

An inequality for completely monotone functions

Vasiliki Bitsouni ^{*1}, Nikolaos Gialelis ^{†1,2}, and Dan-Stefan Marinescu ^{‡3}

¹Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-15784 Athens, Greece

²School of Medicine, National and Kapodistrian University of Athens, GR-11527 Athens, Greece

³National College “Iancu de Hunedoara”, Hunedoara, Romania

Abstract

An inequality, which combines the concept of completely monotone functions with the theory of divided differences, is proposed. It is a straightforward generalization of a result, recently introduced by two of the present authors.

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1 Introduction

In [2], the study of the generalized one-dimensional problem of Population Ecology led to the following multivariate analogue of a basic inequality,

$$\prod_{i=1}^n (1+x_i)^{a_i} \leq e^{\frac{1}{n} \prod_{i=1}^n x_i}, \text{ where}$$

x_1, \dots, x_n are pairwise distinct non-negative real numbers and

$$a_i := \frac{\prod_{\substack{j=1 \\ j \neq i}}^n x_j}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)}, \quad (1)$$

with the equality holding only when one x_i equals zero.

Here we show that (1) is a special case of an inequality which involves a certain class of smooth non-negative functions, i.e. the completely monotone ones (see Definition 1). In particular, we show the following result.

Proposition 1. *Let $f: (0, \infty) \rightarrow [0, \infty)$ be completely increasing (decreasing).*

1. Then

$$f^{(n-1)}(0) := \lim_{x \rightarrow 0^+} f^{(n-1)}(x) \in \overline{\mathbb{R}} \ni \lim_{x \rightarrow \infty} f^{(n-1)}(x) =: f^{(n-1)}(\infty), \quad \forall n \in \mathbb{N}. \quad (2)$$

2. Let, also, $x_1, \dots, x_n \in (0, \infty)$ be pairwise distinct, with

$$m := \min_{i \in \{1, \dots, n\}} \{x_i\} \text{ and } M := \max_{i \in \{1, \dots, n\}} \{x_i\}.$$

*vbitsouni@math.uoa.gr

†ngialelis@math.uoa.gr

‡marinescuds@gmail.com

Then

$$\frac{(-1)^{n-1}}{(n-1)!} f^{(n-1)}(a) \leq \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} \leq \frac{(-1)^{n-1}}{(n-1)!} f^{(n-1)}(b), \quad \forall (a, b) \in [0, m] \times [M, \infty]$$

$$\left(\frac{(-1)^{n-1}}{(n-1)!} f^{(n-1)}(b) \leq \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} \leq \frac{(-1)^{n-1}}{(n-1)!} f^{(n-1)}(a) \right). \quad (3)$$

If, in addition, f is strictly completely increasing (decreasing), then the inequalities in (3) are strict.

The present short note is organized as follows: In [Section 2](#) we introduce some basic notions and we state a preliminary result, which both are necessary for the statement and the proof of [Proposition 1](#) in [Section 3](#). In [Section 4](#) we give examples of the proposed inequality, one of which is (1).

2 Preliminaries

First we give the definition of completely monotone functions.

Definition 1. 1. A function $f \in C^\infty((0, \infty); [0, \infty))$ is completely increasing (decreasing) iff

$$(-1)^n f^{(n)}(x) \leq 0 \quad ((-1)^n f^{(n)}(x) \geq 0), \quad \forall (x, n) \in (0, \infty) \times \mathbb{N}.$$

Moreover, f is strictly completely increasing (decreasing) iff the above inequality is strict.

2. A function $f \in C^\infty((0, \infty); [0, \infty))$ is (strictly) completely monotone iff it is either (strictly) completely increasing or (strictly) completely decreasing.

We note that the notion of complete monotonicity as suggested in [Definition 1](#) does not appear in bibliography, where completely increasing functions are called Bernstein functions (see, e.g., [\[6\]](#)), and completely decreasing functions are called completely (totally/absolutely) monotone (monotonic) functions (see, e.g., [\[3\]](#) and [\[6\]](#)). To the authors' knowledge, the notion of strict complete monotonicity is also new.

Additionally, we state a generalization of a well known result to higher derivatives, the mean value theorem for divided differences (see, e.g., [\[4\]](#), [\[5\]](#), or [\[1\]](#)).

Theorem 1. Let $x_1, \dots, x_n \in \mathbb{R}$ be pairwise distinct, with

$$m := \min_{i \in \{1, \dots, n\}} \{x_i\} \text{ and } M := \max_{i \in \{1, \dots, n\}} \{x_i\},$$

as well as $f \in C([m, M]; \mathbb{R}) \cap C^{n-1}((m, M); \mathbb{R})$. Then $\exists x_0 \in (m, M)$, such that

$$[x_1, \dots, x_n; f] := \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_i - x_j)} = \frac{f^{(n-1)}(x_0)}{(n-1)!}.$$

3 Proof of the main result

We proceed by proving [Proposition 1](#).

Proof. 1. Let $n \in \mathbb{N}$ be abstract. We define $g := (-1)^n f^{(n-1)}$ and we have

$$g'(x) = (-1)^n f^{(n)}(x) \leq 0 \quad (g'(x) \geq 0), \quad \forall x \in (0, \infty),$$

hence g is decreasing (increasing), which means that

$$\lim_{x \rightarrow 0^+} g(x) \in \overline{\mathbb{R}} \ni \lim_{x \rightarrow \infty} g(x)$$

and (2) then follows.

2. From [Theorem 1](#), $\exists x_0 \in (m, M)$ such that

$$-\sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} = (-1)^n [x_1, \dots, x_n; f] = \frac{(-1)^n f^{(n-1)}(x_0)}{(n-1)!}.$$

In the light of point 1., we then get

$$\frac{(-1)^n}{(n-1)!} f^{(n-1)}(b) \leq -\sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} \leq \frac{(-1)^n}{(n-1)!} f^{(n-1)}(a), \quad \forall (a, b) \in [0, m] \times [M, \infty]$$

$$\left(\frac{(-1)^n}{(n-1)!} f^{(n-1)}(a) \leq -\sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} \leq \frac{(-1)^n}{(n-1)!} f^{(n-1)}(b) \right)$$

and (3) then follows. □

4 Examples and corollaries

The first example is that of the positive constant functions, which of course are both completely increasing and decreasing. Therefore, considering $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := 1, \quad \forall x \in (0, \infty),$$

we have

$$\lim_{x \rightarrow 0^+} f^{(n)}(x) = 0 = \lim_{x \rightarrow \infty} f^{(n)}(x), \quad \forall n \in \mathbb{N}$$

and we deduce from [Proposition 1](#) that

$$\sum_{i=1}^n \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} = 0, \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

We now pass to more complicated examples. For this purpose, we have to refer to a catalogue of completely monotone functions. Extended lists of completely increasing and decreasing functions are contained in [\[6\]](#) and [\[3\]](#), respectively. For example, the function $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := \frac{\ln(1+x)}{x}, \quad \forall x \in (0, \infty),$$

is completely decreasing, and in particular, strictly completely decreasing. Besides, we have

$$\lim_{x \rightarrow 0^+} f^{(n-1)}(x) = \frac{(-1)^n (n-1)!}{n} \quad \text{and} \quad \lim_{x \rightarrow \infty} f^{(n-1)}(x) = 0, \quad \forall n \in \mathbb{N}. \quad (4)$$

Therefore, from [Proposition 1](#) we deduce

$$0 < \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} < \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

and (1) then follows.

To sum up, the main result proposes an elegant, systematic and unified approach for the proof of (1) and relevant inequalities. The price we pay is the evaluation of quantities such as (4).

Below follow more examples. We note that the choice of these examples is based on

- the finite behavior at 0 and ∞ of the corresponding completely monotone functions and their derivatives, in order to avoid a trivial right-hand side inequality of the following type $[0, \infty) \ni A \leq B \leq \infty$, as well as
- the simplicity of the evaluation of the right/left-hand side of (3), in order to keep the presentation as compact as possible.

1. Strictly completely increasing functions:

- i. Let $\alpha \in (0, \infty)$ and $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := \frac{x}{\alpha + x}, \quad \forall x \in (0, \infty).$$

Then

$$\lim_{x \rightarrow 0^+} f^{(n)}(x) = \frac{(-1)^{n+1} n!}{\alpha^n} \text{ and } \lim_{x \rightarrow \infty} f^{(n)}(x) = 0, \quad \forall n \in \mathbb{N},$$

hence

$$0 < - \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} < \frac{1}{\alpha^{n-1}}, \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

- ii. Let $\beta \in (0, \infty)$, $\alpha \in (0, \beta)$ and $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := \ln \left(\frac{\beta(x + \alpha)}{\alpha(x + \beta)} \right), \quad \forall x \in (0, \infty).$$

Then

$$\lim_{x \rightarrow 0^+} f^{(n)}(x) = \frac{(-1)^{n+1} n!}{n} \left(\frac{1}{\alpha^n} - \frac{1}{\beta^n} \right) \text{ and } \lim_{x \rightarrow \infty} f^{(n-1)}(x) = 0, \quad \forall n \in \mathbb{N},$$

hence

$$0 < - \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} < \frac{1}{n-1} \left(\frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} \right), \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

2. Strictly completely decreasing functions:

- i. Let $\alpha \in (0, \infty)$ and $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := e^{-\alpha x}, \quad \forall x \in (0, \infty).$$

Then

$$\lim_{x \rightarrow 0^+} f^{(n-1)}(x) = (-1)^{n-1} \alpha^{n-1} \text{ and } \lim_{x \rightarrow \infty} f^{(n-1)}(x) = 0, \quad \forall n \in \mathbb{N},$$

hence

$$0 < \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} < \frac{\alpha^{n-1}}{(n-1)!}, \quad \forall n \in \mathbb{N}.$$

- ii. Let $\alpha, \beta, \gamma \in (0, \infty)$ and $f \in C^\infty((0, \infty); (0, \infty))$ where

$$f(x) := (\alpha + \beta x)^{-\gamma}, \quad \forall x \in (0, \infty).$$

Then

$$\lim_{x \rightarrow 0^+} f^{(n-1)}(x) = \alpha^{\gamma+n-1} \beta^{n-1} \binom{-\gamma}{n-1} (n-1)! \text{ and } \lim_{x \rightarrow \infty} f^{(n-1)}(x) = 0, \quad \forall n \in \mathbb{N},$$

hence

$$0 < \sum_{i=1}^n \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} < \alpha^{\gamma+n-1} \beta^{n-1} \binom{\gamma+n-2}{n-1}, \quad \forall n \in \mathbb{N}.$$

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