Nonlocality of the area functional for two-codimensional graphs of nonsmooth maps in $\mathbb{R}^4$

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Definition of the area functional

Let $\Omega \subset\subset \mathbb{R}^2$ and $N = 1, 2$. The classical area functional $A(u, \Omega)$ measures the 2-dimensional area (in $\mathbb{R}^{2+N}$) of the graph

$$\mathrm{graph}(u) := \{(x, y) \in \Omega \times \mathbb{R}^N : y = u(x)\},$$

of a smooth (say C^1) function $u : \Omega \to \mathbb{R}^N$. By the area formula

$$\text{if } N = 1 \qquad A(u, \Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx;$$

$$\text{if } N = 2 \qquad A(u, \Omega) = \int_\Omega \sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2 + |Ju|^2}\, dx,$$

where $u = (u_1, u_2)$, $Ju = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}$.

[M. Giaquinta, G. Modica and J. Souček, Cartesian Currents in the Calculus of Variations, 1998]

The problem: the area when $N = 2$ (vector case)

Extend and compute the area of the graph of u when $N = 2$ and u is not smooth, for instance when

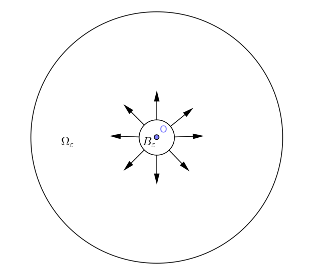
$$u \quad \text{has} \begin{cases} \text{point singularities: vortex map } u = \mathbf{u}_{\text{vor}} \\ \text{jump curves: triple junction map } u = \mathbf{u}_{\text{triple}}. \end{cases}$$

While when $N = 1$ (scalar case) everything is known, almost nothing is known in the vector case $N = 2$: in this case the graph is a surface of codimension two in $\mathbb{R}^4$. Here we shall concentrate on the vortex map $\mathbf{u}_{\text{vor}}$.

Codimension two: the vortex map

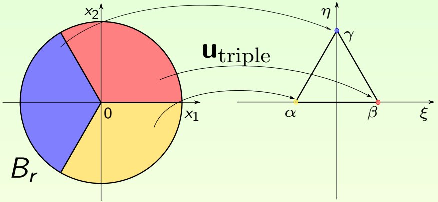
A prototypical example of "simple" nonsmooth map is the vortex map $\mathbf{u}_{\mathrm{vor}} : \Omega = \mathrm{B}_\ell \setminus \{0\} \subset \mathbb{R}^2_{\mathrm{source}} \to \mathbb{S}^1 \subset \mathbb{R}^2_{\mathrm{target}}$,

$$\mathbf{u}_{\mathrm{vor}}(x) := \frac{x}{|x|}$$



Codimension two: the triple junction map

Another example of “simple” map is the triple junction map
$\mathbf{u}_{\mathrm{triple}} : \Omega = \mathrm{B}_r \subset \mathbb{R}^2_{\mathrm{source}} \to \{\alpha, \beta, \gamma\} \in \mathbb{R}^2_{\mathrm{target}}$



What could be the area of the graph of $\mathbf{u}_{\mathrm{vor}}$ and $\mathbf{u}_{\mathrm{triple}}$?

Generalized area of the graph of an arbitrary function

For $u \in L^1(\Omega, \mathbb{R}^N)$ the generalized area of the graph of u, $N = 1, 2$, is defined as

$$\overline{A}(u, \Omega) := \min \left\{ \lim_{k \to +\infty} A(u_k, \Omega), u_k \text{ smooth}, \ u_k \xrightarrow{L^1} u \right\}$$

This coincides with the lower-semicontinuous envelope of A in L^1 and it extends A. The choice of convergence $u_k \xrightarrow{L^1} u$ is crucial here. We do not discuss the issue of other convergences. Just mention that the L^1-convergence is source of interesting non-local phenomena.

Geometric idea: fill in the most "economic" way the "holes" of a **nonsmooth** "Cartesian" 2D-surface sitting in $\mathbb{R}^{2+N}$ (hence, when $N = 2$, with codimension two) using L^1-limits of smooth Cartesian $2D$-surfaces.

Motivations in the scalar case $N = 1$

The generalized area has been useful to solve the Plateau problem in Cartesian form: given $\varphi : \partial\Omega \to \mathbb{R}$, solve

$$\min \left\{ A(u, \Omega) : u = \varphi \text{ on } \partial\Omega \right\}$$

A weak environment is needed for existence, and also to impose the boundary condition $u = \varphi$ on $\partial\Omega$. Next, show that minimizers are smooth inside Ω, and attain the boundary condition.

- Applications: minimal surfaces (see e.g. [Giusti, Minimal Surfaces and Functions of Bounded Variation, 1984]); capillarity problems

Motivations in the vector case $N = 2$

- The Plateau problem in dimension two and codimension two in Cartesian form;
- the relaxation of polyconvex functionals with nonstandard growth.

[J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, 1977]

[N. Fusco and J.E. Hutchinson, *A direct proof for lower semicontinuity of polyconvex functionals*, 1995]

[E. Acerbi and G. Dal Maso, *New lower semicontinuity results for polyconvex integrals*, 1994]

Remark: in the computation of the generalized area when $N = 2$, scalar area-minimizing problems arise: entangled Plateau problems and problems with partial free boundaries, in codimension one.

Recall:

$$\overline{A}(u,\Omega) := \min\left\{ \lim_{k\to+\infty} A\Big(u_k,\Omega\Big),\ u_k \text{ smooth},\ u_k \xrightarrow{L^1} u \right\}.$$

Any sequence (u_k) converging to u such that

$$\overline{A}(u,\Omega) = \lim_{k\to+\infty} A(u_k,\Omega)$$

is called a **recovery sequence**.

The analysis of $\overline{A}(u,\Omega)$ consists of two parts, once we have a guess for what could be the value of the generalized area:

• Lower bound: one has to show that if we pick an **arbitrary** sequence (u_k) of smooth maps converging to u in L^1 then

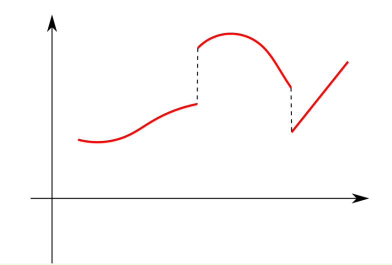
$$\lim_{k \to +\infty} A(u_k, \Omega) \geq \text{our guess}$$

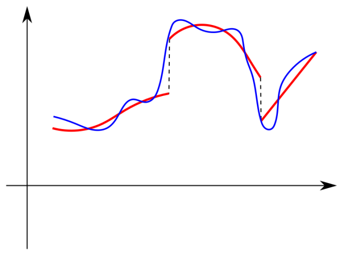
• The upper bound (constructive): we need to exhibit a **particular** sequence (u_k) of smooth maps converging to u in L^1 such that

$$\lim_{k \to +\infty} A(u_k, \Omega) \leq \text{our guess}$$

Meaning of the generalized area

We start with a graph of a real function u on an interval I, possibly with jumps, and we approximate it by smooth graphs.

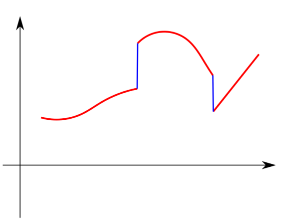




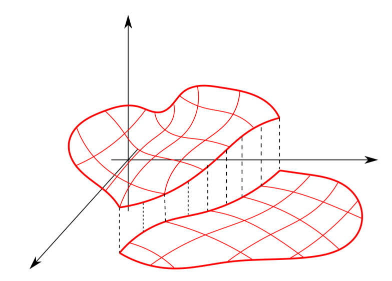
When the smooth maps tend to u, they fill the holes, providing a “closed graph”, called generalized graph.

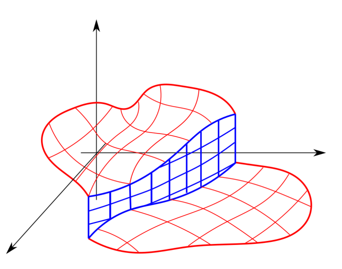
This can be associated to a unique Cartesian current defined on $I \times \mathbb{R}$ whose vertical parts are walls over the discontinuities of u.

Meaning of the generalized area



In two dimensions and $N = 1$, the situation is similar:





Main questions

- What is the domain $\mathrm{Dom}(\overline{A})$ of $\overline{A}$?
- What is the expression of $\overline{A}(u, \Omega)$ for $u \in \mathrm{Dom}(\overline{A})$?

When $N = 2$ the answers are not known: as a starting point, the idea is to try to estimate from above and below the generalized area of the graph of our favourite "simple" nonsmooth maps: here, in particular, of $\mathbf{u}_{\mathrm{vor}}$.

Theorem (scalar case)

Let $N = 1$. *Then* $\mathrm{Dom}(\overline{A}) = BV(\Omega)$. *If* $u \in BV(\Omega)$ *then* $\overline{A}(u, \Omega)$ *has an integral representation:*

$$\overline{A}(u, \Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx + |D^s u|(\Omega)$$

$$\text{singular part} = |D^s u|(\Omega) = \int_{J_u \cap \Omega} |u^+ - u^-| \, ds + \text{Cantor part}$$

J_u *is the jump set (a curve in* Ω*) of* u*, and* $u^\pm$ *are the two traces of* u *on* J_u *from the two sides.*

Geometric idea: the most "economic" way to fill the "holes" in a **nonsmooth** 2D-Cartesian surface sitting in $\mathbb{R}^{2+1}$ is just to take the area of the "vertical walls" over the jump set J_u of u. Our previous pictures were morally correct.

Vector case $N = 2$: known facts on $\overline{A}(\cdot, \Omega)$

- $\mathrm{dom}(\overline{A}) \subsetneq BV(\Omega, \mathbb{R}^2)$;
- $\overline{A}(\cdot, \Omega) = A(\cdot, \Omega)$ on smooth maps;
- on "trivial" (*i.e.*, without triple or multiple junctions) piecewise constant maps $v \in BV(\Omega; \mathbb{R}^2)$

$$\overline{A}(v, \Omega) = |\Omega| + \int_{J_{v \cap \Omega}} |v^+ - v^-| \, ds;$$

- no integral representation of $\overline{A}(v, \Omega)$ is possible, since $\overline{A}(v, \cdot)$ is not subadditive. Proven for $v = \mathbf{u}_{\mathrm{vor}}, \mathbf{u}_{\mathrm{triple}}$

[E. De Giorgi, *On the relaxation of functionals defined on cartesian manifolds*, 1992],

[Acerbi and Dal Maso]

Questions in codimension two: the singular part

When $N = 2$ what is the singular part of $\overline{A}(u, \Omega)$? We have a one more degree of freedom to construct "vertical walls". Take $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ with a straight jump set J_u. The "vertical wall" we want to build now is a 2-dimensional minimal surface in $J_u \times \mathbb{R}^2 \cong \mathbb{R}^3$, with boundary the images of the curves $(t, u^{\pm}(t))$, $t \in J_u$. Such a minimal surface "fills the hole" in graph of u.

Problems:

• which are the correct boundary conditions?

• Singularities can interact with each other, and with $\partial\Omega$, trying to reduce the value of the generalized area: nonlocal phenomena.

• Once we have such a minimal surface, how to construct, with it, a sequence (u_k) of smooth maps converging to u and giving, in the limit, $\overline{A}(u, \Omega)$?

In the case of the vortex map $\mathbf{u}_{\rm vor}(x) = \frac{x}{|x|}$, $x \in \mathrm{B}_\ell \setminus \{0\}$, the "vertical wall" should live over the origin 0, hence in $\{0\} \times \mathbb{R}^2$, and has as boundary a circle. The minimal surface is a 2-dimensional disk. However, π is not always the value of the singular part of the area: true only if ℓ is large enough.

Theorem (E. Acerbi and G. Dal Maso)

If ℓ is large enough then

$$\text{singular part of } \overline{A}(\mathbf{u}_{\mathrm{vor}}, \mathrm{B}_\ell) = \pi.$$

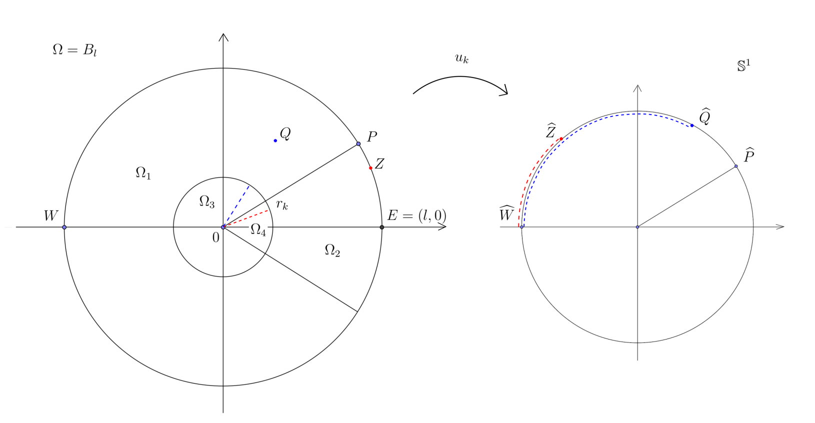
Theorem (E. Acerbi and G. Dal Maso)

For any $\ell > 0$, the singular part of $\overline{A}(\mathbf{u}_{\mathrm{vor}}, \mathrm{B}_\ell)$ is $\leq 2\pi\ell$

Indeed there exists a sequence (u_k) of smooth maps converging to $\mathbf{u}_{\mathrm{vor}}$ in L^1, and such that

$$\text{singular part of } \lim_{k \to +\infty} A(u_k, \mathrm{B}_\ell) = 2\pi\ell$$

Remark: $2\pi\ell$ is the lateral area of a cylinder with height ℓ and basis the target unit disk: a sort of phantom bridge whose projection on $\mathbb{R}^2_{\mathrm{source}}$ is a radius, joining the singular point 0 with $\partial \mathrm{B}_\ell$



$\Omega = B_l$
u_k
$\mathbb{S}^1$
Q
P
Z
Ω_1
Ω_3
r_k
W
$E = (l, 0)$
0
Ω_4
Ω_2
$\widehat{Z}$
$\widehat{Q}$
$\widehat{P}$
$\widehat{W}$

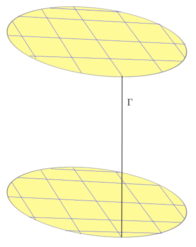
Guess of the singular part

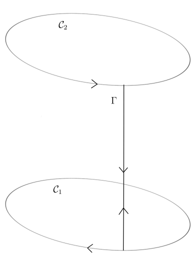
Such a sequence is not an optimal one: indeed, it is possible to **decrease** the value of the singular part, approximating $\mathbf{u}_{\rm vor}$ with a "better" approximating sequence.

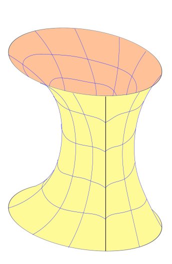
Let us try to imagine how: the singular part pops up as a consequence of the behaviour of u_k in the thin region Ω_2, around the right horizontal radius and inside the annulus $\mathrm{B}_\ell \setminus \mathrm{B}_{r_k}$. In that region, it is possible to construct an optimal sequence (u_k) patching a suitable area-minimizing surface sitting in $\mathbb{R}^3 = \mathbb{R}_{||\ell} \times \mathbb{R}^2_{\rm target}$.

Appearence of a “catenoid” containing a segment

This turns out to be related to the following Plateau problem:







This is nonstandard since the curve boundary of the spanning disk is not simple.

Main result

Theorem (B., A. Elshourbagy and R. Scala)

For $\ell > 0$ sufficiently small,

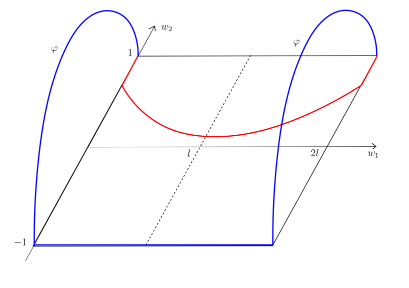
$$\text{singular part of } A(\mathbf{u}_{\mathrm{vor}}, \mathrm{B}_\ell) = \frac{1}{2}\mathrm{area}(C_\ell),$$

where C_ℓ is the "catenoid" containing a segment.

Main result

The "catenoid" with segment can be characterized as the doubling of a disk-type Cartesian minimal surface in $R = (0, \ell) \times (-1, 1)$ with suitable boundary conditions,

$$\frac{1}{2}\mathrm{area}(C_\ell) = \min \left\{ \int_R \sqrt{1 + |\nabla \psi|^2} dx : \psi ... \right\}$$

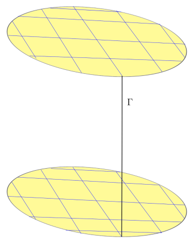


Main result

If $\ell > 0$ is large enough, it is convenient to fill the two disks with a very thin strip joining them; this provides the optimal value

$$\text{singular part of} \quad \overline{A}(\mathbf{u}_{\mathrm{vor}}, \mathrm{B}_\ell) = \pi,$$

consistently with the result of Acerbi and Dal Maso.



However, if $\ell > 0$ is small, then it is better to use the "catenoid containing a segment".

Lower bound

Fix a recovery sequence $v_k \to \mathbf{u}_{\rm vor}$ and divide the domain $\mathrm{B}_\ell = D_k \cup (\mathrm{B}_\ell \setminus D_k)$, $D_k := \{x \in \mathrm{B}_\ell : |v_k(x) - \mathbf{u}_{\rm vor}(x)| \text{ is large}\}$. We split

$$A(\mathbf{u}_{\rm vor}, \mathrm{B}_\ell) = \lim_{k\to+\infty} \big(A(v_k, \mathrm{B}_\ell \setminus D_k) + A(v_k, D_k)\big),$$

and one has to show that

$$\lim_{k\to+\infty} A(v_k, \mathrm{B}_\ell \setminus D_k) \geq \int_{\mathrm{B}_\ell} \sqrt{1 + |\nabla \mathbf{u}_{\rm vor}|^2}\, dx,$$

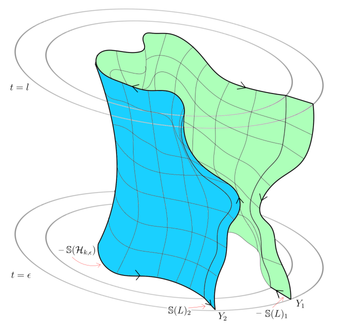
$$\lim_{k\to+\infty} A(v_k, D_k) \geq \frac{1}{2}\mathrm{area}(C_\ell).$$

We use that $A(v_k, D_k) \geq |\mathcal{D}_k| =$ mass of the integral currents in the cylinder $B_1 \times [0, \ell] \subset \mathbb{R}^3$, image of D_k through the map

$$x \to (|x|, v_k(x)).$$

Lower bound: symmetrization

We exploit a Steiner-type symmetrization technique, adapted to a cylindrical enviroment $\Rightarrow$ the currents $\mathcal{D}_k$ are cylindrically symmetric, and up to several errors to be estimated,



the areas of these surfaces tend to the area of a surface which is a competitor for the Plateau-type minimum problem whose solution is (half) the catenoid containing the segment.

Upper bound

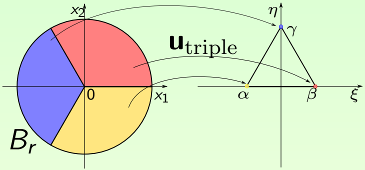
We have to construct a recovery sequence $u_k \to \mathbf{u}_{\mathrm{vor}}$. This requires:

- existence and regularity of a solution of the Plateau problem;
- existence of a Cartesian smooth approximation of this solution;
- definition of u_k so that in D_k we parametrize the minimal surface and in $\mathrm{B}_\ell \setminus D_k$ we glue it with $\mathbf{u}_{\mathrm{vor}}$.

[B., A. Elshourbagy and R. Scala, *The L^1-relaxed area of the graph of the vortex map*, 2021], $\sim$ 110 pages

Final remarks

Also the value of $\overline{A}(\mathbf{u}_{\mathrm{triple}}, \mathrm{B}_r)$ has been computed:



[G. B. and M. Paolini, *On the area of the graph of a singluar map from the plane to the plane taking three values*, 2010]

[R. Scala, *Optimal estimates for the triple junction function and other surprising aspects of the area functional*, 2020]

Bounds for nonsymmetric triple junction maps:

[G. B., A. Elshorbagy, M. Paolini and R. Scala, 2019]

Final remarks

Partial results for piecewise smooth maps with a crack:

[G. B., M. Paolini and L. Tealdi, *On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity*, 2015]

G. B., M. Paolini and L. Tealdi, *Semicartesian surfaces and the relaxed area of maps from the plane to the plane with a line discontinuity*, 2016]

Some perspectives

- Extend the study to the case of multivortices: [S. Carano (SISSA)]
- Extend the study to piecewise constant maps
- Parametric Plateau problem in codimension 1 with partial free boundary [G.B., R. Scala and R. Marziani, in progress]
- Entangled Plateau problem in codimension 1: soap films clusters [G.B. and R. Scala, in progress]