On the shape of almost constant mean curvature hypersufaces

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Alexandrov's Theorem ('62).
Let $S = \partial\Omega$ be a closed hypersurface embedded in $\mathbb{R}^{n+1}$ of class C^2, $\Omega \subset \mathbb{R}^{n+1}$ bounded and connected.
The mean curvature $\boldsymbol{H_S}$ **is constant** *if and only if* $\boldsymbol{S}$ **is a sphere.**

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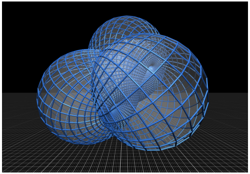
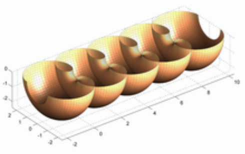
$\Omega \subset \mathbb{R}^{n+1}$ *bounded and connected.*
The mean curvature $\boldsymbol{H_S}$ **is constant** *if and only if* S **is a sphere.**

- The **embeddedness** assumption can not be weakened (Wente *Pacific J. Math.* '86, Kapouleas *Ann. Math.* '90 e *JDG* '91).
- There exist CMC hypersurfaces which are not close (Delaunay, Kapouleas *Ann. Math.* '90)





Main goals – Questions

- If H_S is **close to a constant** then S is **close to a sphere**?
- Is it close to something else?
- Can we **quantify** in terms of some deficit?

Main goals – Questions and Answers

- If H_S is **close to a constant** then S is **close to a sphere**? **No**
- Is it close to something else? **Bubbling may appear**
- Can we **quantify** in terms of some deficit? **Yes**

Bibliography:

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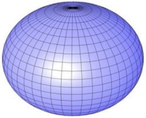
- **C. - Roncoroni - Vezzoni** *Quantitative stability for hypersurfaces with almost constant curvature in space forms.* To appear in **Ann. Mat. Pura e Appl.**
- **C. - Maggi**, *On the shape of compact hypersurfaces with almost constant mean curvature*, **Comm. Pure Appl. Math.**, 70 (2017), 665-716.
- **C. - Figalli - Maggi - Novaga**, *Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature*, **J. Reine Angew. Math. (Crelle's Journal)**, 2018, Issue 741, 275-294.
- **C.**, *Quantitative estimates for almost constant mean curvature hypersurfaces*. **Boll. Unione Mat. Ital.**, 14 (2021), 137-150.

Question: What happens if H is **almost** constant?

- Quantitative studies for almost-CMC - Ω convex:



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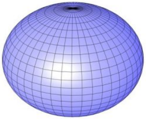

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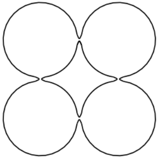
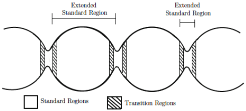
- Examples and gluing techiniques - Bubbling?:





  - Kapouleas *(Ann. Math. '90, JDG '91, Invent. Math. '95, ...)*: examples of closed immersed hypersurfaces.
  - Butscher *(Pacific J. Math. 2011)* e Butscher-Mazzeo *(Ann. SNS Pisa 2012)*: examples where H is close to a constant in any C^k norms.

A related problem - Capillarity theory

Local and global minima, stationary points of

$$\mathcal{F}(E) = P(E) + \int_E g(x)dx\,.$$

$\Rightarrow \delta\mathcal{F} = 0$ for volume preserving fluxes.

Small mass regime: if $|E| = m \ll 1$ then $P(E) \approx m^{\frac{n}{n+1}} \gg m \approx \int_E g$.

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$\Rightarrow$ what happens for $g = 0$?

- Gobal minima $\Rightarrow$ Isoperimetric problem.
- Local minima and stationary points $\Rightarrow$ Alexandrov Thm.

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Question: if $g \neq 0$ and m small, global minima look like balls?

- Global minima $\approx$ balls *[Figalli-Maggi, ARMA 11]* (via quantitative isoperimetric inequality).
- Local minima and stationary points $\Rightarrow$ Quantitative estimates Alexandrov Thm.

Proofs :

- Alexandrov ('62): method of moving planes.
- Reilly *(Indiana '77)*: Reilly's identity.
- Ros *(Rev. Math. Ib. '87)* and Montiel - Ros *(Proc. Do Carmo '91)*: Reilly's identity and Heintze-Karcher inequality.
- Brendle - Eichmair *JDG '13* and Brendle *Publ. Math. IHÈS '13* Heintze-Karcher inequality and flows techiniques.

Strategy – Study these proofs **quantitatively**:

- Alexandrov's proof: proximity to a single ball and optimal quantitative estimates;
- Ros' proof: describe bubbling and (not optimal) quantitative estimates.

Sharp estimates for proximity to a single ball

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Theorem ***[C.-Vezzoni, JEMS '18]***

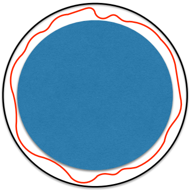
Ω satisfies an int/ext touching ball condition of radius $\rho_0 > 0$.
There exist $\delta_0(n, \rho_0, |\partial\Omega|) > 0$ and $C(n, \rho_0, |\partial\Omega|)$ s.t.

if $\operatorname{osc} H \leq \delta_0$ then

$$B_{r_i}(O) \subseteq \Omega \subseteq B_{r_e}(O)$$

with

$$r_e - r_i \leq C \operatorname{osc} H \, .$$



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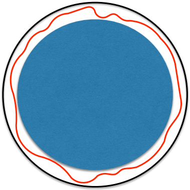

Moreover, $\partial\Omega$ is diffeomorphic to a sphere and there exists a C^1 map $F = Id + \Psi\nu : \partial B_{r_i} \to S$ s.t. $\|\Psi\|_{C^1(\partial B_{r_i})} \leq C \operatorname{osc} H\,.$

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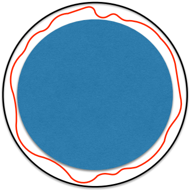

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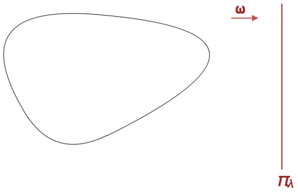
Remark: the stability estimates are **optimal**!!

Proof of Alexandrov's Theorem by moving planes

1. Moving planes.

For each direction $\omega \in \mathbb{R}^{n+1}$ there exists a hyperplane π_ω orthogonal to ω such that Ω is symmetric with respect to π_ω.



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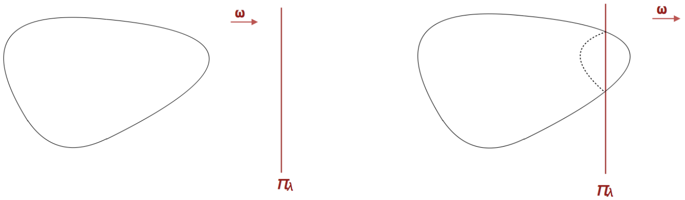

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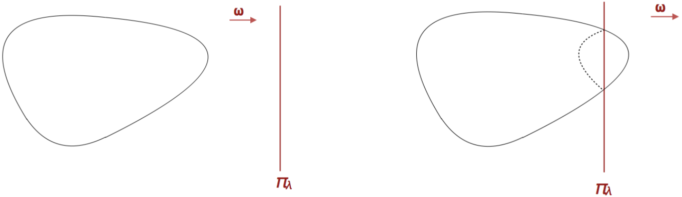
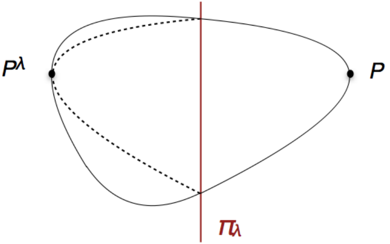
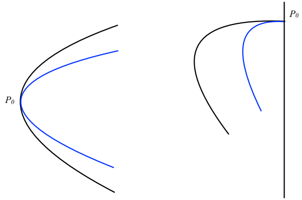




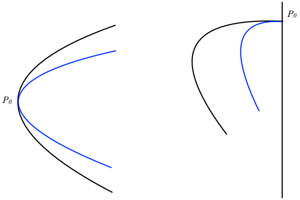
Critical positions: there exist two critical position for which the reflected *cap* is tangent to the hypersurface itself:



The symmetry case:

- $u^1,\, u^2 :\; B_r(P) \cap T_pS \to \mathbb{R}$ s.t. $\mathcal{L}(u^i) := \operatorname{div}\left(\frac{\nabla u^i}{\sqrt{1+|\nabla u^i|^2}}\right) = nH\,.$
- $u^1 - u^2 \geq 0$ with
$$L(u^1 - u^2) = \mathcal{L}(u^1) - \mathcal{L}(u^2) = 0\,.$$
Strong maximum principle and ***Hopf's lemma*** imply that $u^1 = u^2$.
- The set of tangency points is open and closed $\Rightarrow$ S is symmetric with respect to ω.

Critical positions: there exist two critical position for which the reflected *cap* is tangent to the hypersurface itself:



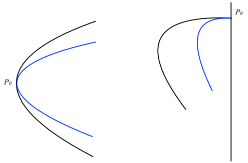
The **ALMOST** symmetry case:

- $u^1,\, u^2 :\; B_r(P) \cap T_pS \to \mathbb{R}$ s.t. $\mathrm{div}\left(\frac{\nabla u^i}{\sqrt{1+|\nabla u^i|^2}}\right) = nH\,.$
- $u^1 - u^2 \geq 0$ with
$$L(u^1 - u^2) = \mathcal{L}(u^1) - \mathcal{L}(u^2) = \mathbf{H_1} - \mathbf{H_2}\,.$$
Harnack's inequality and ***quantitative Hopf's lemma*** imply that $u^1 - u^2$ is *small*.
- Propagate the smallness $\Rightarrow S$ is **ALMOST** symmetric with respect to ω.

IDEA OF THE STABILITY PROOF

Quantitative study of the method of moving planes:

Harnack inequalities and elliptic regularity



$$|L(u_1 - u_2)| \leq \operatorname{osc}(H)$$

$$\sup(u_1 - u_2) \leq C \inf(u_1 - u_2) + \operatorname{osc}(H).$$

$$\|u_1 - u_2\|_{C^1} \leq C \operatorname{osc}(H).$$

- Harnack, Carleson, Hopf: quantitative versions of the maximum principle and Hopf's Lemma $\Rightarrow$ around P: the cap is close to S.
- Approximate symmetry in one direction: starting from the tangency point P, propagate the information. Delicate argument when we are close to $\partial\Omega \cap \pi$.
- Approximate symmetry with respect to O: approximate symmetry in $N+1$ directions.
- Conclusion: every critical hyperplane is close to O.

More general versions of Alexandrov's theorem

[C.-Vezzoni, *IMJ*] and [C.-Roncoroni-Vezzoni, *AMPA*]:

$$\mathsf{H}_S = f(\kappa_1, \ldots, \kappa_{n-1}) \text{ in } \mathbb{M}^n_+ .$$

Theorem: $\Omega \subset \mathbb{M}^n_+$ satisfies an int/ext touching ball condition of radius $\rho_0 > 0$. There exist $\delta_0(n, \rho_0, |\partial\Omega|) > 0$ and $C(n, \rho_0, |\partial\Omega|)$ s.t. if $\delta \leq \delta_0$, then

$$B_{r_i}(O) \subseteq \Omega \subseteq B_{r_e}(O) \quad \text{with} \quad r_e - r_i \leq C \operatorname{osc} \mathsf{H}_S .$$

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OTHER QUANTITATIVE STUDIES OF THE MMP:

- C.-Magnanini-Sakaguchi (*J. Anal. Math. '16*) for an overdeterimed problem in PDEs.
- Serrin's overdetermined problem: Aftalion-Busca-Reichel (*Adv. Diff. Eq. '99*), C.-Magnanini-Vespri (*AMPA '16*)
- C., A. Roncoroni (*Bruno Pini Math. Anal. Semin. '18*) - a rewiew

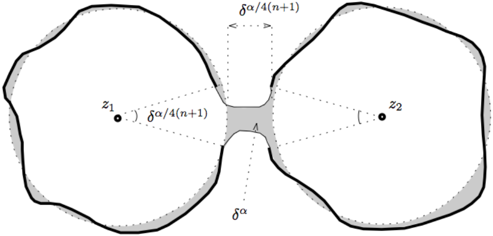
REMOVING THE TOUCHING BALL CONDITION: BUBBLING?

If H is constant, then $H = H_0 = \dfrac{nP(\Omega)}{(n+1)|\Omega|}$:

$$nP(\Omega) = \int_{\partial\Omega} H_0 x \cdot \nu = H_0 \int_{\partial\Omega} x \cdot \nu = (n+1)|\Omega| H_0$$

Rescale Ω s.t. $H_0 = n$, and define $\delta(\Omega) = \|H/n - 1\|_{C^0(\partial\Omega)}$.

C.-Maggi, CPAM '17: Ω is close to a collection of tangent spheres of radius n/H_0 and most of $\partial\Omega$ can be $C^{1,\beta}$ parametrized.



$\alpha = \frac{1}{2(n+2)}$

C.-Maggi, CPAM '17 - Part I

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there exists a family of disjoint balls $\{B_{z_j,1}\}_{j\in J}$ with $\# J \le L$ and s.t.

$$G = \bigcup_{j\in J} B_{z_j,1}$$

satisfies

$$\frac{|\Omega \Delta G|}{|\Omega|} \quad \le \quad C(n)\,L^2\,\delta(\Omega)^\alpha\,,$$

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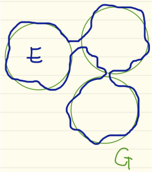
$$\begin{aligned}
\frac{|\Omega \Delta G|}{|\Omega|} &\le C(n)\,L^2\,\delta(\Omega)^\alpha\,,\\
\frac{|P(\Omega) - \#J\,P(B)|}{P(\Omega)} &\le C(n)\,L^2\,\delta(\Omega)^\alpha\,,\\
\frac{\mathrm{hd}(\partial\Omega, \partial G)}{\mathrm{diam}(\Omega)} &\le C(n)\,L^{3/n}\,\delta(\Omega)^{\alpha/4n^2(n+1)}\,.
\end{aligned}$$



C.-Maggi, CPAM '17 - Part II

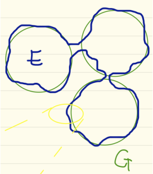
There exist $\Sigma \subset \partial G$ and $\psi : \Sigma \to \mathbb{R}$ s.t.:
$\partial G \setminus \Sigma$ consists at most of $C(n)$ L-spherical caps with diameters bounded by $C(n)\,\delta(\Omega)^{\alpha/4(n+1)}$.
The function ψ is such that $(I + \psi\,\nu_G)(\Sigma) \subset \partial\Omega$ and

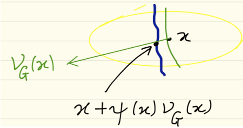
$$\|\psi\|_{C^{1,\gamma}(\Sigma)} \le C(n,\gamma)\,, \qquad \forall \gamma \in (0,1)\,,$$

$$\frac{\|\psi\|_{C^0(\Sigma)}}{\mathrm{diam}(\Omega)} \le C(n)\,L\,\delta(\Omega)^{\alpha}$$

$$\|\nabla\psi\|_{C^0(\Sigma)} \le C(n)\,L^{2/n}\,\delta(\Omega)^{\alpha/8n(n+1)}\,,$$

$$\frac{\mathcal{H}^n(\partial\Omega \setminus (I + \psi\,\nu_G)(\Sigma))}{P(\Omega)} \le C(n)\,L^{4/n}\,\delta(\Omega)^{\alpha/4n(n+1)}\,,$$





where $(I + \psi\,\nu_G)(x) = x + \psi(x)\,\nu_G(x)$ and ν_G is the outward normal to G.

C.-Maggi, CPAM '17. Part III

Moreover:

(i) if $\# J \geq 2$, then $\forall\, j \in J$ exists $\ell \in J$, $\ell \neq j$, s.t.

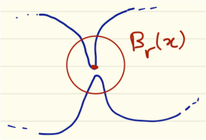
$$\frac{\mathrm{dist}(\partial B_{z_j,1}, \partial B_{z_\ell,1})}{\mathrm{diam}(\Omega)} \leq C(n)\,\delta(\Omega)^{\alpha/4(n+1)}\,,$$

i.e. every ball $\{B_{z_j,1}\}_{j\in J}$ is almost tangent to another ball of G.

(ii) if there exists $\kappa \in (0,1)$ s.t.

$$|B_{x,r} \setminus \Omega| \geq \kappa\,|B|\,r^{n+1}\,, \qquad \forall x \in \partial\Omega\,, r < \kappa\,,$$

and $\delta(\Omega) \leq c(n, L, \kappa)$, then $\# J = 1$, i.e. Ω is close to a single sphere.



Some remarks:

- The proof is based on integral identities, in particular Ros' proof of Alexandrov's Theorem / Heintze-Karcher inequality.
- The examples available in litterature are the *only* possible ones which are not close to a single sphere.
- The theorem is qualitatively optimal.
- Apply our estimates for capillarity problems: local minima are close to a ball.
- Anisotropic version in Delgadino, Maggi, Mihaila, Neumayer (Arch. Rat. Mech. Anal. 2018).
- The case of closeness to a single sphere has been improved in Krummel-Maggi (Calc Var PDE '17).

IDEA OF THE PROOF - H CONSTANT

Proof by integral identities (Reilly, Ros, Montiel-Ros)

- We introduce an auxilliar problem:

$$\begin{cases} \Delta f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega\,. \end{cases}$$

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- Heintze-Karcher inequality:

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1)|\Omega|\,,$$

equality if and only if Ω is a ball.

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- We introduce an auxilliar problem:

$$\begin{cases} \Delta f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega\,. \end{cases}$$

- Reilly's identity for f: $\int_\Omega (\Delta f)^2 - |\nabla^2 f|^2 = \int_{\partial\Omega} H f_\nu^2\,.$
- Heintze-Karcher inequality:

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1)|\Omega|\,,$$

equality if and only if Ω is a ball.

- Conclusion. If H is costant:

$$\frac{n}{H} P(\Omega) = \int_{\partial\Omega} \frac{n}{H} \geq (n+1)|\Omega| = \frac{1}{H} \int_{\partial\Omega} H x \cdot \nu = \frac{n}{H} P(\Omega)\,.$$

Idea... of the idea.. of the proof

- Starting point: Proof by Ros / Heintze-Karcher inequality.
  If $\Delta f = 1$ in Ω and $f = 0$ on $\partial\Omega$, then

$$\int_\Omega \left|\nabla^2 f - \frac{\mathrm{Id}}{n+1}\right| \leq C(n,\Omega)\delta^{1/2}, \quad \int_{\partial\Omega} \left|\frac{n/H_0}{n+1} - |\nabla f|\right|^2 \leq C(n,\Omega)\delta .$$

- Compactness argument: Reilly's identity, $P-$function, Pohozaev's identity, Allard's regularity thm..
- Quantitative version...

NONLOCAL THEORY

Nonlocal Perimeter: introduced by Caffarelli-Souganidis (CPAM 2008) and Caffarelli-Roquejoffre-Savin (CPAM 2010).

$$P_s(\Omega) = \int_\Omega \int_{\Omega^c} \frac{dx\,dy}{|x-y|^{n+2\,s}}\,, \quad \Omega^c = \mathbb{R}^n \setminus \Omega\,, \quad s \in (0, 1/2)\,.$$

The nonlocal mean curvature H_s^Ω is the first variation of the nonlocal perimeter.

If $\partial\Omega \in C^{1,\alpha}$, $\alpha > 2s$, then

$$H_s^\Omega(p) = \frac{1}{\omega_{n-2}} P.V. \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_\Omega(x)}{|x-p|^{n+2s}}\,dx\,, \qquad \widetilde{\chi}_\Omega(x) = \chi_{\Omega^c}(x) - \chi_\Omega(x)\,.$$

Remark. $(1-2s)$ nonlocal $\to$ local:

$$\lim_{s\to\frac{1}{2}^-}(1-2s)P_s(\Omega) = P(\Omega) \qquad \lim_{s\to\frac{1}{2}^-}(1-2s)H_s(\Omega) = H(\Omega)\,.$$

Nonlocal rigidity

C.-Figalli-Maggi-Novaga, Crelle '18 - Part I: simmetry

$$H_s^{\Omega}(p) = \frac{1}{\omega_{n-2}} \, P.V. \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_{\Omega}(x)}{|x-p|^{n+2s}} \, dx \, ,$$

where

$$\widetilde{\chi}_{\Omega}(x) = \chi_{\Omega^c}(x) - \chi_{\Omega}(x) \, .$$

Theorem (Nonlocal Alexandrov Thm)

If Ω is a bounded open set of class $C^{1,2s}$ and H_s^{Ω} is constant, then $\partial\Omega$ is a sphere.

See also X. Cabré, M. Fall, J. Sola-Morales, T. Weth (Crelle '18).

C.-Figalli-Maggi-Novaga, Crelle '18 - Part II: stability

$$\text{Deficit:} \quad \delta_s(\Omega) = \sup_{p,q\in\Sigma,\, p\neq q} \frac{|H_s^\Omega(p) - H_s^\Omega(q)|}{|p-q|} \,.$$

$$\text{Rescaled distance:} \quad \rho(\Omega) = \inf\left\{\frac{t-s}{\mathrm{diam}(\Omega)} : p\in\Omega\,, B_s(p)\subset\Omega\subset B_t(p)\right\}.$$

C.-Figalli-Maggi-Novaga, Crelle '18 - Part II: stability

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Theorem (Stability Alexandrov Theorem)

Ω bounded open set of class $C^{2,\alpha}$, $\alpha > 2s$. There exists $\hat{C}(n) > 0$ s.t.

$$\rho(\Omega) \le \hat{C}(n)\,\eta_s(\Omega)\,, \quad \eta_s(\Omega) = \frac{\mathrm{diam}(\Omega)^{2n+2s+1}}{|\Omega|^2}\,\delta_s(\Omega)\,.$$

Moreover, there exists $\eta(n,s) > 0$ s.t. if $\eta_s(\Omega) \le \eta(n,s)$ then there exists a map $F : \partial B_1(O) \to \mathbb{R}^n$ of class $C^{2,\tau}$ for any $\tau < 2s$, such that $F(\partial B_1(O)) = \partial\Omega$ and

$$\|F - I\|_{C^{2,\tau}(\partial B_1(O))} \le C(n,s,\tau)\,\eta_s(\Omega).$$

In particular, if $\eta_s(\Omega)$ is sufficiently small, then Ω is convex.

Remarks

Proofs.

- Symmetry: method of moving planes.
- Stability: quantitative study of the nonlocal method of moving planes.

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Remark: Local vs NonLocal

- Disjoint union of spheres vs single sphere.
- Convexity for small deficit.
- Local results are not obtained as a limit of the nonlocal one for $s \to 1/2$.

Thanks!