Solvability of some integro-differential equations with anomalous diffusion and transport

Vitali Vougalter, University of Toronto

Vitaly Volpert, University of Lyon 1

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Existence of stationary solutions of nonlocal reaction- diffusion equations: existence of biological species.

1. Introduction

The integro-differential equations: nonlocal consumption of resources, intra-specific competition. Here $0 < s < \frac{1}{4},\ b \in \mathbb{R},\ b \neq 0.$

$$\frac{\partial u}{\partial t} = -D\left(-\frac{\partial^2}{\partial x^2}\right)^s u + b\frac{\partial u}{\partial x} + \int_{-\infty}^{\infty} K(x-y)g(u(y,t))dy + f(x) \quad (1)$$

from cell population dynamics. Cell genotype is x, cell density as a function of the genotype and time is $u(x,t)$. The evolution of cell density is due to cell proliferation, mutations, transport and cell influx/efflux. The change of genotype due to small random mutations-anomalous diffusion term. Large mutations is the integral term. $g(u)$ is the rate of cell birth, depends on u (density dependent proliferation). $K(x-y)$ is the proportion of newly born cells changing their genotype from y to x, depends on the distance between the genotypes. $f(x)$ is the influx/efflux of cells for different genotypes. We proved the existence of a stationary solution in $H^1(\mathbb{R})$. The space variable corresponds to the cell genotype. Without the transport term: V. Vougalter, V. Volpert, Springer (2018)

To the 70th Anniversary of Valentin Afraimovich

Anomalous diffusion problem with $\left(-\frac{\partial^2}{\partial x^2}\right)^s$: defined via the spectral calculus, namely

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(p) e^{ipx} dp, \quad \left(-\frac{\partial^2}{\partial x^2}\right)^s f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |p|^{2s} \widehat{f}(p) e^{ipx} dp.$$

Anomalous diffusion: plasma physics and turbulence.

B.Carreras, V.Lynch, G.Zaslavsky, Phys. Plasmas (2001).

Surface diffusion.

J.Sancho, A. Lacasta, K.Lindenberg, I.Sokolov, A.Romero, Phys. Rev. Lett. (2004).

Semiconductors.

H.Scher, E.Montroll, Phys. Rev. B (1975).

Physical meaning: the random process occurs with longer jumps in comparison with normal diffusion.

Normal diffusion: finite moments of jump length distribution.

Anomalous diffusion: not the case.

R. Metzler, J. Klafter, Phys. Rep. (2000).

The existence of stationary solutions

$$-\left(-\frac{d^2}{dx^2}\right)^s u + b\frac{du}{dx} + \int_{-\infty}^{\infty} K(x-y)g(u(y))dy + f(x) = 0 \qquad (2)$$

with $0 < s < \dfrac{1}{4}$ as before. Set $D = 1,\ K(x) = \varepsilon\mathcal{K}(x),\ \varepsilon \geq 0$ small parameter.

The advantages of introducing the transport term

Influx/efflux $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Do not need to assume now $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} f(x) \in L^2(\mathbb{R})$.

Kernel $\mathcal{K}(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Do not impose the regularity condition now $\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}(x) \in L^2(\mathbb{R})$.

Fractional Sobolev norm

$$\|u\|^2_{H^{2s}(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \left(-\frac{d^2}{dx^2}\right)^s u \right\|^2_{L^2(\mathbb{R})}, \quad 0 < s \le 1.$$

Particular case of $s = \frac{1}{2}$:

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \frac{du}{dx} \right\|^2_{L^2(\mathbb{R})}.$$

Standard Sobolev inequality in one dimension

$$\|u\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \|u\|_{H^1(\mathbb{R})}.$$

E. Lieb, M. Loss, Analysis, (1997).

When the parameter ε vanishes, we obtain the generalized Poisson equation with transport

$$\left(-\frac{d^2}{dx^2}\right)^s u - b\frac{du}{dx} = f(x), \quad f(x) \in L^2(\mathbb{R}). \tag{3}$$

The nonselfadjoint operator in the left side of (3)

$$L_{b,s} = \left(-\frac{d^2}{dx^2}\right)^s - b\frac{d}{dx}: \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad 0 < s \le \frac{1}{2},$$

$$L_{b,s} = \left(-\frac{d^2}{dx^2} \right)^s - b\frac{d}{dx} : \quad H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}), \quad \frac{1}{2} < s < 1.$$

Its essential spectrum: $\lambda_{b,s}(p) = |p|^{2s} - ibp, \quad p \in \mathbb{R}$ contains the origin. Non Fredholm operator.

The standard Fourier transform

$$\widehat{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx. \tag{4}$$

Upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})}. \tag{5}$$

2. Solvability conditions for the linear equation (3).

Sufficient to solve (3) in $L^2(\mathbb{R})$. So, $u(x) \in L^2(\mathbb{R})$ and $f(x) \in L^2(\mathbb{R})$ as assumed.

Then

$$\left(-\frac{d^2}{dx^2}\right)^s u - b\frac{du}{dx} \in L^2(\mathbb{R}).$$

By means of the Fourier transform (4)

$$(|p|^{2s} - ibp)\widehat{u}(p) \in L^2(\mathbb{R}).$$

Hence

$$\int_{-\infty}^{\infty} (|p|^{4s} + b^2p^2)|\widehat{u}(p)|^2 dp < \infty.$$

Let $0 < s \leq \frac{1}{2}$. Clearly

$$\int_{-\infty}^{\infty} p^2|\widehat{u}(p)|^2 dp < \infty.$$

Thus $\frac{du}{dx} \in L^2(\mathbb{R})$ and $u(x) \in H^1(\mathbb{R})$. Similar argument for $\frac{1}{2} < s < 1$.

Uniqueness of solutions of linear problem (3)

Let $0 < s \leq \frac{1}{2}$. Similar reasoning for $\frac{1}{2} < s < 1$.

Suppose $u_{1,2}(x) \in H^1(\mathbb{R})$ both solve (3). The difference $w(x) = u_1(x) - u_2(x) \in H^1(\mathbb{R})$ satisfies

$$\left(-\frac{d^2}{dx^2}\right)^s w - b\frac{dw}{dx} = 0.$$

Operator $L_{b,s}$: no nontrivial zero modes in $H^1(\mathbb{R})$. Hence $w(x) \equiv 0$ on the real line.

Apply Fourier transform (4) to linear equation (3).

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - ibp}, \quad p \in \mathbb{R}, \quad 0 < s < 1.$$

Evidently

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|>1\}}.$$

Second term

$$\left|\frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|>1\}}\right| \le \frac{|\widehat{f}(p)|}{\sqrt{1+b^2}} \in L^2(\mathbb{R}).$$

First term

$$\left\|\frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\le 1\}}\right\|^2_{L^2(\mathbb{R})} \le \frac{\|f(x)\|^2_{L^1(\mathbb{R})}}{\pi(1-4s)} < \infty, \quad 0 < s < \frac{1}{4}.$$

Argument above: unique solution $u(x) \in H^1(\mathbb{R})$ of (3). For $0 < s < \dfrac{1}{4}$ the orthogonality conditions are not needed in the work.

Case $\dfrac{1}{4} \le s < 1$ is more singular, need $\widehat{f}(0) = 0$ for the solvability.

Orthogonality condition: $(f(x), 1)_{L^2(\mathbb{R})} = 0$.

Unique, nontrivial solution of linear equation (3)

$$u_0(x) \in H^1(\mathbb{R}), \quad 0 < s < \frac{1}{4}.$$

No transport term:

$$u_0(x) \in H^{2s}(\mathbb{R}), \quad 0 < s < \frac{1}{4},$$

so that $u_0(x) \in H^1(\mathbb{R})$ under extra regularity assumption on influx/efflux $f(x)$.

3. Fixed point argument

Seek the resulting solution of the stationary nonlinear problem (2) as

$$u(x) = u_0(x) + u_p(x). \tag{6}$$

Perturbative equation with $0 < s < \frac{1}{4}$:

$$\left(-\frac{d^2}{dx^2}\right)^s u_p - b\frac{du_p}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y)+u_p(y))dy. \qquad (7)$$

The Fixed Point argument in a closed ball in the Sobolev space:

$$B_\rho = \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \le \rho\}, \quad 0 < \rho \le 1. \qquad (8)$$

Seek the solution of (7) as the fixed point of the auxiliary nonlinear problem with $0 < s < \dfrac{1}{4}$

$$\left(-\frac{d^2}{dx^2}\right)^s u - b\frac{du}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y)+v(y))dy, \qquad (9)$$

in ball (8). Non Fredholm operator in the left side of (9)

$$L_{b,s} : H^1(\mathbb{R}) \to L^2(\mathbb{R}).$$

No bounded inverse.

V.V., V.Volpert, Doc. Math. (2011),

V.V., V.Volpert, Anal. Math. Phys. (2012)

relied on the orthogonality relations.

V.V., Math. Model. Nat. Phenom., (2010).

The fixed point technique to estimate the perturbation to the standing solitary wave

$$\psi(x,t) = \phi(x)e^{i\omega t}$$

of the Nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi + F(|\psi|^2)\psi$$

when small perturbation is applied either to the potential or to the nonlinear term. The Schrödinger operator involved had the Fredholm property.

When $K(x) = \delta(x)$ is Dirac's delta measure, standard nonlinear heat equation.

The existence of stationary solutions in the case of the standard Laplacian, no transport in $H^3(\mathbb{R}^5)$.

V.V., V. Volpert, Dyn. Partial Differ. Equ. (2015).

The operator T_g via the auxiliary nonlinear problem (9), such that $u = T_g v, \quad u$ is a solution. Our main result is as follows.

Theorem 1. Under our technical assumptions problem (9) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon \leq \varepsilon^*$ for a certain $\varepsilon^* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of problem (7) in B_ρ.

The resulting stationary solution of (2) given by (6) is nontrivial: the source term $f(x)$ is nontrivial and $g(0) = 0$ as assumed.

Proof. Choose arbitrarily $v(x) \in B_\rho$, denote $G(x) := g(u_0(x) + v(x))$.

Apply the standard Fourier transform (4) to (9). Thus

$$\widehat{u}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s} - ibp}.$$

The norm

$$\|u\|^2_{L^2(\mathbb{R})} = 2\pi\varepsilon^2 \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{|p|^{4s} + b^2p^2} dp \leq 2\pi\varepsilon^2 \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{|p|^{4s}} dp$$

as before. Express $\int_{-\infty}^{\infty} dp = \int_{|p|\leq R} dp + \int_{|p|>R} dp$ with $R \in (0, +\infty)$, estimate and minimize over R. We derive

$$\|u\|_{H^1(\mathbb{R})} \leq \varepsilon C \leq \rho \tag{10}$$

for all $\varepsilon > 0$ small enough, such that $u(x) \in B_\rho$ as well.

Uniqueness.

Suppose for some $v(x) \in B_\rho$ there are two solutions $u_{1,2}(x) \in B_\rho$ of (9).

Their difference $w(x) := u_1(x) - u_2(x) \in H^1(\mathbb{R})$ solves

$$\left(-\frac{d^2}{dx^2}\right)^s w - b\frac{dw}{dx} = 0, \quad 0 < s < \frac{1}{4}.$$

$L_{b,s} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ no nontrivial zero modes, $w(x) \equiv 0$ on $\mathbb{R}$.

Then (9) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

To show that this map is a strict contraction.

Choose arbitrarily $v_{1,2}(x) \in B_\rho$. Then $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well when $\varepsilon > 0$ is sufficiently small. For $0 < s < \frac{1}{4}$

$$\left(-\frac{d^2}{dx^2}\right)^s u_1 - b\frac{du_1}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy,$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_2 - b\frac{du_2}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy.$$

Introduce $G_1(x) := g(u_0(x) + v_1(x))$, $G_2(x) := g(u_0(x) + v_2(x))$. Apply the standard Fourier transform (4) to equations above. Arrive at

$$\widehat{u}_1(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s} - ibp}, \quad \widehat{u}_2(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s} - ibp}.$$

Write the norm

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R})} = \varepsilon^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s} + b^2 p^2} dp \leq$$

$$\leq \varepsilon^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp$$

as before. Express

$$\int_{-\infty}^{\infty} dp = \int_{|p| \leq R} dp + \int_{|p| > R} dp,$$

estimate, minimize over $R \in (0, +\infty)$. Upper bound on the norm

$$\|u_1 - u_2\|_{H^1(\mathbb{R})} \leq \varepsilon C \|v_1 - v_2\|_{H^1(\mathbb{R})}.$$

The map $T_g : B_\rho \to B_\rho$ defined by (9) is a strict contraction for all $\varepsilon > 0$ small enough. Unique fixed point $u_p(x)$, the only solution of the perturbative equation (7) in B_ρ. By means of (10)

$$\|u_p(x)\|_{H^1(\mathbb{R})} \leq \varepsilon C \to 0, \quad \varepsilon \to 0.$$

The resulting solution of the stationary problem (2):

$$u(x) = u_0(x) + u_p(x) \in H^1(\mathbb{R}),$$

where $u_0(x)$ solves the linear equation (3).

Also proved: $u(x)$ is continuous in the $H^1(\mathbb{R})$ norm with respect to the nonlinear, twice continuously differentiable rate of cell birth function $g(z)$.

4. Discussion of the possible future work.

1. To study the convergence of the solutions $u(x,t)$ of problem (1) to the equilibrium.

2. To generalize the results above to the case when the normal diffusion is combined with the anomalous diffusion in a single integro-differential equation or a system of coupled integro-differential equations. M.Efendiev, V.V., J. Differential Equations (2021).

3. To perform the iterations of the kernels of an integro-differential equation and to show the existence of its stationary solution in the sense of sequences.

4. To work on the preservation of the nonnegativity of solutions of the systems of parabolic equations. M.Efendiev, V.V., Springer chapters (2021).