Localized structures in nonlocal NLS-type systems

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Seminar at Department of Mathematics, University of Athens

This research has been co-financed by the European Union (European Social Fund–ESF) and Greek national funds-Research Funding Program: Thales and IRSES Grant of EU

- joint work with
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- This research has been co-financed by the European Union (European Social Fund–ESF) and Greek national funds-Research Funding Program: Thales (2013-2016) and IRSES Grant of EU (2014-2017)

Outline of the talk

1. Localized Travelling Waves in nonlocal NLS Lattices
   - Nonlinear Waves in Discrete NLS
   - Travelling Waves in DNLS model
   - Quasi-Periodic Travelling Waves in nonlocal DNLS
2. Nonlocal NLS equation
3. Conclusions
4. References

- Discrete nonlinear Schrödinger equation (DNLS) in 1-D

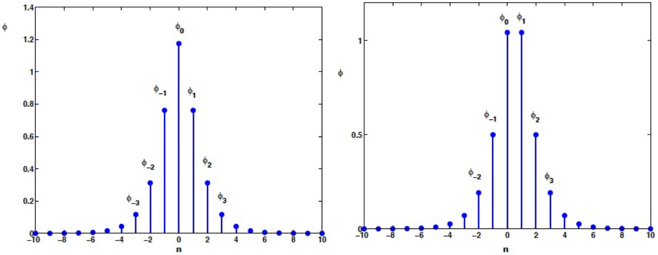
$$i\dot{\psi}_n(t) + \frac{\psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)}{h^2} + f\left(\psi_{n+1}, \psi_n, \psi_{n-1}\right) = 0$$

- General nonlinear term f :
  1. Cubic DNLS, $f = |\psi_n|^2 \psi_n$
  2. Ablowitz-Ladik, $f = |\psi_n|^2 \left(\psi_{n+1} + \psi_{n-1}\right)$
  3. Salerno model
  $$f = 2\alpha |\psi_n|^2 \psi_n + (1-\alpha) |\psi_n|^2 \left(\psi_{n+1} + \psi_{n-1}\right)$$
  4. Saturable nonlinearity
  $$f = \frac{\nu |\psi_n|^2}{1 + \mu |\psi_n|^2} \psi_n$$
  5. Translationally invariant model
  $$\begin{aligned} f =& \alpha_1 |\psi_n|^2 \psi_n + \alpha_2 |\psi_n|^2 \left(\psi_{n+1} + \psi_{n-1}\right) + \alpha_3 \psi_n^2 \left(\bar{\psi}_{n+1} + \bar{\psi}_{n-1}\right) \\ & \ldots + \alpha_{10} \left(|\psi_{n+1}|^2 \psi_{n-1} + |\psi_{n-1}|^2 \psi_{n+1}\right) \end{aligned}$$

Stationary solutions $\psi_n(t) = \phi_n e^{i\omega t}$ satisfy the second-order difference map

$$-\omega\phi_n + \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + f\left(\phi_{n+1}, \phi_n, \phi_{n-1}\right) = 0$$

Two solutions: on-site and inter-site discrete solitons



- Existence Solitons in DNLS with saturable nonlinearity,
- two types of solutions: *periodic* and *vanishing at infinity*,
- Calculus of Variations, Nehari manifolds and Mountain Pass argument.

Problem set-up

Consider the discrete NLS with saturable nonlinearity

$$i\dot{\psi}_n + \psi_{n+1} + \psi_{n-1} - 2\psi_n + \frac{\nu\,|\psi_n|^2}{1+\mu\,|\psi_n|^2}\psi_n = 0$$

where $\mu > 0$ and $\nu \neq 0$.
Standing wave solution : $\psi_n = \exp(-i\omega t)u_n$, $u_n \in \mathbb{R}$ Then,

$$-\Delta u_n - \omega u_n = f(u_n), \quad f(u) = \frac{\nu u^3}{1+\mu u^2}.$$

We consider two types of solutions:

(a) k-periodic, i.e. $u_{n+k} = u_n$;

(b) vanishing at infinity, i.e. $\lim_{n\to\pm\infty} u_n = 0$, $u_n \in l^2$.

Existence Theorem

Theorem

Suppose that either

$$\omega < 0 \quad \text{and} \quad \omega + \nu/\mu > 0 \quad \text{or} \quad \omega > 4 \quad \textit{and} \quad \omega + \nu/\mu < 4.$$

Then for every $k \geq 2$ there exist two nontrivial k-periodic solutions $\pm u^{(k)}$ as well as two nontrivial solution $\pm u \in l^2$, of equation

$$-\Delta u_n - \omega u_n = f(u_n).$$

If $\omega < 0$, then u_k and u are strictly positive. Moreover, the solution u decays exponentially at infinity, i.e.

$$|u_n| \leq C\mathrm{e}^{-a|n|},$$

with $C > 0$ and $a > 0$.

Variational Approach

- Introduce functionals J_k and J on the spaces of k-periodic sequences and l^2, respectively, whose critical points are solutions of equation $-\Delta u_n - \omega u_n = f(u_n)$.
- To produce nontrivial critical points, we use Nehari manifold N_k, N. These are C^1 submanifolds of the spaces X_k, X.
- On Nehari manifolds J_k and J are bounded below by positive constants.
- We minimize J_k and J over N_k, N.
- A minimum point of the functional over the Nehari manifold, this is automatically a solution of equation $-\Delta u_n - \omega u_n = f(u_n)$.
- In the periodic case N_k is finite dimensional. The second part of our Theorem concerning l^2 solutions is more involved because the functional J does not satisfy the Palais–Smale condition. Our idea is to pass to the limit as $k \to \infty$.

Assumptions

We consider the following equation

$$-\Delta u - \omega u = \sigma f(u), \quad \sigma = \pm 1, \quad \omega < 0$$

We suppose that the nonlinearity $f(t)$ satisfies the following assumptions in which

$$F(t) = \int_0^t f(s)\mathrm{d}s$$

is the primitive function of $f(t)$

($h1$) $f(t) = o(t)$ as $t \to 0$,

($h2$) $\lim_{t\to\pm\infty} \frac{f(t)}{t} = l < \infty$,

($h3$) $f \in C^1(\mathbb{R})$ and $f(t)t < f'(t)t^2$ for $t \neq 0$,

($h4$) $\lim[\frac{1}{2}f(t)t - F(t)] = \infty$ as $t \to \pm\infty$.

X_k the space of all k–periodic sequences, $X = l^2$ with $(\cdot,\cdot)_k$ and $(\cdot,\cdot)$ the natural inner products.

The functionals J and J_k

- We define the *bounded* and *self-adjoint* operator $L = -\Delta - \omega$ acting either on X_k or on X.
- we introduce the action functionals

$$J_k(u) = \frac{1}{2}(\, Lu, u\,)_k - \sum_{Q_k} F(u_n), \quad Q_k = \{\, k \in \mathbb{Z} \,:\, -[\frac{k}{2}] \le n \le k - [\frac{k}{2}] - 1 \,\}$$

$$J(u) = \frac{1}{2}(\, Lu, u\,) - \sum_{\mathbb{Z}} F(u_n)$$

- J_k, J are C^1–functionals and the derivatives are given by

$$\begin{aligned} \langle\, J_k'(u), v\,\rangle &= (\, Lu, v\,)_k - \sum_{Q_k} f(u_n)v_n, \\ \langle\, J'(u), v\,\rangle &= (\, Lu, v\,) - \sum_{\mathbb{Z}} f(u_n)v_n \end{aligned}$$

for any $v \in X_k$ and $v \in X$ respectively.

Remark on J and J_k

Critical points of J_k and J are k-periodic and l^2-solutions of

$$-\Delta u - \omega u = \sigma f(u), \quad \sigma = \pm 1, \quad \omega < 0$$

- $\sigma = 1$, a solution of the difference equation is a *ground state* solution if it **minimizes** the action among all solution of the same type.
- $\sigma = -1$ ground states are solutions that **maximize** the action.

Nehari Manifolds associated with functionals J, J_k

Nehari manifolds are defined as follows

$$N_k = \{ v \in X_k : \langle J_k'(u), v \rangle = 0, \ v \neq 0 \} \subset X_k$$

$$N = \{ v \in X : \langle J'(u), v \rangle = 0, \ v \neq 0 \} \subset X$$

Let $I_k(u) = \langle J_k'(u), v \rangle$, $\quad I(u) = \langle J'(u), v \rangle$. These are C^1 functionals and

$$\langle I_k'(u), v \rangle = 2(Lu, v)_k - \sum_{Q_k} [f(u_n) + f'(u_n)u_n] v_n,$$

$$\langle I'(u), v \rangle = 2(Lu, v) - \sum_{\mathbb{Z}} [f(u_n) + f'(u_n)u_n] v_n.$$

Lemma

The sets N_k and N are nonempty closed C^1 submanifolds in X_k and X, respectively. The derivatives I_k' and I' are nonzero on corresponding Nehari manifolds. Moreover, there exists $\beta_0 > 0$ such that $\|u\|_k \geq \beta_0, u \in N_k$, and $\|u\| \geq \beta_0, u \in N$.

Minimization problem

The tangent spaces $\mathrm{T}_u N_k$ and $\mathrm{T}_u N$ at $u \in N_k$ or $u \in N$, respectively, are

$$\mathrm{T}_u N_k = \left\{ v \in X_k \,:\, \langle\, I_k'(u), v \,\rangle = 0 \,\right\}, \quad \mathrm{T}_u N = \left\{ v \in X \,:\, \langle\, I'(u), v \,\rangle = 0 \,\right\},$$

and the line $\mathbb{R}u = \{\, tu \,:\, t \in \mathbb{R} \,\}$ is a transverse line.

Lemma

For $u \in N_k$ the function $J_k(tu)$, $t > 0$, has a unique critical point at $t = 1$ which is, actually, a global maximum. The same statement holds for N and J.

Minimum points of J_k and J on corresponding Nehari manifolds are solutions of equation $-\Delta u - \omega u = \sigma f(u)$. To prove our main result, we consider the following minimization problems

$$m_k = \inf\{J_k(v) : v \in N_k\} \quad \textit{and} \quad m = \inf\{J(v) : v \in N\}\,.$$

Minimization results (Pankov & Rothos, 2008)

Theorem

Assume that the nonlinearity satisfies $(h1) - (h4)$ *and either* $\sigma = 1$, $\omega < 0$ *and* $l + \omega > 0$, *or* $\sigma = -1$, $\omega > 4$ *and* $-l + \omega < 4$. *Then for every* $k > 1$ *equation* $Lu = \sigma f(u)$ *possesses a nontrivial* k*-periodic ground state solution* $u^{(k)} \in X_k$. *Moreover, in case when* f *is odd, i.e.* $f(-u) = -f(u)$, *and* $\sigma = 1$ *there are two nontrivial ground states* $\pm u^{(k)}$ *and* $\pm u^{(k)} > 0$.

Theorem

- *There exists a nontrivial ground state solution* $u \in l^2$ *and* u *decays exponentially fast*

  $$|u_n| \leq Ce^{-\alpha|n|}$$

  for some $\alpha > 0$ *and* $C > 0$. *Moreover, if* f *is odd and* $\sigma = 1$, *then* $-u$ *is also a ground state solution and* $\pm u > 0$.
- *Let* $u^k \in X_k$ *be the solution obtained in that Theorem. Then there exists a ground state solution* $u \in l^2$ *and* $b_k \in \mathbb{Z}$ *such that*

  $$\|u_k(\cdot + b_k) - u(\cdot)\|_k \to 0$$

  as $k \to \infty$.

Two specific nonlinearities

Two nonlinearities not covered by our results:

- $$f(u) = \frac{\nu|u|^{p+1}}{1+\mu|u|^p}, \quad p \neq 2$$

- $$f(u) = \chi(1-\exp(-a|u|^p))u \quad \chi > 0, p > 0, a > 0.$$

Replace

$$(h4) \quad \lim[\frac{1}{2}f(t)t - F(t)] = \infty, \quad as \quad t \to \pm\infty$$

by

$$(h5) \quad \text{the} \quad \text{function} \quad g(t) = f(t) - lt \quad \text{is} \quad \text{bounded.}$$

Results under $(h5)$

Theorem

Assume that the nonlinearity satisfies $(h1)-(h3)$ and $(h5)$, and either $\sigma = 1$, $\omega < 0$, $l + \omega > 0$ and $l + \omega \notin \sigma_k$, or $\sigma = -1$, $\omega > 4$, $-l + \omega < 4$ and $-l + \omega \notin X_k$. Then for every $k > 1$ equation $Lu = \sigma f(u)$ possesses a nontrivial k-periodic ground state solution $u^{(k)} \in X_k$. Moreover, in case when f is odd, i.e. $f(-u) = -f(u)$, and $\sigma = 1$ there are two nontrivial ground states $\pm u^{(k)}$ and $\pm u^{(k)} > 0$.

Theorem

Assume that the nonlinearity satisfies $(h1)-(h3)$ and $(h5)$, and either $\sigma = 1$, $\omega < 0$ and $l + \omega > 0$, or $\sigma = -1$, $\omega > 4$ and $-l + \omega < 4$. Then there exists a nontrivial ground state solution $u \in l^2$ of equation $Lu = \sigma f(u)$, and u decays exponentially fast

$$|u_n| \leq Ce^{-\alpha|n|}$$

for some $\alpha > 0$ and $C > 0$. Moreover, if f is odd and $\sigma = 1$, then $-u$ is also a ground state solution and $\pm u > 0$.

Traveling waves in lattices

DNLS

$$i\dot{\psi}_n(t) + \frac{\psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)}{h^2} + f\left(\psi_{n+1}, \psi_n, \psi_{n-1}\right) = 0$$

Moving into the travelling frame $z = hn - 2ct$ gives a differential advance-delay equation. If $\psi_n(t) = \phi(z)e^{i\omega t}$

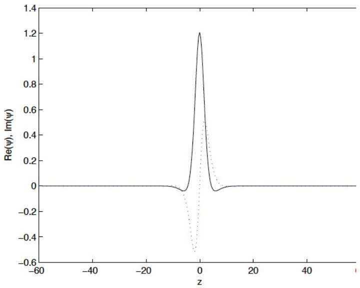
$$\begin{aligned} 2ic\phi'(z) = & \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} - \omega\phi(z) \\ & + f(\phi(z+h), \phi(z)\phi(z-h)) \end{aligned}$$

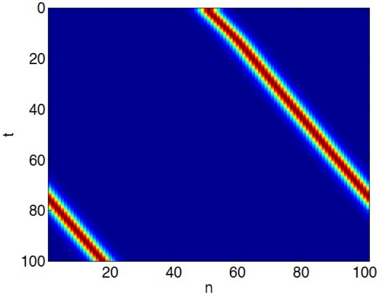
Travelling waves satisfy the constraints:

$$u_1(t) = u_0(t-\tau)e^{i\theta}, \quad u_2(t) = u_0(t-2\tau)e^{2i\theta}, \qquad \text{etc.}$$

Radiationless Solitons

- Localised solutions to a differential difference equation.
- Waves travel across a lattice without shedding any radiation.
- Homoclinic orbit to the zero state in a travelling frame.





Difficulties

- In general, traveling wave solutions are weakly non-local.
- Eigenvalues on the imaginary axis in the linear spectrum give rise to radiation modes.
- Number of eigenvalues is finite for $c \neq 0$ but increases as $c \to 0$
- In general there is at least one resonance.
- Amplitude of radiation modes are generally exponentially small in terms of a bifurcation parameter.

Reformulation of existence problem

Introduce parameters $\kappa \in \mathbb{R}_+, \beta \in [0, \pi]$

$$\omega = \frac{2}{h}\beta c + \frac{2}{h^2}(\cos(\beta)\cosh(\kappa) - 1),$$
$$c = \frac{1}{h\kappa}\sin(\beta)\sinh(\kappa)$$

Scale out h using $\phi(z) = \frac{1}{h}\Phi(Z)e^{i\beta Z}, Z = \frac{z}{h}$. New differential advance-delay equation

$$\begin{gathered} i\sin(\beta)\left(2\frac{\sinh(\kappa)}{\kappa}\frac{d\Phi(Z)}{dZ} - \Phi(Z+1) + \Phi(Z-1)\right) \\ + \cos(\beta)(2\cosh(\kappa)\Phi(Z) - \Phi(Z+1) - \Phi(Z-1)) \\ -f\left(\Phi(Z+1)e^{i\beta}, \Phi(Z), \Phi(Z-1)e^{-i\beta}\right) = 0 \end{gathered}$$

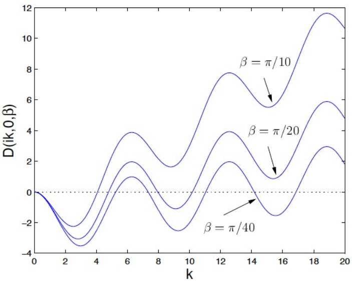
Linear Spectrum

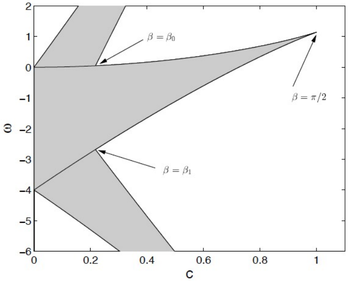
Dispersion relation for the linear equation is obtained using $\Phi(Z) = e^{pZ}$

$$D(p;\kappa,\beta) \equiv 2\cos(\beta)(\cosh(p) - \cosh(\kappa)) + 2i\sin(\beta)\left(\sinh(p) - \frac{\sinh(\kappa)}{\kappa}p\right) = 0$$

- there are finitely many imaginary roots $p = ik_n, n = 1, \ldots, m, \forall\kappa > 0$ and $\beta \in (0, \pi)$
- if $\kappa = 0$, there exists a double root $k = 0$ of $D(ik; 0, \beta)$
- if $\kappa = 0$ and $\beta = \pi/2$ the zero root $k = 0$ is a triple root of $D(ik; 0, \beta)$
- if $\kappa = 0$ and $\beta \in \left(\beta_0, \frac{\pi}{2}\right)$ with $\beta_0 \approx \frac{\pi}{13}$, there exists only one imaginary root besides the double zero root.

Linear Spectrum





Methods

1. Analysis of the normal form near $\kappa = 0$ and $\beta = \pi/2$ (D.P.,V.Rothos, Physica D 202, 16 (2005)).
2. Analysis of persistence of homoclinic orbits near the line $\kappa > 0$ and $\beta = \pi/2$ (D.P.,T.Melvin, A. Champneys, Physica D 236, 22 (2007)).
3. Stokes constant computation. Analysis of Stokes phenomena in a beyond all orders expansion for $\kappa = 0$ and $\beta \neq \pi/2$ (O. Oxtoby, I. Barashenkov, nlin/0610059 (2006)).
4. Pseudo-spectral decomposition (for FK-model (Rothos et al Physica D 2003)) Numerical solutions of the differential advance-delay equation for $\kappa > 0$ and any β (D.P.,T.Melvin, A. Champneys, Physica D 236, 22 (2007)).

- Problem set-up
- Variational Methods to prove the existence of periodic traveling waves
- Bifurcation of periodic traveling waves in nonlocal DNLS equations

Problem set-up. Advance-Delay Eqn

In some approximation the equation of motion is the nonlocal discrete NLS

$$\imath \dot{u}_n = \sum_{m \neq n} J_{n-m}(u_n - u_m) + |u_n|^2 u_n, \quad n \in \mathbb{Z}\,, \tag{1.1}$$

where the long-range dispersive coupling is taken to be either exponentially $J_n = J\mathrm{e}^{-\beta|n|}$ with $\beta > 0$, or algebraically $J_n = J|n|^{-s}$ with $s > 0$, decreasing with the distance n between lattice sites. In both cases the constant J is normalized such that $\sum_{n=1}^{\infty} J_n = 1$, for all β or s. The parameters β and s are introduced to cover different physical situations from the nearest-neighbor approximation ($\beta \to \infty, s \to \infty$) to the quadrupole-quadrupole ($s = 5$) and dipole-dipole ($s = 3$) interactions.

The Hamiltonian H and the number of excitations N

$$H = \frac{1}{2} \sum_{n,m\in\mathbb{Z}} J_{n-m}|u_n - u_m|^2 - \frac{1}{2} \sum_{n\in\mathbb{Z}} |u_n|^4, \quad \text{and} \quad N = \sum_{n\in\mathbb{Z}} |u_n|^2 \tag{1.2}$$

are conserved quantities corresponding to the set of (1.1).
It should be also noted that the derivation of a discrete equation from the Gross-Pitaevskii equation produces at the intermediate step a fully nonlocal discrete NLS equation for the coefficients of the wave function expansion over the complete set of the Wannier functions. (Panayotaros et al, 2014)

We consider the DNLS equation with nonlocal linear part:

$$\mathrm{i}\dot{u}_n = \sum_{j\in\mathbb{N}} a_j \Delta_j u_n + f(u_{n+1}, u_n, u_{n-1}), \quad u_n : \mathbb{R}_+ \to \mathbb{C}, n \in \mathbb{Z} \tag{1.3}$$

where $f \in C(\mathbb{R}_+, \mathbb{R})$ for $\mathbb{R}_+ := [0, \infty)$, $f(0) = 0$. $\exists\ s > 0$, $\mu > 1$, $c_1 > 0$, $c_2 > 0$ and $\bar{r} > 0$ such that

$$(H1) \quad |f(w)| \leq c_1(w^s + 1), \quad c_2(w^{s+1} - 1) \leq F(w), \quad \mu F(w) - \bar{r} < f(w)w$$

for any $w \geq 0$, where $F(w) = \int_0^w f(z)dz$. Furthermore, $\limsup_{w\to 0_+} f(w)/w^{\tilde{s}} < \infty$ for a constant $\tilde{s} > 0$.

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for any $w \geq 0$, where $F(w) = \int_0^w f(z)dz$. Furthermore, $\limsup_{w\to 0_+} f(w)/w^{\tilde{s}} < \infty$ for a constant $\tilde{s} > 0$.

Furthermore, (1.3) can be rewritten into a standard form

$$\mathrm{i}\dot{u}_n = \sum_{m\neq n} a_{|m-n|}\left(u_m - u_n\right) + f(|u_n|^2)u_n, \quad n \in \mathbb{Z}. \tag{1.4}$$

It is well known that (1.4) conserves two dynamical invariants

$$\sum_{n\in\mathbb{Z}} |u_n|^2 \quad \text{– the norm,}$$

$$\sum_{n\in\mathbb{Z}} \left[-\frac{1}{2}\sum_{m\neq n} a_{|m-n|}\left|u_m - u_n\right|^2 + F(|u_n|^2) \right] \quad \text{– the energy.}$$

We are interested in the existence of traveling wave solutions $u_n(t) = U(n-\nu t)e^{\mathrm{i}t}$ of (1.3) or (1.4) with a quasi periodic function $U(z)$, $z = n - \nu t$ and some $\nu \neq 0$.

- We study the existence of traveling wave solutions of the form $u_n(t) = U(n - \nu t)e^{\mathrm{i}t}$, i.e. we are interested in the equation

$$-\nu \mathrm{i} U'(z) = \sum_{j\in\mathbb{N}} a_j \partial_j U(z) + f(|U(z)|^2)U(z)\,, \tag{1.5}$$

where $z = n - \nu t$, $\nu \neq 0$ and $\partial_j U(z) := U(z+j) + U(z-j) - 2U(z)$. We are interested in the existence of quasi periodic solutions $U(z)$ of (1.5).

- We introduce a function

$$\Phi(x) := \frac{4}{x} \sum_{j\in\mathbb{N}} a_j \sin^2\left[\frac{x}{2} j\right]\,.$$

where $\Phi \in C(\mathbb{R}\setminus\{0\}, \mathbb{R})$, Φ is odd, and $\Phi(x) \to 0$ as $|x| \to \infty$.

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Critical Point Theory

Let H be a Hilbert space and let $J \in C^1(H, \mathbb{R})$. Suppose $H = H_1 \oplus H_2$ for closed linear subspaces, and let $e_1, e_2, \cdots$ be the orthonormal basis of H_1.
Let us put $H_n^1 := \operatorname{span}\{e_1, e_2, \cdots, e_n\}$ and $H_n := H_n^1 \oplus H_2$.
Let P_n be the orthogonal projection of H onto H_n. Set $J_n := J/H_n$ - the restriction of functional J on subspace H_n - and so $\nabla J_n(x) = P_n \nabla J(x)$ if $x \in H_n$.

Definition

If there are two positive constants α *and* β *such that*

$$\begin{aligned} J(x) &\geq 0 \quad \forall x \in \{x \in H_1 \mid \|x\| \leq \beta\}, \\ J(x) &\geq \alpha \quad \forall x \in \{x \in H_1 \mid \|x\| = \beta\}, \\ J(x) &\leq 0 \quad \forall x \in \{x \in H_2 \mid \|x\| \leq \beta\}, \\ J(x) &\leq -\alpha \quad \forall x \in \{x \in H_2 \mid \|x\| = \beta\}, \end{aligned}$$

then J *is said to satisfy the local linking condition at* 0.

Definition

We shall say that J satisfies the Palais-Smale $(PS)^*$-condition if any sequence $\{x_n\}_{n\in\mathbb{N}}$ in H such that $x_n \in H_n$, $J(x_n) \le c < \infty$ and $P_n\nabla J(x_n) = \nabla J_n(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Theorem (Li & Szulkin, J Diff Eqn **112**, 1994)

Suppose

(I_1) $J \in C^1(H, \mathbb{R})$ *satisfies $(PS)^*$-condition.*

(I_2) J *satisfies the local linking condition at* 0.

(I_3) $\forall n$, $J_n(x) \to -\infty$ *as* $\|x\| \to \infty$ *and* $x \in H_n$.

(I_4) $\nabla J = A + C$ *for a bounded linear self-adjoint operator* A *such that* $AH_n \subset H_n$, $\forall n \in \mathbb{N}$ *and* C *is a compact mapping.*

Then J *possesses a critical point* $\bar{x}$ *with* $|J(\bar{x})| \ge \alpha$.

Existence Results

Theorem (Feckan & Rothos, 2009)

Let (H1) hold and $T > 0$. Then for almost each $\nu \in \mathbb{R} \setminus \{0\}$ and any rational $r \in \mathbb{Q} \cap (0,1)$, there is a nonzero periodic traveling wave solution $u_n(t) = U(n - \nu t)e^{\mathrm{i}t}$ of (1.3) with $U \in C^1(\mathbb{R}, \mathbb{C})$ and such that

$$U(z+T) = e^{2\pi r \mathrm{i}} U(z), \ \forall z \in \mathbb{R}. \tag{1.6}$$

Moreover, for any $\nu \in \mathbb{R} \setminus \{0\}$ there is at most a finite number of $\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_m \in (0,1)$ such that equation

$$-\nu = \Phi\left(\frac{2\pi}{T}(\bar{r}_j + k)\right)$$

has a solution $k \in \mathbb{Z}$. Then for any $r \in (0,1) \setminus \{\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_m\}$ there is a nonzero quasi periodic traveling wave solution $u_n(t) = U(n - \nu t)e^{\mathrm{i}t}$ with the above properties. In particular, for any $|\nu| > \bar{R}$ and $r \in (0,1)$, there is such a nonzero quasi periodic traveling wave solution.

Bifurcation Results

When a nonresonance condition of Theorem 2 fails, then we have the following bifurcation results.

Theorem (Feckan & Rothos, 2012)

Suppose $f \in C^2(\mathbb{R}_+, \mathbb{R})$ *with* $f(0) = 0$. *If there are* $\bar{r}_1 \in (0,1)$, $\nu \in \mathcal{R}\Phi \setminus \{0\}$ *and* $T > 0$ *such that all solutions* $k_1, k_2, \cdots, k_{m_1} \in \mathbb{Z}$ *of equation*

$$-\nu = \Phi\left(\frac{2\pi}{T}(\bar{r}_1 + k)\right)$$

are either nonnegative or negative, and $m_1 > 0$. *Then for any* $\varepsilon > 0$ *small there are* m_1 *branches of nonzero quasi periodic traveling wave solutions* $u_{n,j,\varepsilon}(t) = U_{j,\varepsilon}(n - \nu_\varepsilon t)e^{\mathrm{i}t}$ *of* (1.3) *with* $U_{j,\varepsilon} \in C^1(\mathbb{R}, \mathbb{C})$, $j = 1, 2, \cdots, m_1$, *and nonzero velocity* ν_ε *satisfying* $U_{j,\varepsilon}(z + T) = e^{2\pi\bar{r}_1 ß}U_{j,\varepsilon}(z)$, $\forall z \in \mathbb{R}$ *along with* $\nu_\varepsilon \to \nu$ *and* $U_{j,\varepsilon} \rightrightarrows 0$ *uniformly on* $\mathbb{R}$ *as* $\varepsilon \to 0$.

Suppose that a_j is decaying exponentially to 0. Let $a_j = e^{-j}$, hence we have the discrete Kac-Baker interaction kernel. Then

$$\sum_{j\in\mathbb{N}} e^{-j} e^{xj\imath} = \sum_{j\in\mathbb{N}} e^{(x\mathrm{i}-1)j} = \frac{e^{x\mathrm{i}-1}}{1-e^{x\mathrm{i}-1}}$$
$$= \frac{\cos x + \mathrm{i}\sin x}{e - \cos x - \mathrm{i}\sin x} = \frac{e\cos x - 1 + e\mathrm{i}\sin x}{e^2 + 1 - 2e\cos x}.$$

Then, we derive

$$\Phi(x) = \frac{2}{x}\left[\sum_{j\in\mathbb{N}} e^{-j} - \frac{e\cos x - 1}{e^2 + 1 - 2e\cos x}\right] = \frac{2e(e+1)(1-\cos x)}{(e-1)x(e^2+1-2e\cos x)}.$$

Applying our results for the specific function Φ we could prove the existence of quasi-periodic traveling waves for DNLS equation with Kac-Baker interaction kernel.

Nonlocal NLS equation: Problem set-up

The 1d nonlocal NLS equation (cubic and quintic terms):

$$i\partial_t\psi + \mu\psi = \mathcal{L}\psi + sC[\psi,\bar{\psi}]\psi + \delta Q[\psi,\bar{\psi}]\psi \tag{2.1}$$

$$C[\psi,\bar{\psi}] := \int_{-\infty}^{+\infty} R_1(x-x')|\psi(x')|^2 dx', \quad Q[\psi,\bar{\psi}] := \int_{-\infty}^{+\infty} R_2(x-x')|\psi(x')|^4 dx' \tag{2.2}$$

with $s,\delta = \pm 1$ and the linear operator will be of the standard Schrödinger type

$$\mathcal{L} \ = -(1/2)\partial_x^2 + V(x).$$

This encompasses the double-well potential of the form:

$$V(x) = (1/2)\hat{\Omega}^2 x^2 + V_0 \mathrm{sech}^2(\mathrm{x}/\omega)$$

with $\hat{\Omega}$ being the normalized strength of the parabolic trap and it is $\hat{\Omega} \ll 1$ in a quasi-1d situation in BECs.

- The competition of cubic and quintic terms can be systematically quantified.
- Fully nonlocal interactions both for the cubic and the quintic terms, rendering the local case a straightforward special-case scenario.

For the kernels R_1, R_2 we will focus our considerations on either the Gaussian

$$R_i(x) = \frac{1}{\sigma\sqrt{\pi}}\exp(-\frac{x^2}{\sigma^2})$$

or the exponential

$$R_i(x) = \frac{1}{2\sigma}\exp(-\frac{|x|}{\sigma}).$$

While the latter is more specifically relevant to the thermal nonlocal (optical) media and to quadratic nonlinear materials, we also use the former due to the mathematical simplicity of its kernel.
The key parameter here is the range of the nonlocal interaction parametrized by σ.
Notice that both kernels in the limit of $\sigma \to 0$ tend to a genuinely local interaction (i.e., $R_i(x) \to \delta(x)$).

Two-mode Approximation

We develop the two-mode approximation in order to obtain a decomposition of the solution ψ over the minimal basis of fundamental states.
We use an orthonormal basis composed by the wave functions

$$\{\phi_L, \phi_R\} \equiv \{(u_0 - u_1)/\sqrt{2}, (u_0 + u_1)/\sqrt{2}\},$$

where u_0 and u_1 are the ground state and the first excited state, respectively, corresponding to the first two eigenvalues of $\mathcal{L}$ that are $\omega_0 = 0.13282$ and $\omega_1 = 0.15571$

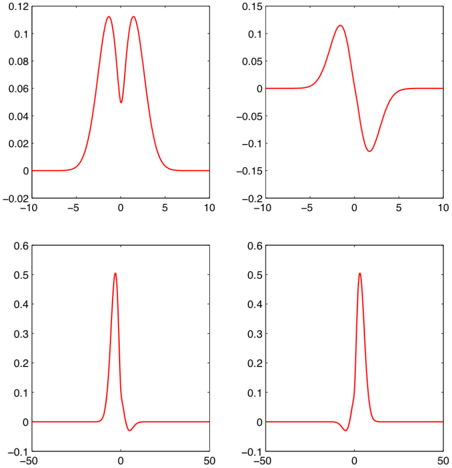

Figure: The ground state u_0 and first excited state u_1 of the potential are shown in the top panel. The rotated orthonormal basis of ϕ_L and ϕ_R (with support, respectively, on the left and right well) is shown in the bottom panels.

The two-mode approximation is then defined as

$$\psi(x,t) = c_L(t)\phi_L(x) + c_R(t)\phi_R(x) \tag{2.3}$$

where c_L and c_R are complex time-dependent amplitudes and the approximation consists of the truncation of the higher modes within the expansion.
We notice that the action of the linear operator $\mathcal{L}$ on our basis elements is as follows:

$$\mathcal{L}\psi = (\Omega c_L - \omega c_R)\phi_L + (\Omega c_R - \omega c_L)\phi_R$$

where $\Omega = (\omega_0 + \omega_1)/2$ and $\omega = (\omega_1 - \omega_0)/2$ are linear combinations of the two eigevalues of $\mathcal{L}$ respectively to the solutions u_0, u_1.

$$i\dot{c}_L\phi_L + i\dot{c}_R\phi_R = (\Omega c_L - \mu c_L - \omega c_R)\phi_L + (\Omega c_R - \mu c_R - \omega c_L)\phi_R + \mathcal{N}(\phi_L, \phi_R)$$

Project the above equation onto the bases of $\phi_{L,R}$. This involves some intgralls which play a fundamental role.

$$\eta_0 = \int\int R_1(x-x')\phi_L^2(x')\phi_L^2(x)dx'dx,$$
$$\eta_1 = \int\int R_1(x-x')\phi_L^2(x')\phi_R^2(x)dx'dx,$$
$$\eta_2 = \int\int R_1(x-x')\phi_L^2(x')\phi_L(x)\phi_R(x)dx'dx,$$
$$\eta_3 = \int\int R_1(x-x')\phi_L(x')\phi_R(x')\phi_L(x)\phi_R(x)dx'dx,$$
$$\eta_4 = \int\int R_2(x-x')\phi_L^4(x')\phi_L^2(x)dx'dx,$$

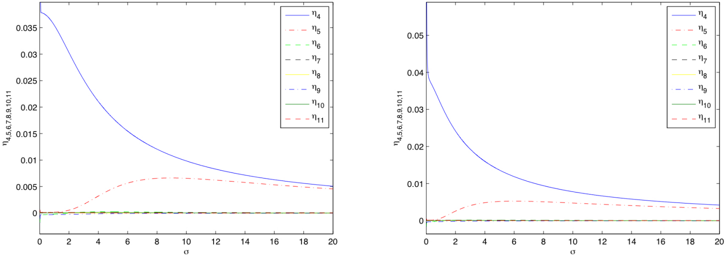

Figure: The overlap integrals $\eta_{4,5,\ldots,11}$ are given here as a function of the interaction range σ, for the two kernels in order to appreciate the dominance of η_4 with respect to the remaining terms for the range $\sigma < \sigma_c$, where the term with prefactor η_4 is not negligible with respect to the overall dominant term η_0.

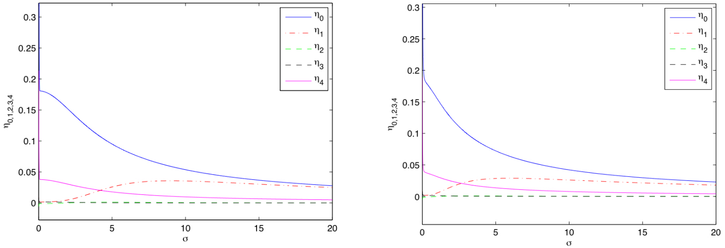

Figure: The overlap integrals η_0, η_1, η_2, η_3 and η_4 are shown as a function of the interaction range σ for the Gaussian (left) and exponential (right) kernels.

Case I: The terms η_0 and η_4 are considered for $\sigma < \sigma_b = 2.96$ (for Gaussian kernel)

$$i\dot{c}_L = (\Omega - \mu)c_L - \omega c_R + s\eta_0 |c_L|^2 c_L + \delta\eta_4 |c_L|^4 c_L$$
$$i\dot{c}_R = (\Omega - \mu)c_R - \omega c_L + s\eta_0 |c_R|^2 c_R + \delta\eta_4 |c_R|^4 c_R,$$

or in action-angle decomposition

$$\left\{ \begin{array}{l} \dot{\rho}_L = \omega \rho_R \sin\theta \\ \\ \dot{\theta}_L = \mu - \Omega + \omega \frac{\rho_R}{\rho_L} \cos\theta - s\eta_0 \rho_L^2 - \delta\eta_4 \rho_L^4, \end{array} \right\} \tag{2.4}$$

for $\theta = 0$ (symmetric) and $\theta = \pi$ asymmetric solutions one has to solve ($N = \rho_L^2 + \rho_R^2$)

$$\delta^3\eta_4^3 N^4 + 3s\eta_4^2\eta_0 N^3 + (3\delta\eta_4\eta_0^2 - \eta_4^2(\mu - \Omega))N^2 +$$
$$+(s^3\eta_0^3 - 2s\delta\eta_0\eta_4(\mu - \Omega))N - \delta\eta_4\omega^2 - \eta_0^2(\mu - \Omega) = 0.$$

Case II. $\sigma > \sigma_b$ the integrals η_0, η_1, η_4

$$\left\{ \begin{array}{l} \dot{\rho}_L = \omega \rho_R \sin\theta \\ \\ \dot{\theta}_L = \mu - \Omega + \omega \frac{\rho_R}{\rho_L} \cos\theta - s\eta_0 \rho_L^2 - s\eta_1 \rho_R^2 - \delta\eta_4 \rho_L^4. \end{array} \right\}$$

$\theta = 0$ (symmetric), $\theta = \pi$ (asymmetric) will be given by

$$\delta^3 \eta_4^3 N^4 + (3s\eta_4^2 \eta - s\eta_4^2 \eta_1) N^3 + (3\delta\eta_4 \eta^2 - \eta_4^2(\mu - \Omega) - 2\delta\eta_4 \eta\eta_1) N^2 + \\ +(s^3 \eta^3 - 2s\delta\eta\eta_4(\mu - \Omega) - s\eta_1 \eta_4^2) N - \delta\eta_4 \omega^2 - \eta^2(\mu - \Omega) = 0$$

with η here standing for $\Delta\eta = \eta_0 - \eta_1$.
Case III. $\sigma > \sigma_c$ the effect of the quintic terms is deemed to be negligible.

The Bifurcation Analysis

We introduce the population imbalance between the two wells,

$$z = (N_L - N_R)/N = (|c_L|^2 - |c_R|^2)/N, \tag{2.5}$$

where $N_{L,R} = |c_{L,R}|^2 = \rho^2_{L,R}$ and $N = N_L + N_R$. The reduced system is Hamiltonian with

$$\mathcal{H} = 2\omega\sqrt{1 - z^2}\cos\theta - \frac{1}{2}s\eta N z^2 - \frac{1}{2}\delta\eta_4 N^2 z^2.$$

Note that η stands either for η_0 $(\sigma < \sigma_b)$ or for $\Delta\eta = \eta_0 - \eta_1$ $(\sigma_b < \sigma < \sigma_c)$.
Critical points $(z, \theta) = (0, 0)$ (*symmetric*) and $(0, \pi)$ (*asymmetric*).
The stationary solutions representing the asymmetric branches are given by:

$$z^2 = 1 - \frac{4\omega^2}{(s\eta N + \delta\eta_4 N^2)^2}, \quad \theta = 0, \ \pi.$$

Taking $z = 0$, we get that

$$N = (-s\eta \pm \sqrt{\eta^2 + 8\delta\eta_4\omega})/2\delta\eta_4, \quad N = (-s\eta \pm \sqrt{\eta^2 - 8\delta\eta_4\omega})/2\delta\eta_4. \tag{2.6}$$

with N_0^{cr}, N_2^{cr} (correspond to the (-) signs in the left and right expressions of Eq. (2.6),), N_1^{cr} and N_3^{cr} (correspond to the (+) signs).

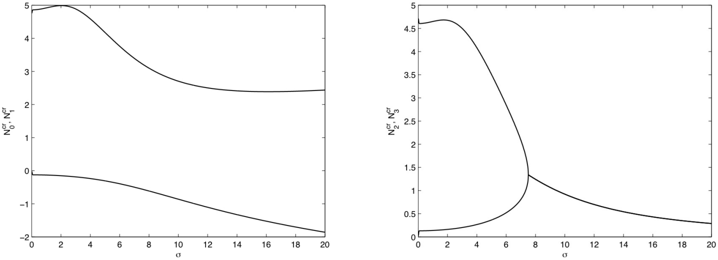

Figure: The critical values N_0^{cr}, N_1^{cr} (left panel) and N_2^{cr}, N_3^{cr} (right panel) whenever $(s, \delta) = (1, -1)$ show when the bifurcations appear. More specificaly, the left panel corresponds to the bifurcations that occur on the symmetric branch $\sigma < 7.52$ and the right panel for those that occur on the antisymmetric one $\sigma \geq 7.52$.

From the Hamiltonian system

$$\mathcal{H} = 2\omega\sqrt{1-z^2}\cos\theta - \frac{1}{2}s\eta N z^2 - \frac{1}{2}\delta\eta_4 N^2 z^2,$$

one can reduce the dynamical evolution to a single second-order ODE:

$$\ddot{z} = -4\omega^2 z - (s\eta N z + \delta\eta_4 N^2 z)\sqrt{4\omega^2 - 4\omega^2 z^2 - \dot{z}^2}$$

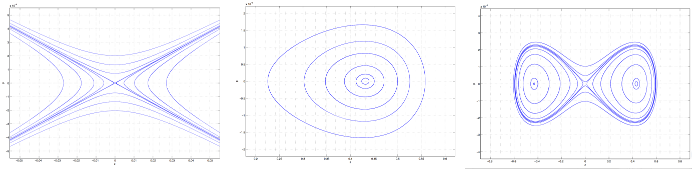

Figure: TThe phase space diagrams of the Hamiltonian system when $s = 1$ and $\delta = -1$ with the Gaussian kernel, for $\sigma = 1$, $N = 5$, and with $N_1^{cr} = 4.9862$ (after the new fixed points are created at $(\pm 0.4318, 0)$). The left panel displays the region of phase space near the symmetric solution $(0, 0)$ (saddle) and the center panel the one near one of the asymmetric fixed points $(0.4318, 0)$ (center). The right panel shows the full phase space diagram of the system for $N = 5$.

Numerical Simulations

The linear stability is analyzed by considering the standard linearization around the stationary solutions ψ_0 in the form

$$\psi(x,t) = \psi_0 + \epsilon(a(x)e^{\lambda t} + b^*(x)e^{\lambda^* t}).$$

This yields the eigenvalue problem

$$\begin{pmatrix} L_1 & L_2 \\ -L_2^* & -L_1^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = i\lambda \begin{pmatrix} a \\ b \end{pmatrix},$$

where the operators are defined as

$$\begin{aligned} L_1\phi &= \left[\mathcal{L} - \mu + sC[\psi_0, \bar{\psi}_0] + \delta Q[\psi_0, \bar{\psi}_0]\right]\phi + s\int_{-\infty}^{+\infty} K(x-x')\psi_0(x)\psi_0^*(x')\phi(x')dx' \\ &+2\delta \int_{-\infty}^{+\infty} K(x-x')\psi_0(x)\psi_0(x'){\psi_0^*}^2(x')\phi(x')dx' \end{aligned}$$

and

$$\begin{aligned} L_2\phi &= s\int_{-\infty}^{+\infty} K(x-x')\psi_0(x')\psi_0(x)\phi(x')dx + \\ &\quad 2\delta \int_{-\infty}^{+\infty} K(x-x')\psi_0(x)\psi_0^*(x')\psi_0^2(x')\phi(x')dx' \end{aligned}$$

for any real function ϕ.

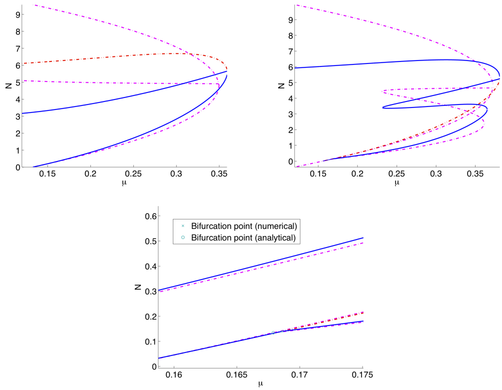

Figure: The stationary solution branches for the case $s = 1$, $\delta = -1$ when the interaction range is $\sigma = 0.1$ expressed in terms of the normalized N as a function of μ. The analytical predictions are denoted with the purple dash–dotted line while the numerically determined solutions are denoted with the solid line that is blue when it is stable and red otherwise. The top left panel shows the symmetric solutions, while the top right presents the antisymmetric ones, both including the asymmetric bifurcations that emerge from them. The bottom panel presents a detail of the symmetry-breaking effect, showcasing the quality of its approximation by the two-mode expansion.

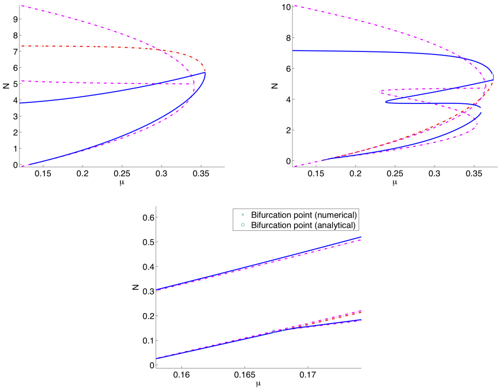

Figure: This figure shows the same features as the previous one for the symmetric branch (top left panel), the anti-symmetric branch (top right panel) and a zoom-in of the symmetry breaking (bottom panel). However, the interaction range here is an order of magnitude larger, namely $\sigma = 1$.

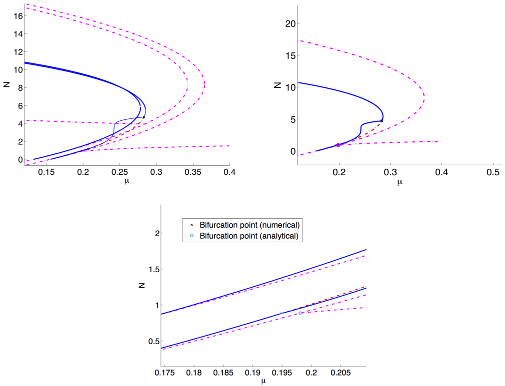

Figure: Same as the previous two figures, but now for large nonlocality interaction range in the case of $\sigma = 8$.

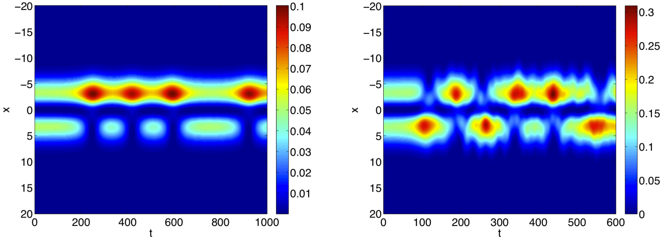

Figure: Spatio-temporal contour plot of the density of the unstable solutions when $\sigma = 1$, for $s = 1$ and $\delta = -1$. In both cases, the weak perturbation added on top of the exact numerical solution in the initial conditions has a projection along the unstable eigenmode. This projection, for sufficiently long times (about 200 in the left panel and about 100 in the right panel), gets amplified and eventually leads to a visible symmetry breaking in the profile of the state, $\mu = 0.19$ and 0.25 (left and right, respectively).

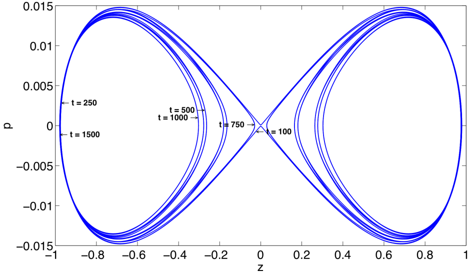

Figure: The numerically obtained trajectory of the solution for $\mu = 0.19$, for times between 0 and 1500.

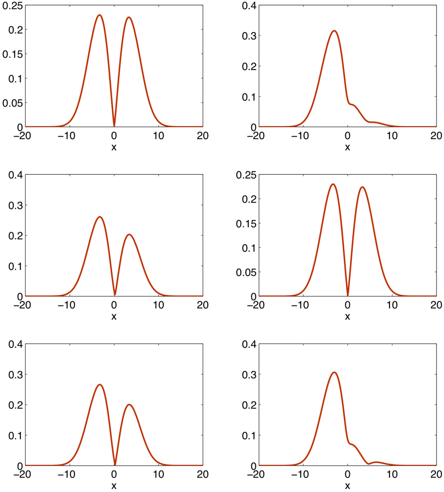

Figure: The profiles of the solution for $t = 100$, 250 (first row), 500, 750 (second row), 1000 and 1500 (third row).

Conclusions

- Discrete Solitons for Discrete NLS with saturable nonlinearities (Variational Methods)
- Periodic Traveling Waves in Discrete NLS (Variational and Topological Methods)
- NLS with nonlocal cubic and quintic terms
  - We systematically developed (two-mode approximation) two cubic-quintic ordinary differential equations and assess the contributions of the long-range interactions in each of the relevant prefactors, gauging how to simplify the ensuing dynamical system.
  - We proceed to a dynamical systems analysis of the resulting pair of ordinary differential equations.
  - The relevant bifurcations, the stability of the branches and their dynamical implications are examined both in the reduced (ODE) and in the full (PDE) setting.

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