Optimal management of stochastic shallow lakes

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Athens, May 7, 2021

European Union
European Social Fund
OPERATIONAL PROGRAMME
EDUCATION AND LIFELONG LEARNING
investing in knowledge society
MINISTRY OF EDUCATION & RELIGIOUS AFFAIRS, CULTURE & SPORTS
MANAGING AUTHORITY
Co-financed by Greece and the European Union
NSRF 2007-2013
programme for development
EUROPEAN SOCIAL FUND
H.F.R.I.
Hellenic Foundation for Research & Innovation

oligotrophic vs eutrophic lakes

Modelling the nutrient content

The nutrient content is usually measured in terms of P concentration.

$$\begin{aligned} \dot{P}(t) = L(t) & \qquad \text{(P loading by natural and human activity)} \\ - sP(t) & \qquad \text{(sedimentation, outflow)} \\ + \Phi\big(P(t)\big) & \qquad \text{(recycling from sediments)} \end{aligned}$$

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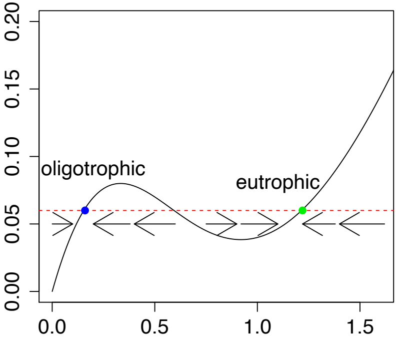
With a change of variables ($x = \frac{P}{m}, \; a = \frac{L}{r}, \; b = \frac{sm}{r}$) the equation becomes

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}.$$

Equilibrium under constant load

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}.$$

When b is not too high the lake may have 2 stable equilibria.



Solid black line: $y = bx - \frac{x^2}{1+x^2}$. Dashed red line: $y = a$.

A welfare function

Farmers or industry have an interest to increase P loading, a.

Visitors prefer a clean lake, i.e. small x.

Suppose a community balances these needs and assigns value to the state of the lake

$$U(a,x) = \ln a - cx^2.$$

Given the current P concentration x, we are interested in the optimal loading $\{a(t) : t \geq 0\}$ to maximise the welfare function

$$J(x, a(\cdot)) = \int_0^\infty e^{-\rho t}\, U(a(t), x(t))dt$$

where $\{x(t) : t \geq 0\}$ solves

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}, \qquad x(0) = x.$$

The problem

Add multiplicative noise [Grass, Kiseleva, Wagener 2015]

$$\begin{cases} dx(t) = \left(u(t) - bx(t) + \dfrac{x^2(t)}{x^2(t)+1} \right) dt + \sigma x(t) dW(t), \\ x(0) = x \end{cases} \tag{1}$$

and the value function

$$V(x) = \sup_{u \in \mathfrak{U}_x} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left[\ln u(t) - cx^2(t) \right] dt \right].$$

Admissible controls $u \in \mathfrak{U}_x$ should be positive, adapted processes in some filtered probability space such that

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \ln u(t) dt \right] < \infty$$

and (1) has a unique strong solution.

DPP and HJB equation

The tool to characterise the value function V is the Dynamic Programming Principle (DPP):

$$V(x) = \sup_{u \in \mathfrak{U}_x} \mathbb{E}\left[\int_0^{\theta_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)dt + e^{-\rho\theta_u} V(x(\theta_u))\right].$$

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- Unbounded controls, the Hamiltonian may be infinite.
- Boundary conditions at zero? at infinity?

Elementary properties

Implicit formula

$$x(t) = xZ_t + \int_0^t \frac{Z_t}{Z_s}\left(u(s) + \frac{x^2(s)}{1+x^2(s)}\right) ds,$$

where $Z_t = e^{\sigma W_t - (b+\frac{\sigma^2}{2})t}$.

Elementary properties

- Any solution $\{x(t) : t \geq 0\}$ with $x(0) \geq 0$ remains positive at all times, $\mathbb{P}$-a.s.
- $\mathfrak{U}_x = \mathfrak{U}_y = \mathfrak{U}$ for all $x, y \geq 0$
- If $x_1(0) \leq x_2(0)$ and $u_1(t) \leq u_2(t)$, $\mathbb{P}$-a.s., then $x_2(t) - x_1(t) \geq \big(x_2(0) - x_1(0)\big) Z_t$, $\mathbb{P}$-a.s.

Estimates for V from the SDE

- $\sigma^2 \geq \rho + 2b \Rightarrow V(x) = -\infty, \ \forall x \geq 0.$
  $\sigma^2 < \rho + 2b \Rightarrow V(x) \in \mathbb{R}, \ \forall x \geq 0$ (assume hereafter).

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- If $0 \leq x < y \leq K$, there exist constants $C_1(K), C_2 > 0$ such that
$$-C_1(K) \leq \frac{V(y) - V(x)}{y - x} \leq -C_2.$$
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- Set $A = c(\rho + 2b - \sigma^2)^{-1}$. There exist constants K_1, K_2 s.t.
$$K_1 \ \leq \ V(x) + A\left(x + \frac{1}{b+\rho}\right)^2 + \frac{1}{\rho}\ln\left(x + \frac{1}{b+\rho}\right) \ \leq \ K_2,$$

Further estimates from the SDE

- V is differentiable at zero and

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$$\rho V=\underbrace{\left(\frac{x^2}{x^2+1}-bx\right)V'-\big(\ln(-V')+cx^2+1\big)+\frac{1}{2}\sigma^2x^2V''}_{H(x,V',V'')}$$

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Note: Boundary condition at $x = 0 \rightarrow$ HJB is satisfied at $x = 0$.

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Definition

[Soner 1986, Lasry, Lions 1989,...]

Let $v : [0, \infty) \to \mathbb{R}$ a continuous function. We say that

i) v is a *viscosity subsolution* of the HJB on $[0, \infty)$, if for every $\phi \in C^2[0, \infty)$ such that $v - \phi$ has a local maximum at $x \geq 0$

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iii) v is a *constrained viscosity solution* of the HJB on $[0, \infty)$, if it is both a viscosity supersolution on $(0, \infty)$ and a viscosity subsolution on $[0, \infty)$.

V is the unique constrained v.s. to the HJB

By the DPP and the boundary condition at $x = 0$, the value function V is a constrained viscosity solution to

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Comparison principle: Let

u be a continuous viscosity subsolution of the HJB on $[0, \infty)$ s.t.

$$\frac{u(y) - u(x)}{y - x} \leq -\frac{1}{c_*} < 0, \quad \forall x, y \in [0, \infty)$$

v be a continuous, strictly decreasing viscosity supersolution of the HJB on $(0, \infty)$ such that

$$\liminf_{x \to \infty} \frac{v(x)}{1 + x^2} > -\infty.$$

Then, $u \leq v$ on $[0, \infty)$.

Properties of the value function $(\sigma^2 < 2b + \rho)$

- V decreases and $0 \leq x < y \Longrightarrow \frac{V(y)-V(x)}{y-x} \leq -c < 0,$

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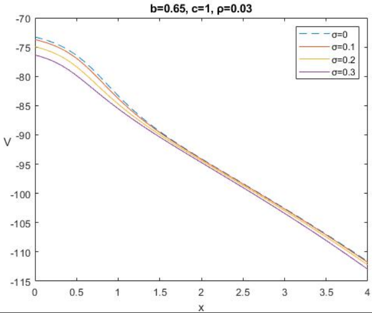
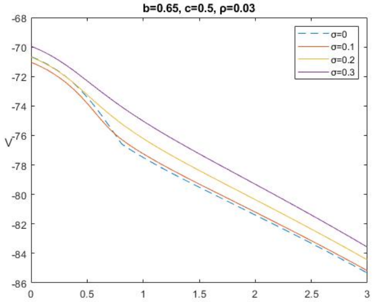
- V is differentiable at 0 and $\ln\left(-V'(0)\right) + \rho V(0) + 1 = 0.$
- $$V(x) + A\left(x + \frac{1}{b+\rho}\right)^2 + \frac{1}{\rho}\ln\left(x + \frac{1}{b+\rho}\right) \xrightarrow{x\to\infty} K$$





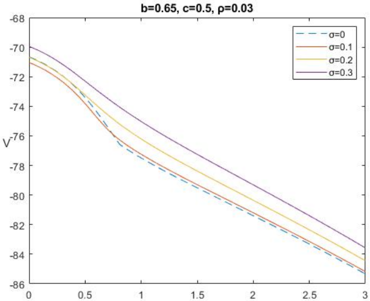
Optimally controlled process

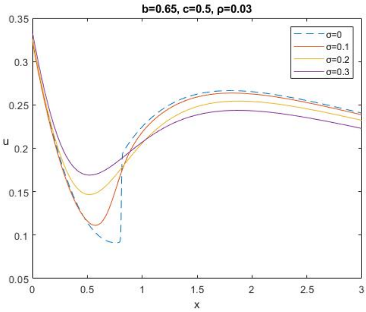
A verification theorem gives the optimal control in feedback form

$$u_*\big(x(t)\big) = -\frac{1}{V'\big(x(t)\big)} \leq \frac{1}{c_*}$$

so the optimally controlled system satisfies

$$dx(t) = \left(-\frac{1}{V'\big(x(t)\big)} - bx(t) + \frac{x^2(t)}{x^2(t)+1}\right)dt + \sigma x(t)dW(t).$$





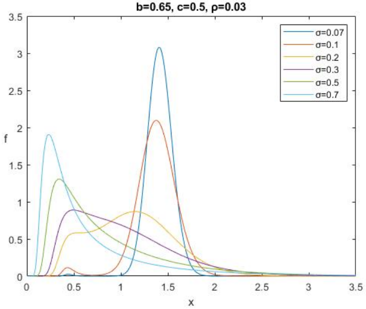
Invariant measure

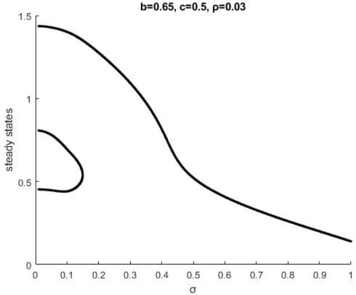
$$\mathcal{L}^*\mu = 0 \Longrightarrow d\mu(x) = \frac{1}{Z}\, x^{-2\left(1+\frac{b}{\sigma^2}\right)}\, e^{-\Psi_\sigma(x)}\, dx.$$

The exponent Ψ_σ is explicitly given in terms of V' and

$$\Psi_\sigma(x) \simeq \frac{2}{\sigma^2|V'(0)|x},\ x \to 0 \qquad \text{and} \qquad \Psi_\sigma(x) \simeq \frac{2}{\sigma^2 x},\ x \to \infty.$$

Polynomial tails at infinity get fatter as σ increases.

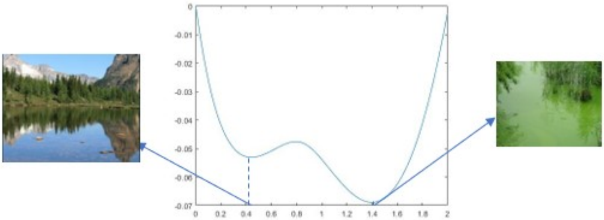




Oligotrophic vs Eutrophic

When σ is small and other parameters are suitable, the invariant distribution may be bimodal. The process $y(t) = \ln\big(x(t)\big)$ is a diffusion in a double-well potential $\Phi_\sigma(y)$:

$$dy(t) = -\Phi_\sigma'\big(y(t)\big)dt + \sigma\, dW(t).$$



with invariant distribution for $\sigma > 0$

$$d\mu_\sigma(x) = \frac{1}{Z_\sigma}\exp\Big(-\frac{2}{\sigma^2}\Phi_\sigma(x)\Big)dx.$$

Deterministic vs Stochastic trajectories

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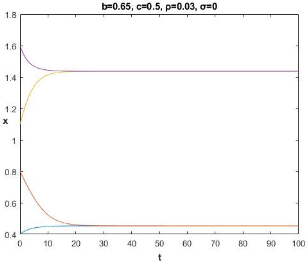
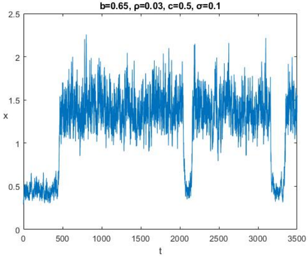




$\sigma \to 0$ asymptotics: metastability

For a diffusion in a double well potential

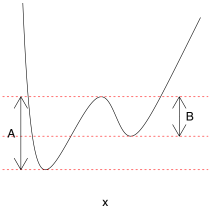
$$dy(t) = -\Phi'\big(y(t)\big)dt + \sigma\, dW(t).$$

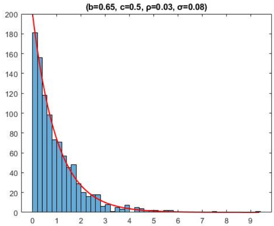
Arrhenius law : $\quad \dfrac{\sigma^2}{2} \log \mathbb{E}\big[\tau_{O\to E}\big] \to A, \quad \dfrac{\sigma^2}{2} \log \mathbb{E}\big[\tau_{E\to O}\big] \to B$

[Siguira 1993, Bovier & den Hollander book 2014]

$$\frac{\tau_{O\to E}}{\mathbb{E}\big[\tau_{O\to E}\big]} \xrightarrow{d} \text{Exp}(1)$$

[Day 1983]





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