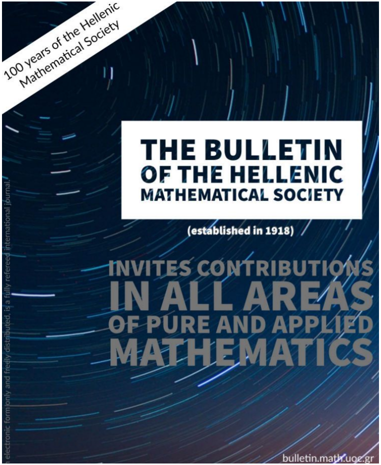
Best Sobolev constants in the presence of sharp Hardy terms

Gerassimos Barbatis
National and Kapodistrian University of Athens

joint work with A. Tertikas
University of Crete

PDE Seminar

"Athens", March 5th, 2021


100 years of the Hellenic
Mathematical Society
THE BULLETIN
OF THE HELLENIC
MATHEMATICAL SOCIETY
(established in 1918)
INVITES CONTRIBUTIONS
IN ALL AREAS
OF PURE AND APPLIED
MATHEMATICS
electronic form only and freely distributed, is a fully refereed international journal.
bulletin.math.uoc.gr

Sobolev inequality

Sobolev inequality

Assume $n \geq 3$. Then

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_n \Big(\int_{\mathbb{R}^n} |u|^{2^*} dx \Big)^{2/2^*}, \qquad u \in C_c^\infty(\mathbb{R}^n),$$

where

$$2^* = \frac{2n}{n-2} \qquad \text{(Sobolev exponent)}$$

Talenti (1976): Best constant

$$S_n = \pi n(n-2) \Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \Big)^{\frac{2}{n}}$$

Extremal

$$u(x) = \big(1 + |x|^2\big)^{-\frac{n-2}{2}}$$

Sobolev inequality

Assume $n \geq 3$. Then

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_n \Big(\int_{\mathbb{R}^n} |u|^{2^*} dx \Big)^{2/2^*}, \qquad u \in C_c^\infty(\mathbb{R}^n),$$

where

$$2^* = \frac{2n}{n-2} \qquad \text{(Sobolev exponent)}$$

Talenti (1976): Best constant

$$S_n = \pi n(n-2)\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{2}{n}}$$

Extremal

$$u(x) = \left(1 + |x|^2\right)^{-\frac{n-2}{2}}$$

The best constant remains the same if $\mathbb{R}^n$ is replaced by a smaller domain; but no extremals in this case.

Hardy inequality

Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \ , \qquad u \in C_c^\infty(\mathbb{R}^n)$$

Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \,, \qquad u \in C_c^\infty(\mathbb{R}^n)$$

- The power $|x|^2$ is optimal
- The constant $\left(\frac{n-2}{2}\right)^2$ is sharp
- No extremals; $|x|^{-\frac{n-2}{2}}$ solves the Euler equation

To prove sharpness use

$$u_\epsilon(x) = \begin{cases} |x|^{-\frac{n-2}{2}+\epsilon}, & |x| < 1, \\ |x|^{-\frac{n-2}{2}-\epsilon}, & |x| > 1. \end{cases}$$

Interpolation between Sobolev and Hardy inequalities gives that for any $2 < p < 2^*$ there holds

$$(p\text{–HS}) \quad \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_{n,p} \Big(\int_{\mathbb{R}^n} |x|^{\frac{p(n-2)}{2} - n} |u|^p dx \Big)^{2/p}, \quad u \in C_c^\infty(\mathbb{R}^n)$$

Interpolation between Sobolev and Hardy inequalities gives that for any $2 < p < 2^*$ there holds

$$(p\text{–HS}) \quad \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_{n,p} \Big(\int_{\mathbb{R}^n} |x|^{\frac{p(n-2)}{2} - n} |u|^p dx \Big)^{2/p}, \quad u \in C_c^\infty(\mathbb{R}^n)$$

Sharp constant computed by Lieb ('83)

$$S_{n,p} = 2p \Big(\frac{n-2}{2}\Big)^{\frac{p+2}{2}} \left[\frac{2\pi^{n/2}\Gamma^2(\frac{p}{p-2})}{(p-2)\Gamma(\frac{n}{2})\Gamma(\frac{2p}{p-2})} \right]^{\frac{p-2}{p}}$$

Extremal

$$u(x) = \big(1 + |x|^{\frac{(p-2)(n-2)}{2}}\big)^{-\frac{2}{p-2}}$$

Problem: Combine Hardy and Sobolev inequalilties

Problem: Combine Hardy and Sobolev inequalilties

More specifically: can we improve a Hardy inequality by adding a Sobolev term to the RHS ?

Problem: Combine Hardy and Sobolev inequalilties

More specifically: can we improve a Hardy inequality by adding a Sobolev term to the RHS ?

$$\int_\Omega |\nabla u|^2 dx \geq c^* \int_\Omega \frac{u^2}{d^2} dx + c\Big(\int_\Omega |u|^{2^*} W(x) dx \Big)^{2/2^*}, \quad u \in C_c^\infty(\Omega)$$

with $d(x)$ some distance function, $W(x)$ some weight and c^* the sharp Hardy constant ?

Problem: Combine Hardy and Sobolev inequalilties

More specifically: can we improve a Hardy inequality by adding a Sobolev term to the RHS ?

$$\int_\Omega |\nabla u|^2 dx \geq c^* \int_\Omega \frac{u^2}{d^2} dx + c\Big(\int_\Omega |u|^{2^*} W(x) dx \Big)^{2/2^*} , \quad u \in C_c^\infty(\Omega)$$

with $d(x)$ some distance function, $W(x)$ some weight and c^* the sharp Hardy constant ?

Most important cases:

$$\text{(i)} \quad d(x) = |x| \text{ with } 0 \in \Omega$$

$$\text{(ii)} \quad d(x) = \mathrm{dist}(x, \partial\Omega)$$

$$\text{(iii)} \quad d(x) = |x| \text{ with } 0 \in \partial\Omega$$

$\longrightarrow$ a number of such results

Problem: Combine Hardy and Sobolev inequalilties

More specifically: can we improve a Hardy inequality by adding a Sobolev term to the RHS ?

$$\int_\Omega |\nabla u|^2 dx \geq c^* \int_\Omega \frac{u^2}{d^2} dx + c\Big(\int_\Omega |u|^{2^*} W(x) dx\Big)^{2/2^*}, \quad u \in C_c^\infty(\Omega)$$

with $d(x)$ some distance function, $W(x)$ some weight and c^* the sharp Hardy constant ?
Most important cases:

$$\text{(i)} \quad d(x) = |x| \text{ with } 0 \in \Omega$$

$$\text{(ii)} \quad d(x) = \mathrm{dist}(x, \partial\Omega)$$

$$\text{(iii)} \quad d(x) = |x| \text{ with } 0 \in \partial\Omega$$

$\longrightarrow$ a number of such results

We are interested in Sobolev improvements involving explicit/sharp Sobolev constant c.

Hardy inequalities involving the distance to the boundary

Hardy inequalities involving the distance to the boundary

$\Omega \subset \mathbb{R}^n$, $d(x) = \mathrm{dist}(x, \partial\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx \ , \qquad u \in C_c^\infty(\Omega)$$

Hardy inequalities involving the distance to the boundary

$\Omega \subset \mathbb{R}^n$, $d(x) = \mathrm{dist}(x, \partial\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx\ , \qquad u \in C_c^\infty(\Omega)$$

If Ω is bounded with smooth boundary then the Hardy inequality is valid and the best constant is $\leq 1/4$.

Hardy inequalities involving the distance to the boundary

$\Omega \subset \mathbb{R}^n$, $d(x) = \mathrm{dist}(x, \partial\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx \;, \qquad u \in C_c^\infty(\Omega)$$

If Ω is bounded with smooth boundary then the Hardy inequality is valid and the best constant is $\leq 1/4$.

If Ω is convex then the best constant is $1/4$.

Hardy inequalities involving the distance to the boundary

$\Omega \subset \mathbb{R}^n$, $d(x) = \mathrm{dist}(x, \partial\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx \;, \qquad u \in C_c^\infty(\Omega)$$

If Ω is bounded with smooth boundary then the Hardy inequality is valid and the best constant is $\leq 1/4$.

If Ω is convex then the best constant is $1/4$.

B., Filippas, Tertikas ('04). If

$$\Delta d \leq 0 \;, \qquad \text{in } \Omega,$$

then the best constant is $1/4$.

Hardy inequalities involving the distance to the boundary

$\Omega \subset \mathbb{R}^n$, $d(x) = \mathrm{dist}(x, \partial\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2}\, dx \ , \qquad u \in C_c^\infty(\Omega)$$

If Ω is bounded with smooth boundary then the Hardy inequality is valid and the best constant is $\leq 1/4$.

If Ω is convex then the best constant is $1/4$.

B., Filippas, Tertikas ('04). If

$$\Delta d \leq 0 \ , \qquad \text{in } \Omega,$$

then the best constant is $1/4$.
Proof. The function $d^{1/2}$ is a positive supersolution to the Euler equation.

A Hardy-Sobolev inequality. Consider $\Omega = \mathbb{R}^3_+$. Then

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq S_3 \Big(\int_{\mathbb{R}^3_+} u^6 dx \Big)^{1/3} , \qquad u \in C_c^\infty(\mathbb{R}^3_+)$$

and

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{x_3^2} dx , \qquad u \in C_c^\infty(\mathbb{R}^3_+).$$

A Hardy-Sobolev inequality. Consider $\Omega = \mathbb{R}^3_+$. Then

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq S_3 \Big(\int_{\mathbb{R}^3_+} u^6 dx \Big)^{1/3}, \qquad u \in C_c^\infty(\mathbb{R}^3_+)$$

and

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{x_3^2} dx, \qquad u \in C_c^\infty(\mathbb{R}^3_+).$$

Benguria, Frank, Loss ('08). There holds

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{x_3^2} dx + S_3 \Big(\int_{\mathbb{R}^3_+} u^6 dx \Big)^{1/3}$$

for all $u \in C_c^\infty(\mathbb{R}^3_+)$!!

Proof. Write points in $\mathbb{R}^n_+$ as $\mathbf{x} = (x, y)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}_+$. Let $H = -\Delta - \frac{1}{4y^2}$, a self-adjoint operator on $L^2(\mathbb{R}^n_+)$.

Proof. Write points in $\mathbb{R}^n_+$ as $\mathbf{x} = (x, y)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}_+$. Let $H = -\Delta - \frac{1}{4y^2}$, a self-adjoint operator on $L^2(\mathbb{R}^n_+)$.
Let $G^H_{par}(\mathbf{x}, \mathbf{x}', t)$ be the corresponding parabolic Green function:

$$\begin{cases} u_t = \Delta u + \frac{1}{4y^2} u \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \end{cases} \quad \Rightarrow \quad u(\mathbf{x}, t) = \int_{\mathbb{R}^n_+} G^H_{par}(\mathbf{x}, \mathbf{x}', t) u_0(x') d\mathbf{x}'$$

Change variables: $u = \sqrt{y}v$. Then the problem is transformed to

$$\begin{cases} v_t = \Delta_x v + v_{yy} + \frac{1}{y} v_y \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) \end{cases}$$

This is the heat equation on

$$\mathbb{R}^{n+1} = \{(x, z) : x \in \mathbb{R}^{n-1}, \ z \in \mathbb{R}^2\}$$

acting on functions that are radial with respect to $z \in \mathbb{R}^2$ (so $y = |z|$).

Write $v(x,y,t) = \hat{v}(x,z,t)$, $\hat{v}$ radial w.r.t. $z \in \mathbb{R}^2$. Then

$$\hat{v}(x,z,t) = (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^2} e^{-\frac{|x-x'|^2+|z-z'|^2}{4t}} \hat{v}_0(x',z')dx'dz'$$

that is

$$v(x,y,t) =$$
$$= (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^{2\pi} e^{-\frac{|x-x'|^2+y^2+y'^2-yy'\cos\theta}{4t}} y' v_0(x',y')dx'dy'd\theta$$

Going back to the functions u, u_0 this gives

$$u(x,y,t) =$$
$$= (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^{2\pi} \sqrt{yy'} e^{-\frac{|x-x'|^2+y^2+y'^2-yy'\cos\theta}{4t}} u_0(x',y')dx'dy'd\theta$$

So

$$G^H_{par}(\mathbf{x},\mathbf{x}',t) = (4\pi t)^{-\frac{n+1}{2}}\sqrt{yy'}e^{-\frac{|x-x'|^2+y^2+y'^2}{4t}}\int_0^{2\pi} e^{\frac{yy'\cos\theta}{2t}}d\theta$$

Hence we can compute the elliptic Green function for H,

$$\begin{aligned} G^H_{ell}(\mathbf{x},\mathbf{x}') &= \int_0^\infty G^H_{par}(\mathbf{x},\mathbf{x}',t)dt \qquad (\mathbf{x},\mathbf{x}'\in\mathbb{R}^n_+) \\ &= c_n\sqrt{yy'}\int_0^{2\pi}\left(|x-x'|^2+y^2+y'^2-2yy'\cos\theta\right)^{-\frac{n-1}{2}}d\theta \end{aligned}$$

Let

$$G^{-\Delta}_{ell}(\mathbf{x},\mathbf{x}') = c'_n|\mathbf{x}-\mathbf{x}'|^{2-n}\,, \quad (\mathbf{x},\mathbf{x}'\in\mathbb{R}^n)$$

denote the Green function for the Laplacian in $\mathbb{R}^n$. It may be seen that **if** $n=3$ then

$$G^H_{ell}(\mathbf{x},\mathbf{x}') \le G^{-\Delta}_{ell}(\mathbf{x},\mathbf{x}')\,, \qquad (\mathbf{x},\mathbf{x}'\in\mathbb{R}^3_+)$$

Completion of proof. The Sobolev inequality in $\mathbb{R}^3$ is

$$\langle -\Delta u, u\rangle_{L^2(\mathbb{R}^3)} \geq S_3 \|u\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)}$$

or equivalently $\quad S_3\|(-\Delta)^{-1/2} g\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)} \leq \|g\|^2_{L^2(\mathbb{R}^3)}$

Completion of proof. The Sobolev inequality in $\mathbb{R}^3$ is

$$\langle -\Delta u, u \rangle_{L^2(\mathbb{R}^3)} \geq S_3 \|u\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)}$$

or equivalently $\quad S_3 \|(-\Delta)^{-1/2} g\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)} \leq \|g\|^2_{L^2(\mathbb{R}^3)}$

By duality $\quad S_3 \|(-\Delta)^{-1/2} g\|^2_{L^2(\mathbb{R}^3)} \leq \|g\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^3)}$

Completion of proof. The Sobolev inequality in $\mathbb{R}^3$ is

$$\langle -\Delta u, u\rangle_{L^2(\mathbb{R}^3)} \geq S_3 \|u\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)}$$

or equivalently $\quad S_3\|(-\Delta)^{-1/2}g\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)} \leq \|g\|^2_{L^2(\mathbb{R}^3)}$

By duality $\quad S_3\|(-\Delta)^{-1/2}g\|^2_{L^2(\mathbb{R}^3)} \leq \|g\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^3)}$

Let $f, g \in C_c^\infty(\mathbb{R}^3_+)$. Then

$$\begin{aligned}
|\langle f, g\rangle_{L^2(\mathbb{R}^3_+)}|^2 &= |\langle H^{1/2}f, H^{-1/2}g\rangle_{L^2(\mathbb{R}^3_+)}|^2 \\
&\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)}\langle H^{-1}g, g\rangle_{L^2(\mathbb{R}^3_+)} \\
&\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)}\langle (-\Delta)^{-1}g, g\rangle_{L^2(\mathbb{R}^3)} \\
&\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)}\frac{1}{S_3}\|g\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^3_+)}
\end{aligned}$$

Completion of proof. The Sobolev inequality in $\mathbb{R}^3$ is

$$\langle -\Delta u, u\rangle_{L^2(\mathbb{R}^3)} \geq S_3 \|u\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)}$$

or equivalently $\quad S_3\|(-\Delta)^{-1/2} g\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3)} \leq \|g\|^2_{L^2(\mathbb{R}^3)}$

By duality $\quad S_3\|(-\Delta)^{-1/2} g\|^2_{L^2(\mathbb{R}^3)} \leq \|g\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^3)}$

Let $f, g \in C_c^\infty(\mathbb{R}^3_+)$. Then

$$\begin{aligned} |\langle f, g\rangle_{L^2(\mathbb{R}^3_+)}|^2 &= |\langle H^{1/2} f, H^{-1/2} g\rangle_{L^2(\mathbb{R}^3_+)}|^2 \\ &\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)} \langle H^{-1} g, g\rangle_{L^2(\mathbb{R}^3_+)} \\ &\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)} \langle (-\Delta)^{-1} g, g\rangle_{L^2(\mathbb{R}^3)} \\ &\leq \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)} \frac{1}{S_3} \|g\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^3_+)} \end{aligned}$$

Hence

$$\|f\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^3_+)} \leq \frac{1}{S_3} \langle Hf, f\rangle_{L^2(\mathbb{R}^3_+)} \qquad \textit{QED}$$

Brezis and Vazquez ('97). Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $2 \leq p < 2^*$. There exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + c\left(\int_\Omega |u|^p dx\right)^{2/p}$$

Brezis and Vazquez ('97). Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $2 \leq p < 2^*$. There exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + c\Big(\int_\Omega |u|^p dx\Big)^{2/p}$$

Filippas and Tertikas ('02). Let

$$X(t) := \frac{1}{1 - \ln t}\,, \qquad t \in (0,1).$$

Let Ω be a bounded domain and $D = \sup_\Omega |x|$. Then there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + c\Big(\int_\Omega X^{\frac{2(n-1)}{n-2}}(|x|/D)|u|^{2^*} dx\Big)^{2/2^*}$$

for all $u \in C_c^\infty(\Omega)$. The exponent of X is sharp.

Adimurthi, Filippas and Tertikas ('09). Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $D = \sup_\Omega |x|$. There holds

$$\int_\Omega |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx$$

$$+(n-2)^{-\frac{2(n-1)}{n}} S_n \Big(\int_\Omega X^{\frac{2(n-1)}{n-2}} (\frac{|x|}{D}) |u|^{2^*} dx \Big)^{2/2^*}$$

for all $u \in C_c^\infty(\Omega)$. The constant is the best possible.

Adimurthi, Filippas and Tertikas ('09). Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $D = \sup_\Omega |x|$. There holds

$$\int_\Omega |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx$$
$$+(n-2)^{-\frac{2(n-1)}{n}} S_n \Big(\int_\Omega X^{\frac{2(n-1)}{n-2}} (\frac{|x|}{D}) |u|^{2^*} dx \Big)^{2/2^*}$$

for all $u \in C_c^\infty(\Omega)$. The constant is the best possible.

Our aim: Find an explicit Sobolev improvement for a sharp Hardy inequality

Adimurthi, Filippas and Tertikas ('09). Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $D = \sup_\Omega |x|$. There holds

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx$$

$$+(n-2)^{-\frac{2(n-1)}{n}} S_n \Big(\int_\Omega X^{\frac{2(n-1)}{n-2}} (\frac{|x|}{D}) |u|^{2^*} dx \Big)^{2/2^*}$$

for all $u \in C_c^\infty(\Omega)$. The constant is the best possible.

Our aim: Find an explicit Sobolev improvement for a sharp Hardy inequality

We obtain such improvements in three different contexts:

1. Point singularity in Euclidean space
2. Point singularity in hyperbolic space
3. Boundary point singularity in Euclidean space

1. Point singularity in Euclidean space

1. Point singularity in Euclidean space

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. There exists $\alpha_n > 0$ such that for all $0 < \alpha \leq \alpha_n$ there holds

$$\int_{B_1} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{B_1} |x|^{\frac{p(n-2)}{2} - n} X(\alpha |x|)^{\frac{p+2}{2}} |u|^p dx \right)^{2/p},$$

for all $u \in C_c^\infty(\Omega)$. Moreover the inequality is sharp for any $\alpha \leq \alpha_n$.

1. Point singularity in Euclidean space

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. There exists $\alpha_n > 0$ such that for all $0 < \alpha \leq \alpha_n$ there holds

$$\int_{B_1} |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2} dx$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \Big(\int_{B_1} |x|^{\frac{p(n-2)}{2} - n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \Big)^{2/p},$$

for all $u \in C_c^\infty(\Omega)$. Moreover the inequality is sharp for any $\alpha \leq \alpha_n$.

Note: For any $\alpha, \alpha' > 0$ there holds

$$\lim_{x \to 0} \frac{X(\alpha|x|)}{X(\alpha'|x|)} = 1\,.$$

Two improved versions:

Two improved versions:

Theorem. Let $n \geq 3$, $2 < p \leq 2^*$, $\theta \in (0,2)$. There exist $R > 1$ and $\alpha_{n,\theta} < 1$ such that for any $0 < \alpha \leq \alpha_{n,\theta}$ there holds

$$\int_{B_1} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx - \theta^2 \int_{B_1} \frac{u^2}{|x|^{2-\theta}(R^\theta - |x|^\theta)} dx$$

$$\geq \quad (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{B_1} |x|^{\frac{p(n-2)}{2} - n} X^{\frac{p+2}{2}}(\alpha |x|) |u|^p dx \right)^{2/p}.$$

for all $u \in C_c^\infty(B_1)$. Moreover the inequality is sharp for all $\alpha \leq \alpha_{n,\theta}$.

Two improved versions:

Theorem. Let $n \geq 3$, $2 < p \leq 2^*$, $\theta \in (0,2)$. There exist $R > 1$ and $\alpha_{n,\theta} < 1$ such that for any $0 < \alpha \leq \alpha_{n,\theta}$ there holds

$$\int_{B_1} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx - \theta^2 \int_{B_1} \frac{u^2}{|x|^{2-\theta}(R^\theta - |x|^\theta)} dx$$

$$\geq \quad (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{B_1} |x|^{\frac{p(n-2)}{2} - n} X^{\frac{p+2}{2}}(\alpha|x|)|u|^p dx \right)^{2/p}.$$

for all $u \in C_c^\infty(B_1)$. Moreover the inequality is sharp for all $\alpha \leq \alpha_{n,\theta}$.

$$R^\theta = 1 + \frac{1}{\sqrt{n-2}} \; , \qquad -\ln \alpha_{n,\theta} = R^{2\theta} - 1 + \int_0^1 \frac{s^{\theta-1}(2R^\theta - s^\theta)}{(R^\theta - s^\theta)^2} ds$$

Theorem. Let $n \geq 3$, $2 < p \leq 2^*$ and $0 \leq \theta < 1/2$. There exists $\alpha_n > 0$ such that for any $0 < \alpha \leq \alpha_n$ there holds

$$\int_{B_1} |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2} dx - \theta(1-\theta) \int_{B_1} \frac{u^2}{|x|^2} X^2(\alpha|x|) dx$$

$$\geq \quad \Big(\frac{1-2\theta}{n-2}\Big)^{\frac{p+2}{p}} S_{n,p} \Big(\int_{B_1} |x|^{\frac{p(n-2)}{2}-n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \Big)^{2/p}.$$

for all $u \in C_c^\infty(B_1)$. Moreover the constant $\big(\frac{1-2\theta}{n-2}\big)^{\frac{p+2}{p}} S_{n,p}$ is sharp for any $\alpha \leq \alpha_n$.

2. Point singularity in hyperbolic space

2. Point singularity in hyperbolic space

Hyperbolic space $\mathbb{H}^n$: use the Poincaré ball model

Equip the unit ball with the Riemannian metric

$$ds^2 = \Big(\frac{1-|x|^2}{2}\Big)^{-2} dx^2.$$

Under this model we have

$$|\nabla_{\mathbb{H}^n} v|^2 = \Big(\frac{1-|x|^2}{2}\Big)^2 |\nabla_{\mathbb{R}^n} v|^2 \quad , \qquad dV = \Big(\frac{1-|x|^2}{2}\Big)^{-n} dx$$

and Riemannian distance to the origin is

$$\rho(x) = \ln\Big(\frac{1+|x|}{1-|x|}\Big).$$

In $\mathbb{H}^n$ both Hardy and Sobolev inequalities are valid:

E. Hebey. If $n \geq 3$ then

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} u^2 dV \geq S_n \Big(\int_{\mathbb{H}^n} |u|^{2^*} dV \Big)^{2/2^*}$$

for all $u \in C_c^\infty(\mathbb{H}^n)$.

In $\mathbb{H}^n$ both Hardy and Sobolev inequalities are valid:

E. Hebey. If $n \geq 3$ then

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} u^2 dV \geq S_n \Big(\int_{\mathbb{H}^n} |u|^{2^*} dV \Big)^{2/2^*}$$

for all $u \in C_c^\infty(\mathbb{H}^n)$.

More generally, for any $2 \leq p \leq 2^*$,

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} u^2 dV \geq S_{n,p} \Big(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2} - n} |u|^p dV \Big)^{2/p}$$

for all $u \in C_c^\infty(\mathbb{H}^n)$.

Hardy inequality is also valid:

$$\int_{\mathbb{H}^n} |\nabla u|^2 dV \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV \ , \qquad u \in C_c^\infty(\mathbb{H}^n).$$

Hardy inequality is also valid:

$$\int_{\mathbb{H}^n} |\nabla u|^2 dV \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV \ , \qquad u \in C_c^\infty(\mathbb{H}^n).$$

Several related results and improvements: Carron; Yang and Kong; Kombe; Berchio, Ganguly, Grillo, Pinchover; Kristaly, ...

Hardy inequality is also valid:

$$\int_{\mathbb{H}^n} |\nabla u|^2 dV \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV \ , \qquad u \in C_c^\infty(\mathbb{H}^n).$$

Several related results and improvements: Carron; Yang and Kong; Kombe; Berchio, Ganguly, Grillo, Pinchover; Kristaly, ...

Q: What about Sobolev improvement?

Consider the inequality

$$\int_{B_1} |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2} dx$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \Big(\int_{B_1} |x|^{\frac{p(n-2)}{2}-n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \Big)^{2/p}$$

(valid for $\alpha \leq \alpha_n$ and all $u \in C_c^\infty(B_1)$) and "translate" it to the hyperbolic context.

Consider the inequality

$$\int_{B_1} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{B_1} |x|^{\frac{p(n-2)}{2}-n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \right)^{2/p}$$

(valid for $\alpha \leq \alpha_n$ and all $u \in C_c^\infty(B_1)$) and "translate" it to the hyperbolic context. Obtain

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. For any $0 < \alpha \leq \alpha_n$ there holds

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2}-n} X^{\frac{p+2}{2}} (\alpha \tanh(\frac{\rho}{2})) |u|^p dV \right)^{2/p},$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. Moreover the inequality is sharp for all $0 < \alpha \leq \alpha_n$.

Consider the inequality

$$\int_{B_1} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{B_1} |x|^{\frac{p(n-2)}{2}-n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \right)^{2/p}$$

(valid for $\alpha \leq \alpha_n$ and all $u \in C_c^\infty(B_1)$) and "translate" it to the hyperbolic context. Obtain

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. For any $0 < \alpha \leq \alpha_n$ there holds

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2}-n} X^{\frac{p+2}{2}}(\alpha \tanh(\frac{\rho}{2})) |u|^p dV \right)^{2/p},$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. Moreover the inequality is sharp for all $0 < \alpha \leq \alpha_n$.

Can we have more?

Indeed we may also obtain

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. For any $0 < \alpha \leq \alpha_n$ we have

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} u^2 dV - \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\sinh^2 \rho} dV$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2} - n} X^{\frac{p+2}{2}} (\alpha \tanh(\rho/2)) |u|^p dV \right)^{2/p}$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. Moreover the constant $(n-2)^{-\frac{p+2}{p}} S_{n,p}$ is sharp for all $0 < \alpha \leq \alpha_n$.

Indeed we may also obtain

Theorem. Let $n \geq 3$ and $2 < p \leq 2^*$. For any $0 < \alpha \leq \alpha_n$ we have

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} u^2 dV - \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{u^2}{\sinh^2 \rho} dV$$

$$\geq (n-2)^{-\frac{p+2}{p}} S_{n,p} \left(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2} - n} X^{\frac{p+2}{2}} (\alpha \tanh(\rho/2)) |u|^p dV \right)^{2/p}$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. Moreover the constant $(n-2)^{-\frac{p+2}{p}} S_{n,p}$ is sharp for all $0 < \alpha \leq \alpha_n$.

We would like $(n-1)^2/4$ instead of $n(n-2)/4$ in front of the L^2 term.

Berchio, Ganguly, Grillo ('17). There holds

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} u^2 dV$$
$$\geq \frac{1}{4} \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV + \frac{(n-1)(n-3)}{4} \int_{\mathbb{H}^n} \frac{u^2}{\sinh^2 \rho} dV$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. The inequality is optimal, non-improvable.

Berchio, Ganguly, Grillo ('17). There holds

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV - \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} u^2 dV$$
$$\geq \frac{1}{4} \int_{\mathbb{H}^n} \frac{u^2}{\rho^2} dV + \frac{(n-1)(n-3)}{4} \int_{\mathbb{H}^n} \frac{u^2}{\sinh^2 \rho} dV$$

for all $u \in C_c^\infty(\mathbb{H}^n)$. The inequality is optimal, non-improvable.

Also recent article by Berchio, Ganguly, Grillo, Pinchover ('19).

We are interested in Sobolev improvements of the Poincaré-Hardy inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \geq \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} v^2 dV + \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{v^2}{\sinh^2 \rho} dV \,,$$

for $u \in C_c^\infty(\mathbb{H}^n)$.

We are interested in Sobolev improvements of the Poincaré-Hardy inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \geq \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} v^2 dV + \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{v^2}{\sinh^2 \rho} dV \,,$$

for $u \in C_c^\infty(\mathbb{H}^n)$.

To state the result we need to introduce a constant $\overline{S}_{n,p}$ and a function $Y(t)$, $t > 0$.

The constant $\overline{S}_{n,p}$, $2 < p \leq 2^*$.

The constant $\overline{S}_{n,p}$, $2 < p \le 2^*$.

Given $2 < p \le 2^*$ we define $\overline{S}_{n,p}$ to be the best constant for the Poincaré-Sobolev inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \ge \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} v^2 dV$$
$$+\overline{S}_{n,p}\Big(\int_{\mathbb{H}^n} (\sinh\rho)^{\frac{p(n-2)}{2}-n} |v|^p dV\Big)^{2/p}, \qquad v \in C_c^\infty(\mathbb{H}^n)$$

Note. The positivity of $\overline{S}_{n,p}$ follows from the positivity of $\overline{S}_{n,2^*}$ (Mancini and Sandeep '08)

The function $Y(t)$.

The function $Y(t)$.

Consider the following two auxiliary problems:

$$\begin{cases} g''(t) + \frac{1}{4\sinh^2 t} g(t) = 0, \quad t > 0, \\ \lim_{t\to+\infty} g(t) = 1, \end{cases}$$

and, for $n \geq 3$,

$$\begin{cases} h''(t) - \frac{(n-1)(n-3)}{4\sinh^2 t} h(t) = 0, \quad t > 0, \\ \lim_{t\to+\infty} h(t) = 1\,. \end{cases}$$

Prove existence and uniqueness and study asymptotics near zero. Then define a function $\rho = \rho(t)$ by

$$\int_0^{\rho(t)} \frac{dr}{h(r)^2} = \int_0^t \frac{ds}{g(s)^2}\,, \qquad t > 0.$$

The function $Y(t)$.

Consider the following two auxiliary problems:

$$\begin{cases} g''(t) + \frac{1}{4\sinh^2 t} g(t) = 0, \quad t > 0, \\ \lim_{t\to+\infty} g(t) = 1, \end{cases}$$

and, for $n \geq 3$,

$$\begin{cases} h''(t) - \frac{(n-1)(n-3)}{4\sinh^2 t} h(t) = 0, \quad t > 0, \\ \lim_{t\to+\infty} h(t) = 1\,. \end{cases}$$

Prove existence and uniqueness and study asymptotics near zero. Then define a function $\rho = \rho(t)$ by

$$\int_0^{\rho(t)} \frac{dr}{h(r)^2} = \int_0^t \frac{ds}{g(s)^2}\,, \qquad t > 0.$$

The function $Y(t)$ is defined by

$$Y(t) = (n-2)\frac{h\big(\rho(t)\big)^2 \sinh t}{g(t)^2 \sinh \rho(t)}\ , \qquad t > 0.$$

The function $Y(t)$ is defined by

$$Y(t) = (n-2)\frac{h\big(\rho(t)\big)^2 \sinh t}{g(t)^2 \sinh \rho(t)}\ , \qquad t > 0.$$

Theorem (Poicaré-Hardy-Sobolev inequality). Let $n \geq 3$ and $2 < p \leq 2^*$. There holds

$$\begin{aligned}\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \quad &\geq \quad \Big(\frac{n-1}{2}\Big)^2 \int_{\mathbb{H}^n} v^2 dV + \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{H}^n} \frac{v^2}{\sinh^2 \rho} dV \\ &+ (n-2)^{-\frac{p+2}{p}} \overline{S}_{n,p} \Big(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2} - n} Y^{\frac{p+2}{2}}(\rho) |v|^p dV \Big)^{2/p}.\end{aligned}$$

for all $v \in C_c^\infty(\mathbb{H}^n)$. Moreover the inequality is sharp.

Concerning the function $Y(t)$ we have:

Proposition.

(i) There exists $\alpha_n > 0$ such that

$$Y(t) \geq X\Big(\alpha_n \tanh \frac{t}{2}\Big) , \qquad t > 0$$

(ii) There holds

$$\lim_{t \to 0} \frac{Y(t)}{X(t)} = 1$$

Concerning the function $Y(t)$ we have:

Proposition.

(i) There exists $\alpha_n > 0$ such that

$$Y(t) \geq X\big(\alpha_n \tanh \frac{t}{2}\big)\ , \qquad t > 0$$

(ii) There holds

$$\lim_{t \to 0} \frac{Y(t)}{X(t)} = 1$$

What about $\overline{S}_{n,p}$?

The precise value of $\overline{S}_{n,p}$ is not known.

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \geq \Big(\frac{n-1}{2}\Big)^2 \int_{\mathbb{H}^n} v^2 dV$$
$$+\overline{S}_{n,p}\Big(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2}-n} |v|^p dV \Big)^{2/p}$$

The precise value of $\overline{S}_{n,p}$ is not known.

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 dV \geq \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} v^2 dV$$
$$+\overline{S}_{n,p}\left(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p(n-2)}{2}-n} |v|^p dV\right)^{2/p}$$

Using the half-space model for $\mathbb{H}^n$ we find that $\overline{S}_{n,p}$ is the best constant for the Hardy-Sobolev inequality

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx$$
$$+\overline{S}_{n,p}\left(\int_{\mathbb{R}^n_+} \left(\frac{2}{|x-e_n|\;|x+e_n|}\right)^{n-\frac{n-2}{2}p} |u|^p dx\right)^{2/p}$$

for all $u \in C_c^\infty(\mathbb{R}^n_+)$.

For $p = 2^*$ this becomes

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx + \overline{S}_{n,2^*} \left(\int_{\mathbb{R}^n_+} |u|^{2^*} dx \right)^{2/2^*}, \quad u \in C_c^\infty(\mathbb{R}^n_+)$$

The result of Benguria-Frank-Loss states that

$$\overline{S}_{3,2^*} = S_3.$$

For $p = 2^*$ this becomes

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx + \overline{S}_{n,2^*} \left(\int_{\mathbb{R}^n_+} |u|^{2^*} dx \right)^{2/2^*}, \quad u \in C_c^\infty(\mathbb{R}^n_+)$$

The result of Benguria-Frank-Loss states that

$$\overline{S}_{3,2^*} = S_3.$$

We have

Theorem. Let $n = 3$. For any $2 \leq p < 2^*$ there holds

$$\overline{S}_{3,p} = S_{3,p} = \frac{p}{2^{\frac{2}{p}}} \left[\frac{4\pi \Gamma^2(\frac{p}{p-2})}{(p-2)\Gamma(\frac{2p}{p-2})} \right]^{\frac{p-2}{p}}$$

Open problem: compute $\overline{S}_{n,p}$ for $n \geq 4$

In case $n = 3$ the result is sharp also with the logarithmic function X.

In case $n = 3$ the result is sharp also with the logarithmic function X.

Theorem. There exists an $\overline{\alpha}_3 > 0$ such that for all $0 < \alpha \leq \overline{\alpha}_3$, all $2 < p \leq 2^*$ and all $v \in C_c^\infty(\mathbb{H}^3)$ there holds

$$\int_{\mathbb{H}^3} |\nabla_{\mathbb{H}^3} v|^2 dV \geq \int_{\mathbb{H}^3} v^2 dV + \frac{1}{4} \int_{\mathbb{H}^3} \frac{v^2}{\sinh^2 \rho} dV$$
$$+ S_{3,p} \Big(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p-6}{2}} X^{\frac{p+2}{2}} (\alpha \tanh(\rho/2)) |v|^p dV \Big)^{2/p}$$

The constant $S_{3,p}$ is sharp for all $0 < \alpha \leq \overline{\alpha}_3$.

In case $n=3$ the result is sharp also with the logarithmic function X.

Theorem. There exists an $\overline{\alpha}_3 > 0$ such that for all $0 < \alpha \leq \overline{\alpha}_3$, all $2 < p \leq 2^*$ and all $v \in C_c^\infty(\mathbb{H}^3)$ there holds

$$\int_{\mathbb{H}^3} |\nabla_{\mathbb{H}^3} v|^2 dV \geq \int_{\mathbb{H}^3} v^2 dV + \frac{1}{4} \int_{\mathbb{H}^3} \frac{v^2}{\sinh^2 \rho} dV$$

$$+ S_{3,p} \Big(\int_{\mathbb{H}^n} (\sinh \rho)^{\frac{p-6}{2}} X^{\frac{p+2}{2}} \big(\alpha \tanh(\rho/2)\big) |v|^p dV \Big)^{2/p}$$

The constant $S_{3,p}$ is sharp for all $0 < \alpha \leq \overline{\alpha}_3$.

Note. To show optimality use the sharpness of the inequality

$$\int_{B_1} |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2} dx - \theta(1-\theta) \int_{B_1} \frac{u^2}{|x|^2} X^2(\alpha|x|) dx$$

$$\geq \Big(\frac{1-2\theta}{n-2}\Big)^{\frac{p+2}{p}} S_{n,p} \Big(\int_{B_1} |x|^{\frac{p(n-2)}{2}-n} X(\alpha|x|)^{\frac{p+2}{2}} |u|^p dx \Big)^{2/p}$$

3. Boundary point singularity in Euclidean space

3. Boundary point singularity in Euclidean space

Let Ω be a domain in $\mathbb{R}^n$ and assume that $0 \in \partial\Omega$. We are interested in the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{|x|^2} dx \ , \qquad u \in C_c^\infty(\Omega)$$

and related Sobolev improvements.

3. Boundary point singularity in Euclidean space

Let Ω be a domain in $\mathbb{R}^n$ and assume that $0 \in \partial\Omega$. We are interested in the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{|x|^2} dx \,, \qquad u \in C_c^\infty(\Omega)$$

and related Sobolev improvements.

What about the best constant c? We know that

$$c = \Big(\frac{n-2}{2}\Big)^2$$

works, but can we do better ? The geometry plays a role.

I. Cones
II. Bounded domains with nice boundary

Part I. Cones.

Part I. Cones.

A. Nazarov ('06). Let C_Σ be a finite (or infinite) cone with vertex at zero:

$$C_\Sigma = \{ r\omega : \omega \in \Sigma \, , \; 0 < r < 1 \;\; (\text{or } r > 0) \;\; \}$$

where Σ an open subset of $\mathbb{S}^{n-1}$.
Let $\mu_1(\Sigma)$ be the first eigenvalue of the Dirichlet Laplacian on Σ. Then

$$\int_{C_\Sigma} |\nabla u|^2 dx \geq \left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx,$$

for all $u \in C_c^\infty(C_\Sigma)$. Moreover the constant is the best possible.

Proof. The function

$$\phi(x) = r^{-\frac{n-2}{2}} \psi_1(\omega)$$

is a positive solution to the Euler equation.

Q: Is it possible to improve the above inequality by adding more terms ?

Q: Is it possible to improve the above inequality by adding more terms ?

Theorem (B., Filippas, Tertikas '18). There holds

$$\int_{C_\Sigma} |\nabla u|^2 dx \geq \Big[\Big(\frac{n-2}{2}\Big)^2 + \mu_1(\Sigma)\Big] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx + \frac{1}{4} \int_{C_\Sigma} \frac{u^2}{|x|^2} X^2(|x|) dx$$

for all $u \in C_c^\infty(C_\Sigma)$. The inequality is sharp.

Q: Is it possible to improve the above inequality by adding more terms ?

Theorem (B., Filippas, Tertikas '18). There holds

$$\int_{C_\Sigma} |\nabla u|^2 dx \geq \left[\left(\frac{n-2}{2}\right)^2 + \mu_1(\Sigma)\right] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx + \frac{1}{4} \int_{C_\Sigma} \frac{u^2}{|x|^2} X^2(|x|) dx$$

for all $u \in C_c^\infty(C_\Sigma)$. The inequality is sharp.

Proof. The function

$$\phi(x) = r^{-\frac{n-2}{2}} X^{-\frac{1}{2}}(r) \psi_1(\omega)$$

is a positive solution to the Euler equation.

What about Sobolev improvements?

Theorem (B., Filippas, Tertikas '18). Let $n \geq 3$. There exists a positive constant $C = C(\Sigma)$ such that

$$\int_{C_\Sigma} |\nabla u|^2 dx \geq \left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx$$

$$+ C(\Sigma) \left(\int_{C_\Sigma} X^{\frac{2n-2}{n-2}}(|x|) |u|^{2^*} dx \right)^{2/2^*},$$

for all $u \in C_c^\infty(C_\Sigma)$. Moreover

(i) The exponent $(2n-2)/(n-2)$ of $X(|x|)$ is the best possible.
(ii) For the best constant $C(\Sigma)$ we have

$$C(\Sigma) \leq C_n |\Sigma|^{\frac{2}{n}};$$

in particular it cannot be taken to be independent of Σ.

What about additional improvements ?
Define $X_1(t) = X(t)$ and for $k \geq 2$

$$X_k(t) = X_1(X_{k-1}(t)), \qquad t \in (0,1).$$

What about additional improvements ?
Define $X_1(t) = X(t)$ and for $k \geq 2$

$$X_k(t) = X_1(X_{k-1}(t)), \qquad t \in (0,1).$$

Theorem (B., Filippas, Tertikas '18). There holds

$$\int_{C_\Sigma} |\nabla u|^2 dx \;\geq\; \Big[\Big(\frac{n-2}{2}\Big)^2 + \mu_1(\Sigma)\Big] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{C_\Sigma} \frac{u^2}{|x|^2} X_1^2 X_2^2 \dots X_i^2 dx \,,$$

for all $u \in C_c^\infty(C_\Sigma)$.
The inequality is sharp at each step.

Proof. For each fixed m, the function

$$\phi(x) = r^{-\frac{n-2}{2}} X_1(r)^{-1/2} \dots X_m(r)^{-1/2} \psi_1(\omega)$$

is a positive solution to the mth-improved Hardy inequality.

What about Sobolev improvements for the mth improved Hardy inequality?

Theorem Let $n \geq 3$. There exists a constant C that depends only on Σ such that for any $m \in \mathbb{N}$

$$\begin{aligned}\int_{C_\Sigma} |\nabla u|^2 dx \quad &\geq \quad \left[\left(\frac{n-2}{2}\right)^2 + \mu_1(\Sigma)\right] \int_{C_\Sigma} \frac{u^2}{|x|^2} dx \\ &\quad + \frac{1}{4} \sum_{i=1}^{m} \int_{C_\Sigma} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\ &\quad + C \left(\int_{C_\Sigma} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},\end{aligned}$$

for all $u \in C_c^\infty(C_\Sigma)$. The inequality is sharp.

Part II. General bounded domains

Let Ω be a domain with nice boundary and with $0 \in \partial\Omega$. We are interested in the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{|x|^2} dx, \qquad u \in C_c^\infty(\Omega)$$

Part II. General bounded domains

Let Ω be a domain with nice boundary and with $0 \in \partial\Omega$. We are interested in the Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{|x|^2} dx, \qquad u \in C_c^\infty(\Omega)$$

Consider the half-space $\mathbb{R}^n_+ = \{x_n > 0\}$.

This is a special case of a cone, hence the Hardy constant is

$$\Big(\frac{n-2}{2}\Big)^2 + \mu_1(\mathbb{S}^{n-1}_+) = \Big(\frac{n-2}{2}\Big)^2 + (n-1) = \frac{n^2}{4}$$

This is the ‘right’ constant locally for a domain with smooth boundary.

M.M. Fall ('12) Let Ω be bounded Lipschitz boundary and assume that $\partial\Omega$ is C^2 near the origin. There exists an $r = r(\Omega)$ such that for all $u \in C_c^\infty(\Omega \cap B_r)$ there holds

$$\int_{\Omega\cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega\cap B_r} \frac{u^2}{|x|^2} dx$$

M.M. Fall ('12) Let Ω be bounded Lipschitz boundary and assume that $\partial\Omega$ is C^2 near the origin. There exists an $r = r(\Omega)$ such that for all $u \in C_c^\infty(\Omega \cap B_r)$ there holds

$$\int_{\Omega\cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega\cap B_r} \frac{u^2}{|x|^2} dx + C \int_{\Omega\cap B_r} \frac{u^2}{|x|^2} X_1^2 dx$$

M.M. Fall ('12) Let Ω be bounded Lipschitz boundary and assume that $\partial\Omega$ is C^2 near the origin. There exists an $r = r(\Omega)$ such that for all $u \in C_c^\infty(\Omega \cap B_r)$ there holds

$$\int_{\Omega\cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega\cap B_r} \frac{u^2}{|x|^2} dx + C \int_{\Omega\cap B_r} \frac{u^2}{|x|^2} X_1^2 dx$$

Theorem (B., Filippas, Tertikas '18). Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $0 \in \partial\Omega$ admiting an exterior ball of radius ρ at 0. Let $D = \sup_\Omega |x|$. There exist $\sigma_n > 0$ such that if $\rho \geq D/\sigma_n$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx$$

for all $u \in C_c^\infty(\Omega)$.

If in addition Ω satisfies an interior ball condition at 0 then the constant is sharp.

Theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $0 \in \partial\Omega$ having an exterior ball of radius ρ at 0. Let $D = \sup_\Omega |x|$. There exist $\sigma_n > 0$ and $\kappa > 0$ such that if $\rho \geq D/\sigma_n$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_\Omega \frac{u^2}{|x|^2} X_1^2 \ldots X_i^2 dx,$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(\sigma_n |x|/(3\kappa D))$.

If in addition Ω satisfies an interior ball condition at 0 then the estimate is sharp at each step.

What about Sobolev improvements?

What about Sobolev improvements?

Theorem (Hardy-Sobolev inequality) Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $0 \in \partial\Omega$ having an exterior ball of radius ρ at 0. There exist $\sigma_n, C_n > 0$ such that if $\rho \geq D/\sigma_n$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + C_n \Big(\int_\Omega X^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(\Omega)$; here $X = X(|x|/3D)$.

If in addition Ω satisfies an interior ball condition at 0 then the estimate is sharp.

A natural question:

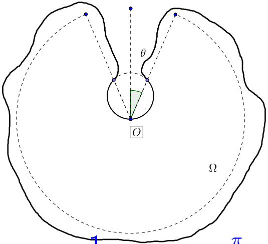
In order to have the Hardy inequality with constant $n^2/4$ is it necessary that the exterior ball is large compared to $D = \sup_\Omega |x|$?

Example. For $\rho \in (0, 1/2)$ and $\theta \in (0, \pi/2)$ define

$$\mathscr{A}_{\rho,\theta} = \{x = (x', x_n) \in B_1 \ : \ x_n < \cot\theta |x'| \text{ and } |x - \rho e_n| > \rho\}.$$

Let $\Omega \supset \mathscr{A}_{\rho,\theta}$ having $B(\rho e_n, \rho)$ as largest exterior ball at 0. Let $\lambda_1(n, \theta)$ be the first Dirichlet eigenvalue of the Laplace operator on the spherical cap

$$\Sigma_\theta = \{(x', x_n) \in S^{n-1} : x_n < \cot\theta \, |x'|\}.$$



If

$$\rho < \frac{1}{2\cos\theta} e^{-\frac{\pi}{\sqrt{n-1-\lambda_1(n,\theta)}}},$$

then the best Hardy constant of Ω is strictly smaller than $n^2/4$.

Back to Hardy-Sobolev inequalities with explicit constants.

Back to Hardy-Sobolev inequalities with explicit constants.

Let $n \geq 3$, $0 \leq \gamma < n/2$ and $2 < p \leq 2^*$. We define $S^*_{n,p,\gamma}$ to be the best constant for the inequality

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \gamma(n-\gamma) \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2} dx \geq S^*_{n,p,\gamma} \Big(\int_{\mathbb{R}^n_+} |x|^{\frac{p(n-2)}{2} - n} |u|^p dx \Big)^{2/p}$$

for all $u \in C_c^\infty(\mathbb{R}^n_+)$.

Theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $0 \in \partial\Omega$ and let $D = \sup_\Omega |x|$. Assume that Ω satisfies an exterior ball condition at zero with exterior ball $B_\rho(-\rho e_n)$. Then for any $2 < p \leq 2^*$ and any $\gamma \in [0, n/2)$ there exist an $r_{n,\gamma}$ and $\alpha^*_{n,\gamma}$ in $(0,1)$ such that, if the radius ρ of the exterior ball satisfies $\rho \geq D/r_{n,\gamma}$ then for all $0 < \alpha \leq \alpha^*_{n,\gamma}$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx$$

$$+(n-2\gamma)^{-\frac{p+2}{p}} S^*_{n,p,\gamma} \left(\int_\Omega |x|^{\frac{p(n-2)}{2}-n} \left(\frac{|x+2\rho e_n|}{2\rho} \right)^{\frac{p(n-2)}{2}-n} X^{\frac{p+2}{2}} |u|^p dx \right)^{\frac{2}{p}}$$

for all $u \in C_c^\infty(\Omega)$; here $X = X(\alpha|x|/D)$.

Theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $0 \in \partial\Omega$ and let $D = \sup_\Omega |x|$. Assume that Ω satisfies an exterior ball condition at zero with exterior ball $B_\rho(-\rho e_n)$. Then for any $2 < p \leq 2^*$ and any $\gamma \in [0, n/2)$ there exist an $r_{n,\gamma}$ and $\alpha^*_{n,\gamma}$ in $(0,1)$ such that, if the radius ρ of the exterior ball satisfies $\rho \geq D/r_{n,\gamma}$ then for all $0 < \alpha \leq \alpha^*_{n,\gamma}$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx$$

$$+(n-2\gamma)^{-\frac{p+2}{p}} S^*_{n,p,\gamma} \left(\int_\Omega |x|^{\frac{p(n-2)}{2}-n} \left(\frac{|x+2\rho e_n|}{2\rho} \right)^{\frac{p(n-2)}{2}-n} X^{\frac{p+2}{2}} |u|^p dx \right)^{\frac{2}{p}}$$

for all $u \in C_c^\infty(\Omega)$; here $X = X(\alpha|x|/D)$.

But no sharpness...

Theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $0 \in \partial\Omega$ and let $D = \sup_\Omega |x|$. Assume that Ω satisfies an exterior ball condition at zero with exterior ball $B_\rho(-\rho e_n)$. Then for any $2 < p \leq 2^*$ and any $\gamma \in [0, n/2)$ there exist an $r_{n,\gamma}$ and $\alpha^*_{n,\gamma}$ in $(0,1)$ such that, if the radius ρ of the exterior ball satisfies $\rho \geq D/r_{n,\gamma}$ then for all $0 < \alpha \leq \alpha^*_{n,\gamma}$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{n^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx$$
$$+(n-2\gamma)^{-\frac{p+2}{p}} S^*_{n,p,\gamma} \left(\int_\Omega |x|^{\frac{p(n-2)}{2}-n} \left(\frac{|x+2\rho e_n|}{2\rho} \right)^{\frac{p(n-2)}{2}-n} X^{\frac{p+2}{2}} |u|^p dx \right)^{\frac{2}{p}}$$

for all $u \in C_c^\infty(\Omega)$; here $X = X(\alpha|x|/D)$.

But no sharpness...

About the proof.

About the proof.

For interior point singularity: Change variables so that the two infima are directly comparable.

About the proof.

For interior point singularity: Change variables so that the two infima are directly comparable. For example the best constant $\overline{S}_{n,p}$ we have

$$\overline{S}_{n,p} = \inf \frac{\int_0^\infty \int_{\mathbb{S}^{n-1}} h(\rho)^2 \left(w_\rho^2 + \frac{1}{\sinh^2 \rho} |\nabla_\omega w|^2 \right) dS\, d\rho}{\left(\int_0^\infty \int_{\mathbb{S}^{n-1}} (\sinh \rho)^{-\frac{p+2}{2}} h(\rho)^p |w|^p dS\, d\rho \right)^{2/p}}.$$

The term involving ∇_ω can be ignored by symmetrization.

About the proof.

For interior point singularity: Change variables so that the two infima are directly comparable. For example the best constant $\overline{S}_{n,p}$ we have

$$\overline{S}_{n,p} = \inf \frac{\int_0^\infty \int_{\mathbb{S}^{n-1}} h(\rho)^2 \left(w_\rho^2 + \frac{1}{\sinh^2 \rho} |\nabla_\omega w|^2 \right) dS \, d\rho}{\left(\int_0^\infty \int_{\mathbb{S}^{n-1}} (\sinh \rho)^{-\frac{p+2}{2}} h(\rho)^p |w|^p dS \, d\rho \right)^{2/p}}.$$

The term involving ∇_ω can be ignored by symmetrization.

For boundary point singularity: use in addition a conformal map from $\mathbb{R}^n_+$ onto $B(\rho)^c$.

The end