Inverse scattering for quasilinear operators of higher order

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The linear Euler-Lagrange equation that arises from vibrating of the beam contains (in the simplest model) derivatives of the 4th and 2nd order

$$u^{(4)}(x) - cu''(x) = p(x),$$

where $u(x)$ denotes the deviation from the equilibrium of the beam at point x and $p(x)$ is the density of the lateral load at x.
If we consider a suspension bridge as a beam of length L with hinged ends then downward deflection is measured by $u(x,t)$ that satisfies the equation of order four (with Navier boundary conditions) :

$$\gamma u_{xxxx}(x,t) + u_{tt}(x,t) = -ku^{+}(x,t) + W + f(x,t),$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0,$$

where γ, W, k are constants, and $f(x,t)$ is the external forcing term.

A multidimensional nonlinear beam-equation (see Gazzola and et al, "Polyharmonic boundary value problems", Springer, 2010) :

$$\partial_t^2 U(x,t) + \Delta^2 U(x,t) + m(x)U(x,t)|U(x,t)|^p = 0,$$

where $p \geq 0$, under time-harmonic assumption $U(x,t) = u(x)e^{-i\omega t}$ leads to the biharmonic equation

$$\Delta^2 u(x) + m(x)u(x)|u(x)|^p = \omega^2 u(x).$$

The wave parameter ω is fixed (in general), but nevertheless we can consider it fixed but very big in order to apply limiting process and appropriate numerical methods. This allows to consider some scattering problems with high frequency for this potential equation.

We deal with quasilinear operators of fourth order of the form

$$H_4 u(x) := \Delta^2 u(x) + \vec{W}(x, |u|)\nabla u(x) + V(x, |u|)u(x), \quad x \in R^n, n = 1, 2, 3,$$

where $\vec{W}$ and V are complex-valued (in general) and s.t. $V(x,1), \vec{W}(x,1)$ belong to $L^p_{loc}(R^n)$ with p depending on n, and in addition, for any $\rho > 0$

$$|V(x, s_1) - V(x, s_2)| \le C_\rho \beta_V(x)|s_1 - s_2|,$$

$$|\vec{W}(x, s_1) - \vec{W}(x, s_2)| \le C_\rho \beta_W(x)|s_1 - s_2|, \quad 0 \le s_1, s_2 \le \rho,$$

and both (together with β_V, β_W) have behaviour at the infinity

$$|\vec{W}(x,1)|, \quad |V(x,1)| \le \frac{C}{|x|^\mu}, \quad |x| > R, \quad \mu > n.$$

These conditions include the power-type nonlinearities of the nonlinear beam equation described above, and most other physically relevant nonlinearities, such as the saturation and sinc nonlinearities.
The scattering problems are connected with the special solutions of the differential equation $H_4 u(x) = k^4 u(x)$, i.e., the solutions in the form

$$u(x,k,\theta) = u_0(x,k,\theta) + u_{sc}(x,k,\theta), \quad u_0(x,k,\theta) = e^{ik(x,\theta)}, \quad \theta \in S^{n-1},$$

where u_0 is the incident (plane) wave with angle θ and the scattered filed u_{sc} satisfies the Sommerfeld radiation conditions

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u_{sc}(x,k,\theta) = o(1), \quad r = |x| \to \infty,$$

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) \Delta u_{sc}(x,k,\theta) = o(1), \quad r = |x| \to \infty.$$

These solutions give us the data for inverse problem.

Under the Sommerfeld radiation conditions these solutions ($\equiv$ scattering solutions) are the unique solutions of the following integral equation (analogue of the Lippmann-Schwinger equation for linear Schrödinger operator)

$$u(x,k,\theta) = u_0 - \int\limits_{R^n} G_k^+(|x-y|)\left(\vec{W}(y,|u|)\nabla + V(y,|u|)\right) u(y,k,\theta)\,dy,$$

where G_k^+ is the outgoing fundamental solution of the operator $\Delta^2 - k^4$. It is the kernel of the integral operator $(\Delta^2 - k^4 - i0)^{-1}$ and it is equal to

$$G_k^+(k|x|) = \frac{i}{8k^2}\left(\frac{k}{2\pi|x|}\right)^{\frac{n-2}{2}}\left(H^{(1)}_{\frac{n-2}{2}}(k|x|) + \frac{2i}{\pi}K_{\frac{n-2}{2}}(k|x|)\right), \quad k > 0,$$

where $H_\nu^{(1)}$ and K_ν are the Hankel and Macdonald's functions of order ν, respectively. This integral operator maps as follows (due to Agmon's estimates for the operator $-\Delta - k^2$) :

$$\|(\Delta^2 - k^4 - i0)^{-1} f\|_{W^s_{2,-\delta}} \leq \frac{C}{k^{3-s}}\|f\|_{L^2_\delta}, \quad s = 0,1,2, \quad \delta > \frac{1}{2}, \quad k \geq 1.$$

In the linear case these estimates are enough to prove the unique solvability for such solutions in the spaces $W^1_{2,-\delta}(R^n)$ and obtain the estimates

$$\|u_{sc}\|_{W^1_{2,-\delta}(R^n)} \leq \frac{C}{k}, \quad \delta > \frac{1}{2}, \quad k > 1.$$

But in the nonlinear case we need to prove the solvability in the different type of spaces, namely in the Sobolev space $W^1_\infty(R^n)$ (due to nonlinearity).

Theorem (Direct problem)

Suppose that $\vec{W}(\cdot,1), V(\cdot,1), \beta_W, \beta_V$ *belong to* $L^p_{loc}(R^n), n = 1,2,3,$ $\max\{1; \frac{n}{2}\} < p \leq \infty$, *and have special behaviour at the infinity as* $O\left(\frac{1}{|x|^\mu}\right)$, $\mu > n$. *Then for any* $\rho > 0$ *there exists* $k_0 > 0$ *such that for all* $k \geq k_0$ *the Lippmann-Schwinger equation w.r.t.* u_{sc} *has a unique solution in the ball* $B_\rho(0) = \{f \in W^1_\infty(R^n) : \|f\|_{W^1_\infty} \leq \rho\}$ *and*

$$\|u_{sc}\|_{L^\infty(R^n)} \leq \frac{C}{k^{\frac{5-n}{2}}}, \quad \|\nabla u_{sc}\|_{L^\infty(R^n)} \leq \frac{C}{k^{\frac{3-n}{2}}}$$

for all $k \geq k_0$. *For* $n = 1$ *we may consider* $p = 1$.

In addition, the following asymptotic representation for these solutions as $|x| \to \infty$ holds

$$u(x,k,\theta) = u_0 + C_n \frac{k^{\frac{n-7}{2}} e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A(k,\theta,\theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right).$$

Here the function $A(k,\theta,\theta')$ is called the scattering amplitude and is defined by

$$A(k,\theta,\theta') = \int_{R^n} e^{-ik(\theta',y)} \left(\vec{W}(y,|u|)\nabla + V(y,|u|)\right) u(y,k,\theta)\, dy,$$

where $\theta' = \frac{x}{|x|}$ is the angle of observation.

The first result which concerns the inverse problems is the analogue of famous Saito's formula which was proved originally by Y. Saito for linear Schrödinger operator with smooth potentials.

Theorem (Saito's formula)

Suppose that $\vec{W}(\cdot,1), V(\cdot,1), \beta_W, \beta_V$ *belong to* $L^p_{loc}(R^n), n = 2, 3$ $n < p \leq \infty$, *and have special behaviour at the infinity mentioned above. In addition we assume that* $\nabla \vec{W}(\cdot,1) \in L^p_{loc}(R^n), n < p \leq \infty$ *and has special behaviour at the infinity mentioned above. Then the limit*

$$\lim_{k\to+\infty} k^{n-1} \int\limits_{S^{n-1}} \int\limits_{S^{n-1}} e^{-ik(\theta-\theta',x)} A(k,\theta,\theta')\, d\theta\, d\theta' =$$

$$= \frac{(2\pi)^n}{\pi} \int\limits_{R^n} \frac{V(y,1) - \frac{1}{2}\nabla \vec{W}(y,1)}{|x-y|^{n-1}}\, dy$$

holds uniformly for $n = 2$ *and in the sense of tempered for* $n = 3$.

In $1D$ it is also valid under special interpretation of the left-hand side.

In 1D case uniformly in $x \in R$ we have that

$$\lim_{k\to+\infty}\Big[e^{-2ikx}\int_R e^{iky}\left(W(y,|u_+|)u_+' + V(y,|u_+|)u_+\right)\,dy +$$

$$e^{2ikx}\int_R e^{-iky}(W(y,|u_-|)u_-' + V(y,|u_-|)u_-)\,dy +$$

$$\int_R e^{-iky}(W(y,|u_+|)u_+' + V(y,|u_+|)u_+)\,dy +$$

$$\int_R e^{iky}(W(y,|u_-|)u_-' + V(y,|u_-|)u_-)\,dy\Big] = \int_R (V(y,1) - \frac{1}{2}W'(y,1))\,dy,$$

where $u_\pm$ are the scattering solutions of $H_4 u = k^4 u$ that behave as $u_\pm(y,|u_\pm|) \approx e^{\pm iky}$ with $u_\pm'(y,|u_\pm|) \approx \pm ike^{\pm iky}$ when $k \to +\infty$.

The significance of Saito's formula for inverse scattering problems is apparent from its corollaries.

Corollary (Uniqueness)

Let $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ be as in Saito's formula. If the corresponding scattering amplitudes $A_1(k,\theta,\theta')$ and $A_2(k,\theta,\theta')$ coincide for some sequence $k_j \to +\infty$ and for all angles θ, θ', then $V_1(\cdot,1) - \frac{1}{2}\nabla\vec{W}_1(\cdot,1) =$ $= V_2(\cdot,1) - \frac{1}{2}\nabla\vec{W}_2(\cdot,1)$ in the sense of tempered distributions (a.e. in R^n).

Corollary (representation formula)

Under the same assumptions as in Saito's formula

$$V(x,1) - \frac{1}{2}\nabla\vec{W}(x,1) =$$

$$= \frac{1}{2^{n+1}\pi^{2n-2}} \lim_{k\to+\infty} k^n \int\limits_{S^{n-1}}\int\limits_{S^{n-1}} e^{-ik(\theta-\theta',x)}|\theta-\theta'|A(k,\theta,\theta')\,d\theta\,d\theta'$$

in the sense of tempered distributions.

In the linear case (i.e., in the case when no dependence on $|u|$ in V and $\vec{W}$) we can use different type of data for inverse problem. Namely, the estimates for the operator $\left(\Delta^2 - k^4 - i0\right)^{-1}$ can be applied to the operator $\left(H_4 - k^4 - i0\right)^{-1}$. This operator exists as the limit $\lim_{\epsilon \to +0} \left(H_4 - k^4 - i\epsilon\right)^{-1}$ in the operator topology from $L^2_\delta(R^n)$ to $W^1_{2,-\delta}(R^n)$ (with the same δ as above) with the norm estimate

$$\| \left(H_4 - k^4 - i0\right)^{-1} f \|_{W^1_{2,-\delta}} \leq \frac{C}{k^2} \|f\|_{L^2_\delta}.$$

Moreover, this operator is an integral operator with the kernel $G(x, y, k)$ which satisfies the integral equation

$$G(x, y, k) = G^+_k(|x-y|) - \int_{R^n} G^+_k(|x-z|) \left(\vec{W}(z)\nabla_z + V(z)\right) G(z, y, k)\, dz.$$

The solvability of this equation (in the weighted Sobolev spaces) for k big enough follows from the norm estimate (see above). But even more is true, the following solvability result holds in "L^∞" space.

We consider now $n = 3$ (for simplicity)

Proposition

Under the same assumption for $\vec{W}$ and V as in Saito's formula there is a constant $k_0 > 1$ such that the function $G(x, y, k)$ can be defined by the series of iterations

$$G(x, y, k) = \sum_{j=0}^{\infty} G^{(j)}(x, y, k), \quad G^{(0)} = G_k^+$$

which solves this integral equation uniquely, when $k \geq k_0$, and

$$|G(x, y, k) - G_k^+(x, y, k)| \leq \frac{c_0}{4\pi^2 k^3 |x - y|},$$

$$|\nabla G(x, y, k) - \nabla G_k^+(x, y, k)| \leq \frac{c_0}{2\pi^2 k^2 |x - y|}.$$

The knowledge of $G(x,y,k)$ for large values of k,x,y leads to the result.

Theorem

Let $\xi \in R^3$ be arbitrary and fixed. Assume that $\vec{W}$ and V are as above. Then for $\xi = -k\left(\frac{x}{|x|} + \frac{y}{|y|}\right)$

$$\mathcal{F}^{-1}\left(V - \frac{1}{2}\nabla\vec{W}\right)(\xi) =$$

$$= 32\sqrt{2\pi} \lim_{x,y,k\to\infty} k^4|x||y|e^{-ik(|x|+|y|)}\left(G_k^+(|x-y|) - G(x,y,k)\right),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in R^3.

As an immediate corollary we have the uniqueness result for this inverse scattering problem. If $G_1(x,y,k)$ and $G_2(x,y,k)$ are two kernels which correspond to two pairs $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ and if $G_1(x,y,k)$ and $G_2(x,y,k)$ coincide for all x,y,k big enough then $V_1 - \frac{1}{2}\nabla\vec{W}_1 = = V_2 - \frac{1}{2}\nabla\vec{W}_2$ a.e. in R^3.

In the inverse scattering theory (for linear equations) there is very effective and very applicable approximate method which is called the Born approximation. We consider this method for the backscattering problem, i.e., for the problem when the angle of observation θ' is equal to minus angle of incident wave $\theta, \theta' = -\theta$. It can be proved in this case that

$$A(k, \theta, -\theta) \approx -\frac{1}{2}\mathcal{F}(\nabla \vec{W})(2k\theta) + \mathcal{F}(V)(2k\theta), \quad k \to +\infty.$$

This fact justifies the following definition

Definition

The inverse backscattering Born approximation $V_B^b(x)$ for the operator H_4 is defined as

$$V_B^b(x) := \mathcal{F}^{-1}\left(A\left(\frac{k}{2}, \theta, -\theta\right)\right)(x) =$$

$$= \frac{1}{(2\pi)^n} \int\limits_{R_+ \times S^2} k^2 e^{-ik(x,\theta)} A\left(\frac{k}{2}, \theta, -\theta\right) d\theta\, dk.$$

In the absence (in general) of uniqueness of inverse backscattering problem the following result is valid.

Theorem (Reconstruction of singularities)

Suppose that $\vec{W}$ belongs to the weighted Sobolev space $W^1_{p,\delta}(R^3)$ and V belongs to $L^p_\delta(R^3)$ where $3 < p < \infty$ and $\delta > \frac{3}{p'}$. Then the difference

$$V_B^b(x) - \left(V(x) - \frac{1}{2}\nabla \vec{W}(x) \right) \in W_2^t(R^3) \quad (mod C^\infty(R^3))$$

for any $t < \frac{3}{2}$.

This theorem means that using the inverse backscattering Born approximation we can uniquely determine all main singularities of the combination $V(x) - \frac{1}{2}\nabla \vec{W}(x)$ since

$$W_2^{\frac{3}{2}}(R^3) \subset W_p^{\frac{3}{p}}(R^3), \quad 3 < p < \infty.$$

Moreover, any smooth bounded domain ($p = \infty$) can be uniquely determined using this Born approximation.

Especially transparent (and in what sense final) results are obtained in the one-dimensional case. We have in this case the equation

$$H_k u(x) := u^{(4)}(x) + q_1(x, |u|)u'(x) + q_0(x, |u|)u(x) = k^4 u(x), \quad x \in R,$$

where $u(x)$ denotes, for example, the deflection (displacement) at the point x of the ideal beam, $k \neq 0$ is real number and the nonlinear potentials q_1 and q_0 are complex-valued (in general) functions which are integrable in the space coordinate and they are Lipschitz in the space of nonlinearities.

We use the usual Lebesgue $L^p(R)$ spaces and Sobolev spaces $H^s(R)$, with the norm

$$\|f\|^2_{H^s(R)} = \int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\mathcal{F}f(\xi)|^2 \, d\xi,$$

where $\mathcal{F}f$ is the Fourier transform of f

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} f(x) \, dx.$$

In the problems which we consider the main role is played by the special solutions of the equation $H_4 u = k^4 u$, i.e., the solutions of the form

$$u(x, k) = e^{ikx} + u_{sc}(x, k),$$

where the scattered part u_{sc} satisfy the Sommerfeld radiation conditions at the infinity in the one-dimensional case. In that case u_{sc} is the unique solution of the so-called Lippmann-Schwinger integral equation

$$u_{sc} = -\int_{-\infty}^{\infty} G_k^+(|x-y|)(q_1(y, |u|)u'(y) + q_0(y, |u|)u(y))\,dy,$$

where G_k^+ is the outgoing fundamental solution of the one-dimensional be-Helmholtz operator $\frac{d^4}{dx^4} - k^4$, i.e., the kernel of the integral operator

$$(\frac{d^4}{dx^4} - k^4 - i0)^{-1} = \frac{1}{2k^2}\left((-\frac{d^2}{dx^2} - k^2 - i0)^{-1} - (-\frac{d^2}{dx^2} + k^2)^{-1}\right).$$

This function $G_k^+(|x|)$ is equal to

$$G_k^+(|x|) = \frac{ie^{i|k||x|} - e^{-|k||x|}}{4|k|^3}.$$

The derivative of G_k^+ with respect to x can be calculated as

$$(G_k^+(|x|))'_x = \frac{-e^{i|k||x|} + e^{-|k||x|}}{4k^2} sign(x), \quad x \neq 0.$$

It can be mentioned here that G_k^+ satisfies for any $k > 0$ the one-dimensional Sommerfeld radiation conditions at the infinity in the form

$$\left(\frac{\partial}{\partial|x|} - ik\right) G_k^+(|x|) = o(1), \quad |x| \to \infty,$$

$$\left(\frac{\partial}{\partial|x|} - ik\right) \frac{d^2}{dx^2} G_k^+(|x|) = o(1), \quad |x| \to \infty.$$

This function $G_k^+(|x|)$ and its derivative satisfy the following uniform estimates

$$|G_k^+(|x|)| \leq \frac{1}{2|k|^3}, \quad |(G_k^+(|x|))'_x| \leq \frac{1}{2k^2}.$$

Using these estimates for G_k^+ we can prove that for $|k| \geq k_0$ there is a unique solution of this Lippmann-Schwinger equation as the limit in $W_\infty^1(R)$ of $u_{sc} = \lim\limits_{j\to+\infty} u_{sc}^{(j)}$, where $u_j = u_0 + u_{sc}^{(j)}$ and

$$u_{sc}^{(j)} = -\int\limits_{-\infty}^{\infty} G_k^+(|x-y|)(q_1(y,|u_{j-1}|)u'_{j-1} + q_0(y,|u_{j-1}|)u_{j-1})\,dy,$$

for $j = 1, 2, ...$ and $u_{sc}^{(0)} = 0$. This solution satisfies the estimates

$$\|u - u_j\|_{L^\infty(R)} \leq \frac{C_j}{|k|^{2+j}}, \quad \|u' - u'_j\|_{L^\infty(R)} \leq \frac{C_j}{|k|^{j+1}}$$

uniformly in $|k| \geq k_0$, where k_0 depends on the norm in $L^1(R)$ of the potentials q_0 and q_1 in the space coordinate uniformly in $|u|$ in some range of its changes. In particular,

$$\|u_{sc}\|_{W_\infty^1(R)} \leq \frac{2C_0}{|k|}.$$

These solutions for fixed positive $k \geq k_0$ admit the following asymptotical representations when $x \to \pm\infty$:

$$u(x,k) = a(k)e^{ikx} + o(1), \quad u(x,k) = e^{ikx} + b(k)e^{-ikx} + o(1),$$

respectively, where the coefficients $a(k)$ and $b(k)$ are defined as

$$a(k) = 1 - \frac{i}{4k^3}\int_{-\infty}^{\infty} e^{-iky}(q_1(y,|u|)u'(y) + q_0(y,|u|)u(y))\,dy$$

$$b(k) = -\frac{i}{4k^3}\int_{-\infty}^{\infty} e^{iky}(q_1(y,|u|)u'(y) + q_0(y,|u|)u(y))\,dy$$

and they are called the "transmission" and the "reflection" coefficients, respectively. Defining the solution $u(x,k)$ for negative k as $u(x,k) := \overline{u(x,-k)}$ with $u'(x,k) := \overline{u'(x,-k)}$ we obtain that $a(k) = \overline{a(-k)}$ and $b(k) = \overline{b(-k)}$ for negative k.

Thus, for negative values of $k \le -k_0$

$$a(k) = 1 - \frac{i}{4k^3} \int_{-\infty}^{\infty} e^{-iky} (\overline{q_1(y, |u|)} u'(y) + \overline{q_0(y, |u|)} u(y))\, dy$$

$$b(k) = -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} (\overline{q_1(y, |u|)} u'(y) + \overline{q_0(y, |u|)} u(y))\, dy$$

We are interested further only the reflection coefficient. We put $b(k) = 0$ for $|k| < k_0$. Hence, we have well-defined the reflection coefficient $b(k)$ for all $k \in R$. The inverse problem that considered here is to extract some information about the potentials q_0 and q_1 (more precisely, about $q_0(x, 1)$ and $q_1(x, 1)$) by the knowledge of the reflection coefficient $b(k)$.

The properties of $u(x,k)$ ($u_{sc} \to 0$) allow us to conclude that for $k \to +\infty$

$$b(k) \approx -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{i2ky}(ikq_1(y,1) + q_0(y,1))\,dy.$$

Using integration by parts in the latter formula we obtain the approximation

$$b(k) \approx -\frac{i\sqrt{2\pi}}{4k^3}\mathcal{F}(\beta)(2k)$$

or

$$b\left(\frac{k}{2}\right) \approx -\frac{i2\sqrt{2\pi}}{k^3}\mathcal{F}(\beta)(k),$$

where $\beta(y) = -\frac{q_1'(y,1)}{2} + q_0(y,1)$. This asymptotic leads to the direct Born approximation

$$u_B(x,k) = e^{ikx} - \frac{i\sqrt{2\pi}}{4k^3}\mathcal{F}(\beta)(2k)e^{-ikx},$$

i.e., we may substitute our scattering solution by this formula which includes only the potential β.

The asymptotical representations for $u(x, k)$ for large x can be considered as the analogue of the one-dimensional Sommerfeld radiation conditions for this operator of order 4. But what is more important, the asymptotic of $u(x, k)$ (or of b) for large k justifies the following definition which plays the crucial role in the inverse scattering problem.

Definition

The inverse scattering Born approximation $V_B(x)$ of the potential β is defined by

$$V_B(x) := \mathcal{F}^{-1}\left(\frac{i}{2\sqrt{2\pi}}k^3 b\left(\frac{k}{2}\right)\right),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on the line.

This equality must be considered in the sense of tempered distributions.

Since $u = \lim\limits_{j\to+\infty} u_j$ in $W^1_\infty(R)$ we may introduce a sequence for the inverse Born approximation V_B by

$$V_{B,j} := \mathcal{F}^{-1}\left(\frac{i}{2\sqrt{2\pi}}k^3 b_j\left(\frac{k}{2}\right)\right), \quad j = 0, 1, 2, \dots$$

where the approximation for reflection coefficient is defined for positive and negative k as

$$b_j(k) = -\frac{i}{4k^3}\int\limits_{-\infty}^{\infty} e^{iky}(q_1(y, |u_j|)u_j'(y) + q_0(y, |u_j|)u_j(y))\,dy, \quad k > 0,$$

and

$$b_j(k) = -\frac{i}{4k^3}\int\limits_{-\infty}^{\infty} e^{iky}(\overline{q_1(y, |u_j|)}u_j'(y) + \overline{q_0(y, |u_j|)}u_j(y))\,dy, \quad k < 0,$$

respectively.

Then the reflection coefficient $b(k)$ can be obtained as the limit

$$b(k) = \lim_{j\to\infty} b_j(k)$$

uniformly in $|k| \geq k_0$. But it is more remarkable that

$$V_B(x) - V_{B,j}(x) \in H^s(R), \quad j = 0, 1, 2, ...,$$

for any $s < j + \frac{1}{2}$. There is also the following formula (it can be easily checked) :

$$V_{B,0}(x) = \Re(\beta)(x) + \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\Im(\beta)(y)}{x-y}\, dy \quad (mod C_0(R)),$$

where $C_0(R)$ denotes the set of all continuous functions that vanish at the infinity. The same formula holds also for $V_{B,1}$ but it is more involved and we need more conditions for the nonlinearities.

We assume now that

$$q_0(x, 1+s) = q_0(x, 1) + q_0^*(x, s_0^*)s,$$

$$q_1(x, 1+s) = q_1(x, 1) + q_1^*(x, s_1^*)s + q_1^{**}(x, s_1^*)s^2, \quad |s_0^*|, |s_1^*| < |s|,$$

where $q_0(x, 1)$, $q_1(x, 1)$ belong to $L^1(R)$ and $W_1^1(R)$, respectively, and

$$|q_0^*(x, s_0^*)| \leq h_0(x), \quad |q_1^{**}(x, s_1^*)| \leq h_1(x)$$

with h_0 and h_1 from $L^1(R)$ and with $q_1^*(x, 1)$ from $L^1(R) \cap L^p(R)$ for some $p > 1$. Under these conditions we have also that

$$V_{B,1}(x) = \Re(\beta)(x) + \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\Im(\beta)(y)}{x-y}\, dy \quad (mod C_0(R)).$$

We are in the position now to formulate the main result.

Theorem

If $q_0(x,|u|)$ $(q_0(x,1) \in L^1)$ *and* $q_1(x,|u|)(q_1(x,1) \in W_1^1)$ *are as before then the inverse scattering Born approximation* V_B *admits the representation*

$$V_B(x) = \Re(\beta)(x) + \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\Im(\beta)(y)}{x-y}\, dy \quad (mod C_0(R))$$

where $\beta(x) = -\frac{q_1'(x,1)}{2} + q_0(x,1)$.

Corollary

If q_0, q_1 *and* β *are as in Theorem and in addition are real-valued, then the difference*

$$V_B(x) - \beta(x) \in C_0(R).$$

In order to obtain this result we were needed to calculate precisely the so-called "first" nonlinear term in the Born sequence and investigate its smoothness (using this precise form). Without it we are not able to obtain the required result. Indeed, let us substitute $u(x,k) = e^{ikx} + u_{sc}(x,k)$ into $b(k)$ then it can be easily seen that

$$V_B(x) = \beta(x) + V_{rest}(x) \quad (mod \quad C^{\infty}(R)).$$

In order to estimate the smoothness of V_{rest} we first remark that

$$|k^3 b_{sc}(k)| \le \frac{C}{|k|}, \quad |k| \ge \sqrt{2c_0},$$

where b_{sc} is a part of $b(k)$ which corresponds to u_{sc}. This implies that

$$\|V_{rest}\|^2_{H^t(R)} \le C \int\limits_{\sqrt{2c_0}}^{\infty} \frac{(1+k^2)^t}{k^2}\, dk < \infty$$

for any $t < \frac{1}{2}$. Using then Sobolev imbedding theorem we obtain that for arbitrary positive ϵ (small enough)

$$V_{rest}(x) \in W_p^{\frac{1}{p}-\epsilon}(R), \quad 2 \le p < \infty.$$

There is an explanation why we need to investigate the first nonlinear term in the Born series (in addition). The smoothness of V_{rest} shows that we are not able to consider $p = \infty$. It means that if we do not use the first nonlinear term then it is possible to reconstruct any singularity from L^p with $p < \infty$ but not from L^∞ (i.e., jumps).
There is one more corollary.

Corollary

If q_0, q_1 and β are as in Theorem and $\Im(\beta) \in H^r(R)$ for $r > \frac{1}{2}$, then the difference

$$V_B(x) - \Re(\beta)(x) \in C_0(R).$$

Remark

According to these corollaries, if q_1 is smooth enough (or $\Im(\beta)$ in the complex case), then we can recover any local L^p-singularities and any jumps of the unknown potential $q_0(x,1)$ using the Born approximation.

Let us consider now linear one-dimensional 4th order equation in the self-adjoint form

$$u^{(4)}(x) + 2iq(x)u'(x) + iq'(x)u(x) + V(x)u(x) = k^4 u(x), \quad x \in R,$$

where q and V are real-valued functions. In that case we can prove the following result (which is much better than the previous theorem and corollaries w.r.t. the reconstruction of the singularities of unknowns)

Theorem

Assume that the potentials $q(x)$ and $V(x)$ belong to the Sobolev space $W_1^1(R)$ and the Lebesgue space $L^1(R)$, respectively. Then

$$V_B(x) - V(x) \in C_0(R),$$

i.e. the difference is continuous everywhere.

Remark

This theorem shows that all singularities and all jumps of the unknown potential $V(x)$ can be obtained exactly by the inverse scattering Born approximation. In particular (this is the most practical application of this result), we can prove that for the function $V(x)$ being the characteristic function of an interval on the line, this interval is uniquely determined by the scattering data. Moreover, in the class of such potentials we have uniqueness result of the inverse scattering problem.

Thanks very much for your attention !