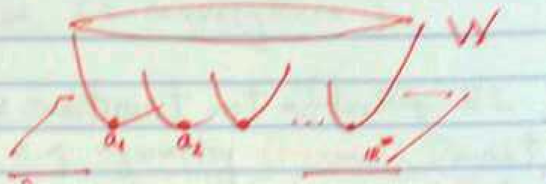
1. Generalities



(1) $\text{Min} \int \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx$

$W > 0$ on $\mathbb{R}^m \setminus \{a_1, \dots, a_N\}$
$W \in C^2(\mathbb{R}^m; \mathbb{R})$, $\partial^2 W(a_i)$ p.d.
$W_u(u) \cdot u \geq 0$ for $|u| \geq M$ (or $\lim_{|u| \to +\infty} W(u) > 0$)

(2) $\Delta u - W_u(u) = 0$, $u: \mathbb{R}^n \to \mathbb{R}^m$

PROBLEM

Construct entire bounded slts "connecting" (IN DIFFERENT DIRECTIONS) the minima (or a subset) of W at infinity, $|x| \to \infty$.

Fact: $\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx < \infty$

(u(x) any critical pt) $\underset{n \geq 2}{\Longrightarrow}$ $u(x) \equiv a_i$, some a_i

$I_\Omega(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx$

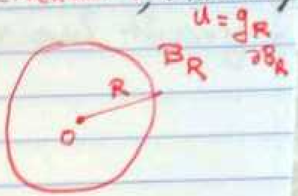
Ω bded

2. The blow-down technique (Γ-convergence)

$\underset{B_R}{\mathrm{Min}}\ J_{B_R}(u)$ (+ some constraint not affecting EL
e.g. symmetry (Neumann), Dirichlet)

u_R minimizer, $|u_R(x)| < M$

$\Delta u_R - W_u(u_R) = 0$, $u_R : B_R \to \mathbb{R}^m$



Plan: Seek $u(x) = \lim u_R(x)$, $R \to \infty$

Difficulty: $u_R \to u$, C^2 on compacts

(Even under symmetry $u \equiv 0$ possibility)

$\varepsilon = \frac{1}{R}$, $x = \frac{y}{\varepsilon}$ $\quad v_\varepsilon(y) = u_R(x)$

$$\int_{B_R} \left(\tfrac{1}{2}|\nabla_x u|^2 + W(u)\right)dx = R^{n-1}\int_{B_1}\left(\tfrac{1}{2R}|\nabla_y v|^2 + R\,W(v)\right)dy$$

$$\mathrm{Min}\int_{B_1}\left(\tfrac{\varepsilon}{2}|\nabla_y v|^2 + \tfrac{1}{\varepsilon}W(v)\right)dy =: \mathrm{Min}\, J^\varepsilon_{B_1}(v), \quad v_\varepsilon \text{ min}$$

Minimizer

$$C > \int_{B_1}\left(\tfrac{\varepsilon}{2}|\nabla_y v_\varepsilon|^2 + \tfrac{1}{\varepsilon}W(v_\varepsilon)\right)dy \geq \sqrt{2}\int_{B_1}|\nabla_y v_\varepsilon|\sqrt{W(v)} = \int_{B_1}|\nabla_y(\varphi\circ v_\varepsilon)|$$

$\|\varphi\circ v_\varepsilon\|_{BV} < C$, $BV \subset\subset L^1$, φ Lip

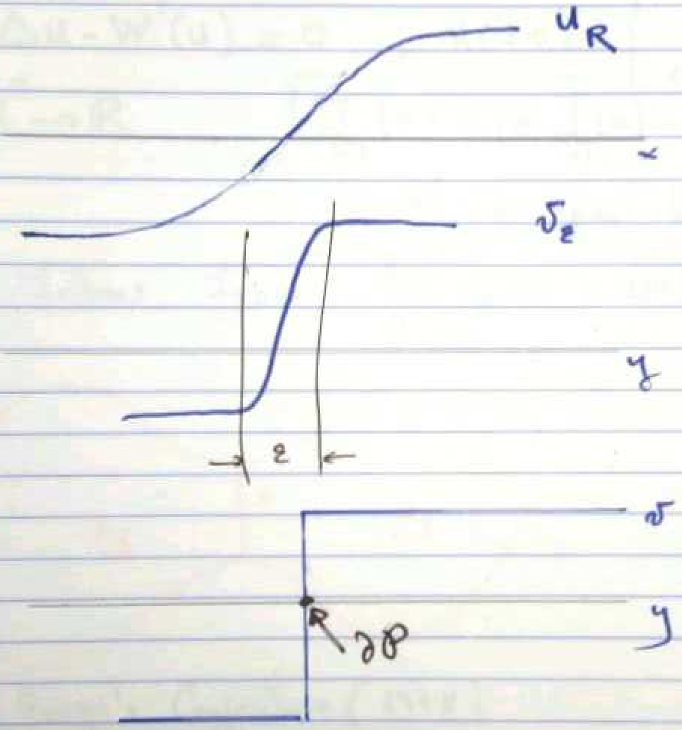
$\therefore\ \|v_\varepsilon - v\|_{L^1(B_1)} \to 0$, $v(y) \in \{a_1, \dots, a_N\}$

$v = \sum_{i=1}^{N} a_i \mathbb{1}_{D_i}$, $\mathcal{P} = \{D_j\}$, minimizing partition of B_1

$$\Gamma\text{-}\lim J^\varepsilon_{B_1} = E(\mathcal{P}) = \sum_{i\neq j} \sigma_{ij}\, \mathcal{H}^{n-1}(\partial D_i \cap \partial D_j)$$

$$\boxed{\begin{array}{ll} v_\varepsilon(y) \xrightarrow{\varepsilon\to 0} v(y), & x = \frac{y}{\varepsilon} \\ u_R(x) \xrightarrow[R\to\infty]{C^2_{loc}} u(x) & v_\varepsilon(y) = u_R(x) \\ \multicolumn{2}{l}{\text{Relationship between } \lim_{|x|\to\infty} u(x) \text{ and } v\ ??} \end{array}}$$

<u>Difficulty</u> : No estimate on $\|v_\varepsilon - v\|_{L^1}$ (in ε) available by general theory!



L^1 convergence NOT A PROBLEM:

By Caffarelli – Cordoba (vector version) Density Est

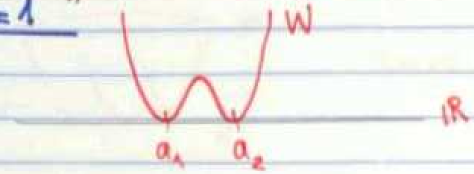
$u_\varepsilon \to a_i$ uniformly $V \subset\subset D_i$

(no need to assume $\mathcal{P}$ minimizing)

NEED ε-Estimates up to the bdry $\partial\mathcal{P} = \bigcup_{0<i<j\leq N} (\partial D_i \cap \partial D_j)$

3. Differences between scalar ($m=1$) and vector ($m \geq 2$)

$m=1$

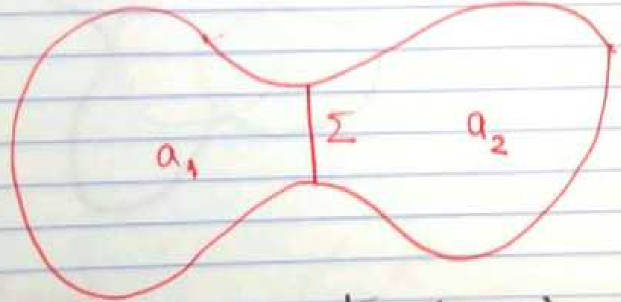


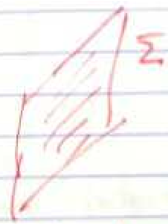
$$\Delta u - W'(u) = 0 \quad , \quad u(\pm\infty) = \begin{cases} a_2 \\ a_1 \end{cases}$$

$u: \mathbb{R}^n \to \mathbb{R}$,

(*) $\boxed{J_{\Omega}(u+v) \geq J_{\Omega}(u) \; , \; \forall v \in C_0^1(\Omega) \quad \forall \Omega \text{ open, bded.}}$

(*) follows from $\lim_{R\to\infty} u_R(x) = u(x)$





heteroclinic

De Giorgi's Conjecture (1978)

Under (*)

$u(x) = U(d(x,\Sigma))$, $\quad \dfrac{d^2U}{dy^2} - W'(U) = 0$

Σ hyperplane in $\mathbb{R}^n$ $\quad U(\pm\infty) = \begin{cases} a_2 \\ a_1 \end{cases}$

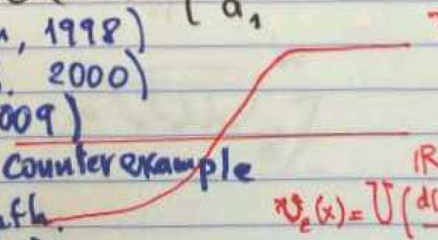
in low dimensions

Ghoussoub - Gui ($n=2$, Math Ann, 1998)

Ambrosio - Cabre ($n=3$, J.A.M.S. 2000)

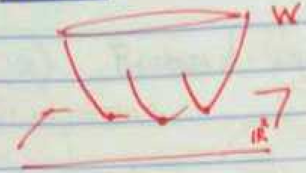
Savin ($n \leq 8$, Ann. Math. 2009)

del Pino - Kowalczyk - Wei ($n \geq 8$, counterexample Ann. Math. 2011)

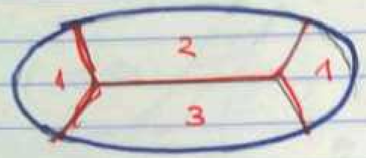


$u_\varepsilon(x) = U\left(\dfrac{d(x,\Sigma)}{\varepsilon}\right)$

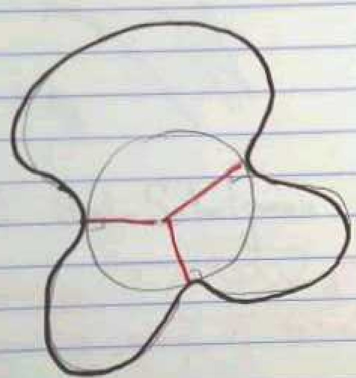
$m \geq 2$



$(m=2,\ n=2,\ N=3)$



(Mass constraint)

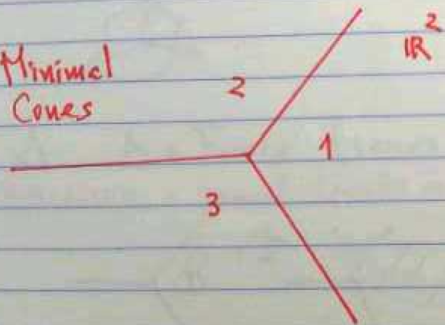


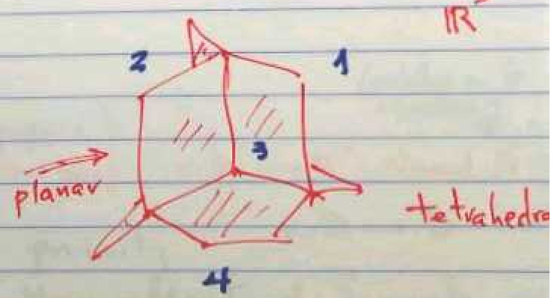
$\boxed{\text{Junctions}}$

(show $\lim_{\varepsilon \to 0} v_\varepsilon(y)$

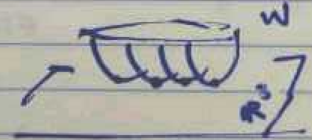
$u_R = v_\varepsilon$ "diffuse" interfaces)

Minimal Cones





$\boxed{\text{ONLY IN } \mathbb{R}^2, \mathbb{R}^3 \text{ KNOWN}}$ (triple)



J. Taylor (Ann. Math. 1976)

(Almgren's Ph.D. student)

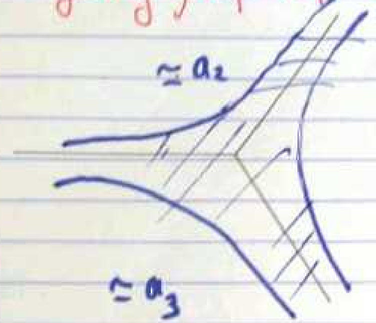
4. What is Known for the Vector Case $m \geq 2$

a) Bronsard-Gui-Schatzman CPAM 1996

Hypotheses: W on $\mathbb{R}^2$, $m = n = 2$, $N = 3$

symmetry group of equilateral triangle, $u : \mathbb{R}^2 \to \mathbb{R}^2$

u equivariant



$$I_\gamma = \left\{ x \in \mathbb{R}^2 \;\middle|\; \min_i |u(x) - a_i| \geq \gamma \right\}$$

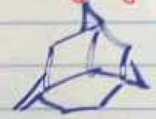
(diffuse interface)

b) Gui-Schatzman Ind. Univ. Math. J. 2008

Hypotheses: W on $\mathbb{R}^3$, $m = n = 3$, $N = 4$

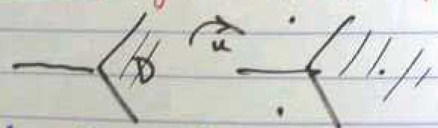
symmetry group of tetrahedron

$u : \mathbb{R}^3 \to \mathbb{R}^3$, u equivariant



c) A + Fusco (series of papers) (relates to G + location)

Hypotheses: general (point) reflection group, $u : \mathbb{R}^n \to \mathbb{R}^m$, General N



$u : D \to D$

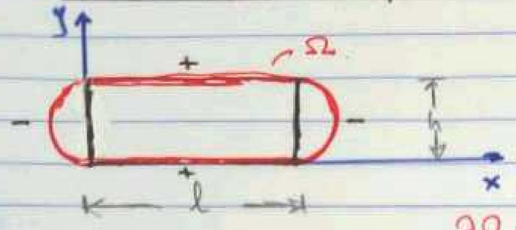
positivity

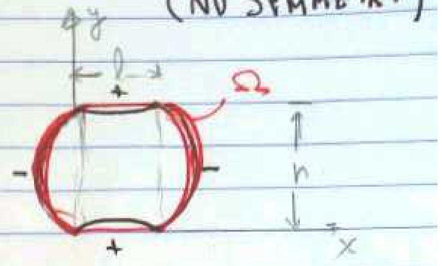
Main effect of Symmetry

d) Bates-Fusco-Smyrnelis ARMA 2017
lattices

A + Fusco + Smyrnelis, book (Birkhauser, Green, 2018)

5. New Results – Simple Geometries (NO SYMMETRY)





$\partial\Omega \in C^{1+\alpha}$

$l > h$ $\qquad$ $l < h$

$\Omega \cong (\underbrace{\text{Rectangle}}_{R}) \cup (\underbrace{\text{half disks}}_{\Omega_L,\ \Omega_R}) \subset \mathbb{R}^2$

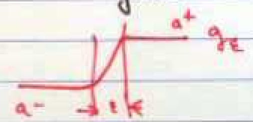
$$\min J^{\varepsilon}_{\Omega}(u) = \iint_{\Omega} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\right) dx\,dy$$

$u : \Omega \to \mathbb{R}^m$

$u = g_\varepsilon$ on $\partial\Omega$

$$g_\varepsilon = \begin{cases} a^-, & \text{half circles} \\ a^+, & \text{flat parts (except on } \varepsilon \text{ transition zones)} \end{cases}$$

$g_\varepsilon \in C^{1+\alpha}$



Th1 ($l < h$, Boundary layers)

For $\varepsilon \ll 1$ $\quad u_\varepsilon \sim a_-$ except in two bdry layers at the flat parts (+), $O(\varepsilon)/o(\varepsilon)$ thick.

Th2 ($l > h$, Internal layers)

For $\varepsilon \ll 1$ $\quad u_\varepsilon \sim \begin{cases} a^- & \text{in } \Omega_L \\ a^+ & \text{in } R = \text{rectangle} \\ a^- & \text{in } \Omega_R \end{cases}$

with two layers $O(\varepsilon)$ thick.

We show that the structure of minimizers $\{u_\epsilon\}$ for $0 < \epsilon << 1$ depend drastically on weather $l < h$ of $l > h$ that is from the geometry of Ω.

We are stating somewhat loosely our main results for convenience of the reader the X

Theorem 1.1. ($l < h$, *The Boundary Layer Case*) *There is $\epsilon_0 > 0$ such that, if u_ϵ, $\epsilon \in (0, \epsilon_0]$ is a minimizer of* (1.5), *then*

(1.9)
$$2\sigma l - C\epsilon \leq \int_R \Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial y}|^2 + \frac{1}{\epsilon}W(u_\epsilon)\Big)dxdy \leq J_\Omega^\epsilon(u_\epsilon) \leq 2\sigma l + C\epsilon|\ln\epsilon|^3,$$

0 C_δ X

$$|u_\epsilon(z) - a_-| \leq \delta e^{-\frac{k}{\epsilon}(d(z,\partial\Omega^+) - C\epsilon^{\frac{1}{2}}|\ln\epsilon|^3)^+}, \quad z \in \Omega$$

where σ is the Action of the connecting orbit between a_- and a_+ and $C, \delta << 1$ and k are positive constants, $\xi^+ = \max\{0, \xi\}$ *and* $\partial\Omega^+ = [0, l] \times \{0, h\}$.

These estimates imply that u_ϵ converges uniformly in compacts in $\bar{\Omega} \setminus \partial\Omega^+$ to a_- and there is a boundary layer in a neighborhood of $\partial\Omega^+$ which can be shown to be strictly thicker that O(ϵ) (O(ϵ)/o(1)), see Theorem 4.9 below.

Theorem 1.2. ($l > h$, *The Internal Layer Case*) *There is $\epsilon_0 > 0$ such that, if u_ϵ, $\epsilon \in (0, \epsilon_0]$ is a minimizer of* (1.5), *then*

(1.10)
$$2\sigma l - C\epsilon \leq \int_D \Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial x}|^2 + \frac{1}{\epsilon}W(u_\epsilon)\Big)dxdy \leq J_\Omega^\epsilon(u_\epsilon) \leq 2\sigma l + C\epsilon,$$

$$|u_\epsilon(z) - a_-| \leq Ke^{-\frac{k}{\epsilon}(d(z,R) - C\epsilon^{\frac{1}{2}})^+}, \quad z \in \Omega \setminus R,$$

$$|u_\epsilon(z) - a_+| \leq Ke^{-\frac{k}{\epsilon}(d(z,\Omega\setminus R) - C\epsilon^{\frac{1}{2}})^+}, \quad z \in R,$$

where D is a strict subset of Ω (cfr. Figure 4), $R = [0, l] \times (0, h)$, K, k and C positive constants.

These estimates imply that

Lemma 1 $(l > h)$

u_ε min, $0 < \varepsilon < \varepsilon_0$

$$(UB)\quad J^\varepsilon_\Omega(u_\varepsilon) = \iint_\Omega \left(\frac{\varepsilon}{2}\left[\left|\frac{\partial u_\varepsilon}{\partial x}\right|^2 + \left|\frac{\partial u_\varepsilon}{\partial y}\right|^2 \right] + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx\,dy \leq 2\sigma h + C_\varepsilon$$

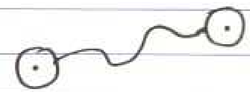
$$(LB)\quad \iint_\Omega \left(\frac{\varepsilon}{2}\left|\frac{\partial u_\varepsilon}{\partial x}\right|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx\,dy \geq 2\sigma h - C_\varepsilon$$

Lemma 2

a) $\sigma = J_{\mathbb{R}}(\bar{u}) = \min\limits_{A} J_{\mathbb{R}}(u)$, $A = \left\{ \lim\limits_{s \to \pm\infty} u(s) = a_\pm \right\}$ (Connection)

b) $\lim\limits_{s \to s_\pm} d(v(s), \partial B_{\delta_\pm}(a_\pm)) = 0 \Rightarrow J_{(s_-, s_+)}(v) \geq \sigma - \frac{1}{2} C_W (\delta_-^2 + \delta_+^2)$

$\left(W(u) \leq \frac{1}{2} C_W^2 \delta^2, \ \delta \leq \delta_W \right)$



Indications of the usefulness of (LB), (UB)

Claim 1:

(1) $$\iint_\Omega \left|\frac{\partial u_\varepsilon}{\partial y}\right|^2 dx\,dy \leq C.$$

Pf: $$\iint_\Omega \frac{\varepsilon}{2}\left|\frac{\partial u_\varepsilon}{\partial y}\right|^2 dx\,dy \underset{(UB)}{\leq} 2\sigma h - \iint_\Omega \left(\frac{\varepsilon}{2}\left|\frac{\partial u_\varepsilon}{\partial x}\right|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx\,dy + C_\varepsilon$$

$$\underset{(LB)}{\leq} 2\sigma h - 2\sigma h + C_\varepsilon + C_\varepsilon = 2C_\varepsilon.$$ ✓

Claim 2: $u_\varepsilon \xrightarrow{L^1} u_0 = a_+ \mathbb{1}_{\Omega^+} + a_- \mathbb{1}_{\Omega^-}$

$\Omega = \Omega^+ \cup \Omega^-$, $\mathcal{P} = \{\Omega^+, \Omega^-\}$ (we will not utilize minimality of $\mathcal{P}$)

(3) $\boxed{\mathcal{L}^n(\Omega^+) \neq 0, \ \mathcal{L}^n(\Omega^-) \neq 0,}$ $\boxed{\| \mathbb{1}_{\Omega^\pm} \|_{BV} \leq 2h}$ (2)

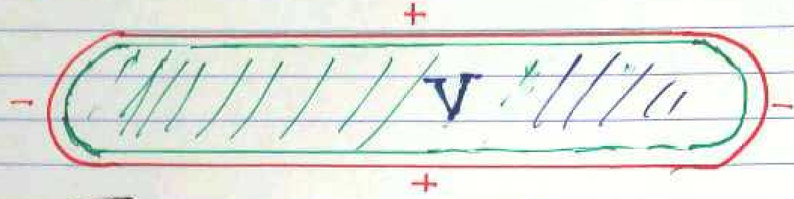
Pf: By the (MM trick), w.l.s.c. of $\|\cdot\|_{BV}$ and (UB), (2) follows

- $\mathcal{L}^n(\Omega^+) \neq, \ \mathcal{L}^n(\Omega^-) \neq 0$

Pf

By contradiction: Suppose $\mathcal{L}^n(\Omega^-) = 0$.

(CC) vector version $\Longrightarrow$ $u_\varepsilon \to a^+$ unif. compacts



$u_\varepsilon \to a^+$ on $V \subset\subset \Omega$

Lemma 2 $\Longrightarrow$ $J^\varepsilon_{\Omega_L}(u_\varepsilon) \geq \pi r_0 \sigma + o(1)$

$J^\varepsilon_{\Omega_R}(u_\varepsilon) \geq \pi r_0 \sigma + o(1)$

contradicting (UB)!

$\mathcal{L}^n(\Omega^+) = 0 \quad \Longrightarrow \quad \otimes$ (similar argument) $\ell > h$

Claim 3:

The bdry of the partition $\mathcal{P} = \{\Omega^+, \Omega^-\}$ consists of a finite # of vertical line segments with endpoints on $\partial\Omega$.

Pf

$u_\varepsilon \longrightarrow u_0$ in $L^1(\Omega)$ as $\varepsilon \longrightarrow 0$

$\dfrac{\partial u_\varepsilon}{\partial y} \longrightarrow \dfrac{\partial u_0}{\partial y}$ in $\mathcal{D}'(\Omega)$ — " —

$\left\| \dfrac{\partial u_\varepsilon}{\partial y} \right\|_{L^2(\Omega)} < C$ by Claim 1

$\Longrightarrow$

$\left\| \dfrac{\partial u_0}{\partial y} \right\|_{L^2(\Omega)} < C$ by w.l.s.c.

$u_0(x,y)$ is a characteristic fn $\Longrightarrow$ independent of y.

From $\|u_0\|_{BV} < \infty$ we conclude.

⋮