k=1}^{\infty} B(0, k\varepsilon) \text{ for some } \varepsilon > 0.$$

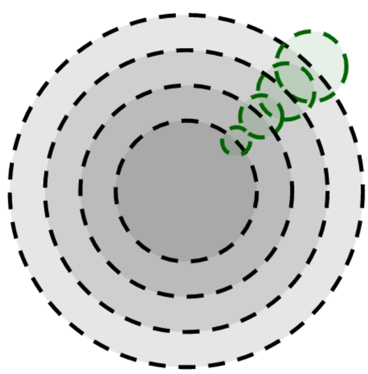

Figure 9: An example of admissible expanding sequence $\{U \cap U_k\}_{k=1}^{\infty}$ for the case $U = \mathbb{R}^n$.

Since

$$\partial B(0,\varepsilon) \in \bigcap_{m=1}^{\infty} C^m(C\varepsilon, K, L),$$

in the light of Proposition 2.1.1 we have that

$$\partial B(0,k\varepsilon) \in \bigcap_{m=1}^{\infty} C^m(kC\varepsilon, K, L).$$

We compute

$$\frac{1}{(kC\varepsilon)^m} \max\left\{1, |B(0,k\varepsilon)|^{\frac{1}{2}}\right\} = O\left(k^{\frac{n}{2}-m}\right), \text{ as } k \nearrow \infty.$$

Hence, we deduce that

$$\frac{1}{(kC\varepsilon)^m} \max\left\{1, |B(0,k\varepsilon)|^{\frac{1}{2}}\right\} \le C \ \forall k \in \mathbb{N}, \text{ if } n \in \{1,2\} \ \& \ m \in \mathbb{N} \setminus \{1\}.$$

Combining this bound with the increasing bound of every "sub solution"

$$\mathbf{u}_k \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\left(\mathbb{R}; H^1_0(B(0,k\varepsilon)) \bigcap_{m=2}^{\infty} H^m(B(0,k\varepsilon))\right)$$

provided by Theorem 3.1.3, we get that the corresponding sequence of norms of $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is uniformly bounded. Thus follows the next already known (see [7]) result.

Theorem 3.2.3. *Let $n \in \{1,2\}$, τ be as in Theorem 3.1.3, $\zeta \in X^1(\mathbb{R}^n)$ with $\||\zeta| - \rho\|_{L^2(\mathbb{R}^n)}$, $u_0 \in H^1(\mathbb{R}^n)$ and $\mathbf{u}$ be the (unique and global) solution of (3.1) that Theorem 3.2.1 provides. If*

1. $\zeta \in \bigcap_{m=1}^{\infty} X^m(\mathbb{R}^n)$ *and*
2. $u_0 \in \bigcap_{m=1}^{\infty} H^m(\mathbb{R}^n)$,

then $\mathbf{u} \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\left(\mathbb{R}; \bigcap_{m=1}^{\infty} H^m(\mathbb{R}^n)\right)$, *with*

$$\left\|\mathbf{u}^{(j)}\right\|_{L^\infty(J_0;H^m(\mathbb{R}^n))} \le C\left(\|u_0\|_{H^m(\mathbb{R}^n)}, \|\zeta\|_{X^{m+2}(\mathbb{R}^n)}, \||\zeta| - \rho\|_{L^2(\mathbb{R}^n)}, |J_0|\right),$$

$\forall\ j \in \mathbb{N}_0$, $m \in \mathbb{N} \setminus \{1\}$, *and* J_0.

- **The case $U = \mathbb{R}^n_+$ and similar ones.**

Here, the extension of the elliptic regularity theory for open sets with almost C^m boundaries of Section 2.2 is crucial, since there is no admissible "smooth" way of choosing the expanding sequence $\{U \cap U_k\}_{k=1}^{\infty}$ for the case $U = \mathbb{R}^n_+$.

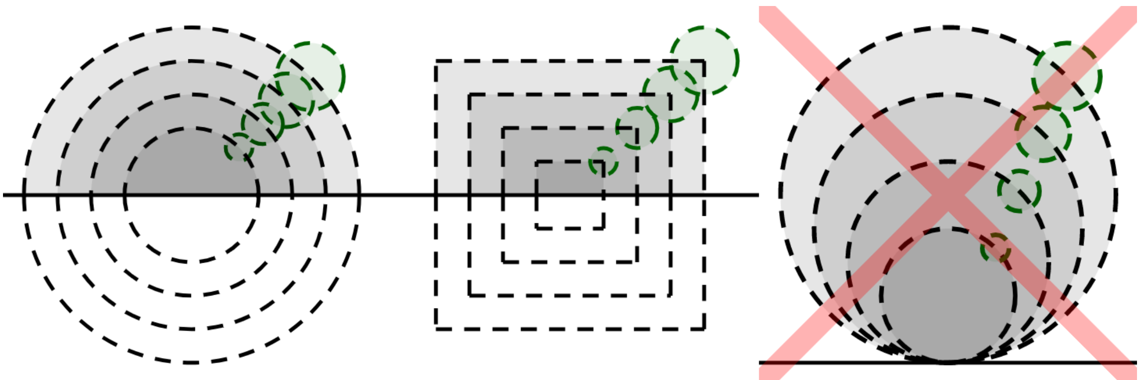

Figure 10: There is no admissible expanding sequence $\{U \cap U_k\}_{k=1}^{\infty}$ for the case $U = \mathbb{R}^n_+$, such that every $\partial(U \cap U_k)$ is of class C^m, as we see at the third case. We also mention the crux at the first two cases: not only $U_k = c_k + \lambda_k U_0$ $\forall k \in \mathbb{N}$ for appropriate $\{c_k\}_{k=1}^{\infty} \subsetneq \mathbb{R}^n$, $\{\lambda_k\}_{k=1}^{\infty}$ and $U_0 \subset\subset \mathbb{R}^n$, but also $U \cap U_k = c_k + \lambda_k (U \cap U_0)$ $\forall k \in \mathbb{N}$ as well.

Therefore, based on the previous discussion, the analogous of Theorem 3.2.3 follows by the same steps.

Theorem 3.2.4. *Let $n \in \{1,2\}$, τ be as in Theorem 3.1.3, $\zeta \in X^1(\mathbb{R}^n_+)$ with $\| |\zeta| - \rho \|_{L^2(\mathbb{R}^n_+)}$, $u_0 \in H^1_0(\mathbb{R}^n_+)$ and $\mathbf{u}$ be the (unique and global) solution of (3.1) that Theorem 3.2.1 provides. If*

1. $\zeta \in \bigcap_{m=1}^{\infty} X^m(\mathbb{R}^n_+)$ *and*

2. $u_0 \in \bigcap_{m=1}^{\infty} H^m(\mathbb{R}^n_+)$, *with* $\left(\Delta^j u_0\right) \in H^1_0(\mathbb{R}^n_+)$ $\forall j \in \mathbb{N}_0$,

then $\mathbf{u} \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\left(\mathbb{R}; H^1_0(\mathbb{R}^n_+) \cap \bigcap_{m=2}^{\infty} H^m(\mathbb{R}^n_+)\right)$, *with*

$$\|\mathbf{u}^{(j)}\|_{L^\infty(J_0;H^m(\mathbb{R}^n_+))} \le C\Big(\|u_0\|_{H^m(\mathbb{R}^n_+)}, \|\zeta\|_{X^{m+2}(\mathbb{R}^n_+)}, \||\zeta| - \rho\|_{L^2(\mathbb{R}^n_+)}, |J_0|\Big),$$

$\forall$ $j \in \mathbb{N}_0$, $m \in \mathbb{N} \smallsetminus \{1\}$, and J_0.

We note that, following exactly the same steps we can generalize the above result also for the case where

$$U = c + \mathbb{R}^n_\pm, \ \forall c \in \mathbb{R}^n, \ \text{for } n \in \{1,2\},$$

as well as for

$$U = \bigcup_{\substack{r>0,\\ \theta \in (\theta_1,\theta_2)}} \left\{x \in \mathbb{R}^2 \,\middle|\, x_1 - c_1 = r\cos\theta \ \& \ x_2 - c_2 = r\sin\theta\right\}, \ \forall\, (c_{1,2}, \theta_{1,2}) \in \mathbb{R}^2 \text{ with } \theta_2 - \theta_1 \in (0, 2\pi).$$

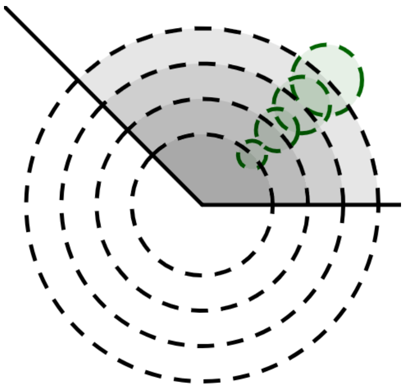

Figure 11: An admissible expanding sequence $\{U \cap U_k\}_{k=1}^{\infty}$ that covers a "broken $\mathbb{R}^2_+$" set.

- **The case where U is an exterior open set.**

If $U \subsetneq \mathbb{R}^2$ is an exterior open set, i.e., a connected open set with bounded $\partial U \neq \varnothing$, then the frontal extension operator is crucial, since we are now able to expand only the exterior front of boundary of $U \cap U_1$. Assuming ∂U is almost regular, then the constant ε associated with ∂U can be eventually considered smaller than ε_k associated with each U_k. In the light of this fact along with point 3 of Remark 2.2.1, we can consider that eventually every $\partial(U \cap U_k)$ is almost regular of constant ε, which allows us to assume that the elliptic regularity estimates do not depend on k.

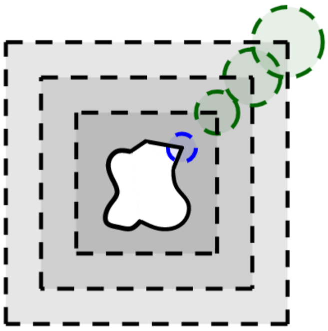

Figure 12: An example of admissible expanding sequence $\{U \cap U_k\}_{k=1}^{\infty}$ for the case of an exterior U. The constant value ϵ associated with ∂U (the blue ball) is eventually smaller than ε_k associated with each U_k (the green balls).

On the other hand, the estimates that involve the frontal extension operator, and in particular the estimates in Proposition 3.1.1, depend on k yet in a certain manageable manner. Evidently, our regularity result is

Theorem 3.2.5. *Let $n = 2$, $U \subsetneq \mathbb{R}^n$ be an exterior set, τ be as in Theorem 3.1.3, $\zeta \in X^1(U)$ with $\||\zeta| - \rho\|_{L^2(U)}$, $u_0 \in H_0^1(U)$ and $\mathbf{u}$ be the (unique and global) solution of (3.1) that Theorem 3.2.1 provides. If*

1. $\partial U \in \bigcap_{m=1}^{\infty} C^m(\varepsilon, K, L_m)$,
2. $\zeta \in \bigcap_{m=1}^{\infty} X^m(U)$ *and*
3. $u_0 \in \bigcap_{m=1}^{\infty} H^m(U)$, *with* $\left(\Delta^j u_0\right) \in H_0^1(U)\ \forall j \in \mathbb{N}_0$,

then $\mathbf{u} \in \bigcap_{j=0}^{\infty} W^{j,\infty}_{\mathrm{loc}}\left(\mathbb{R}; H_0^1(U) \cap \bigcap_{m=2}^{\infty} H^m(U)\right)$, *with*

$$\left\|\mathbf{u}^{(j)}\right\|_{L^\infty(J_0;H^m(U))} \le C\left(\frac{1}{\varepsilon}, K, L_m, \|u_0\|_{H^m(U)}, \|\zeta\|_{X^{m+2}(U)}, \||\zeta| - \rho\|_{L^2(U)}, |J_0|\right),$$

$\forall\ j \in \mathbb{N}_0$, $m \in \mathbb{N} \smallsetminus \{1\}$, *and* J_0.

4 Appendix: Notation

- $k, m, n \in \mathbb{N}$.
- $\mathbb{R}^0$ is the trivial vector space and its (single) element is the 0-dimensional vector.
- Every $f\colon \mathbb{R}^0 \to \mathbb{R}$ is considered as a real constant.
- $x' \in \mathbb{R}^{n-1}$ stands for the $(n-1)$-dimensional vector, which, for $n \ge 2$, is obtained by removing the n-th component of a given n-dimensional vector x, i.e., $\mathbb{R}^n \ni x = (x_i)_{i=1}^n = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.
- We write $\mathfrak{p}\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ for the "projection" defined as $\mathfrak{p}(x) = x', \forall x \in \mathbb{R}^n$.
- $\mathbb{R}^n_{\pm}$ stands for the upper/lower half euclidean space, i.e. for $\left\{x \in \mathbb{R}^n \,\middle|\, x_n \gtrless 0\right\}$.
- $y = \Phi(x) \in \mathbb{R}^n$ stands for the local coordinates (in this case, $x \in \mathbb{R}^n$ stands for the background coordinates) where Φ is a rigid motion, i.e., an affine transformation of the form $\Phi(x) = c + Ax$, where $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is orthogonal.
- $(xA) \in \mathbb{R}^n$ is a simplified way to express $A^{\mathrm{T}}x$, for arbitrary $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. $A^{\mathrm{T}} \in \mathbb{R}^{n \times m}$ denotes the transpose of A.
- S, with or without subscript, is an arbitrary subset of $\mathbb{R}^n$ and S^c stands for its complement. S° denotes the topological interior, ∂S the topological boundary, $\overline{S}$ the topological closure, and S^e the topological exterior (i.e., the set $\overline{S}^c$) of S.
  Further, if S is an (m-dimensional) manifold in $\mathbb{R}^n$ ($m \le n$), then $\operatorname{bd} S$ denotes its boundary ($(m-1)$-dimensional manifold in $\mathbb{R}^n$), and $\operatorname{int} S = S \smallsetminus \operatorname{bd} S$ its interior (m-dimensional manifold in $\mathbb{R}^n$), both in the sense of manifolds.
- We write $S_1 \subset\subset S_2$ when $\overline{S_1} \subseteq S_2^\circ$ and $\overline{S_1}$ is compact.
- U, with or without subscript, is an arbitrary open $\subseteq \mathbb{R}^n$, and $|U|$ its Lebesgue measure.
- We denote by U_δ, for some U and $\delta > 0$, the open subset of U which equals $U = \mathbb{R}^n$ if $\partial U = \varnothing$, and to $\left\{x \in U \,\middle|\, \operatorname{dist}(x, \partial U) > \delta\right\}$ otherwise.

- $B(x,\delta)$ is the ball of centre $x \in \mathbb{R}^n$ and radius $\delta > 0$.
- U^δ for some U and $\delta > 0$ stands for the open superset of U defined as $U^\delta := U \cup \bigcup_{x \in \partial U} B(x,\delta) = \bigcup_{x \in U} B(x,\delta) = U + B(0,\delta)$.
- $\nabla^m f$ for $m \in \mathbb{N}_0$ and $f\colon S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ stands for the vector of components the partial derivatives of order m of f.
- Df for $f\colon S \to \mathbb{R}^n$ with $S \subseteq \mathbb{R}^m$ is the Jacobi matrix $\in \mathbb{R}^{n\times m}$ of f.
- We denote by $\cdot_w$ a differential operator considered in the weak sense.
- C is any generic positive constant, as well as, any increasing function in $C\big([0,\infty)^m; \mathbb{R}_+\big)$ for some $m \in \mathbb{N}$ (assuming the evident definition of increasing multivariate functions).
- $\mathcal{F}$, with or without subscript, stands for an arbitrary space of functions and we write $\mathcal{F}(S)$ when the functions are defined in S.
- $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ means that $\mathcal{F}_1$ and $\mathcal{F}_2$ are normed spaces with $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and also $\|f\|_{\mathcal{F}_2} \leq C\|f\|_{\mathcal{F}_1}$ $\forall f \in \mathcal{F}_1$, where C is independent of the choice of f.
- $C_b^m(U)$ is the Banach space

  $$\left\{ f \in C^m(U) \,\middle|\, D^\alpha f \text{ is bounded everywhere in } U, \text{ for all multi indices } \alpha : 0 \leq |\alpha| \leq m \right\},$$

  (for arbitrary $m \in \mathbb{N}_0$ and for all U), equipped with its natural norm.
- $\mathbf{f}\colon J \to \mathcal{F}(U)$ for an interval $J \subseteq \mathbb{R}$ denotes the associated function to the corresponding one $f\colon J \times U \to \mathbb{C}$, with $f(t,\cdot) \in \mathcal{F}(U)$ $\forall t \in J$, defined by $[\mathbf{f}(t)](x) = f(t,x)$, $\forall\ x \in U$ & $t \in J$.

Acknowledgement The authors thank Dr. Eric Stachura for useful comments.

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