Direct and inverse Sturm-Liouville problems: A method of solution

Vladislav V. Kravchenko

Regional mathematical center of Southern Federal University, Rostov-on-Don, Russia & Department of Mathematics, Cinvestav, Mexico;

May, 2021, Athens
1. Series representations for transmutation (transformation) operator kernels;

2. Series representations for solutions of Sturm-Liouville equations;

3. Efficient methods for solving direct spectral problems on finite and infinite intervals;

\[-y'' + q(x)y = \lambda y,\]

\[\uparrow\]

Liouville transformation

\[-(P(x)Y'(x))' + Q(x)Y(x) = \lambda R(x)Y(x)\]

Sturm-Liouville equation

Jacques Charles Francois Sturm (1803-1855) y Joseph Liouville (1809-1882)
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Sturm-Liouville equation

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Let \( \varphi(\rho, x) \) denote a solution of the Cauchy problem

\[
-\varphi'' + q(x)\varphi = \lambda \varphi, \quad x \in (0, b),
\]

\[
\varphi(\rho, 0) = 1, \quad \varphi'(\rho, 0) = h \in \mathbb{C},
\]

\( \rho := \sqrt{\lambda} \in \mathbb{C}, \quad \text{Im} \rho \geq 0, \quad q \in L_2(0, b) \). Then there exists a continuous function \( G(x, t) \) such that

\[
\varphi(\rho, x) = \cos \rho x + \int_0^x G(x, t) \cos \rho t \, dt, \quad \forall \rho \in \mathbb{C}.
\]

Construction of the kernel $G$

**Theorem**

$$G(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n} \left( \frac{t}{x} \right), \quad 0 \leq t \leq x$$

where $P_k$ - Legendre polynomials. The series converges w.r.t $t$ in $L_2$-norm (if $q \in C[0, b]$ it converges uniformly). The coefficients $g_n$ are constructed by a simple recurrent integration procedure

$$g_n(x) = \int_0^x c_n(s)g_{n-1}(s)ds,$$

where $c_n$ are known, starting with

$$g_0(x) = \phi(0, x) - 1.$$

Neumann series of Bessel functions

\[ G(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n} \left( \frac{t}{x} \right) \]

\[ \varphi(\rho, x) = \cos \rho x + \int_{0}^{x} G(x, t) \cos \rho t \, dt. \]

\[ \varphi(\rho, x) = \cos \rho x + \sum_{n=0}^{\infty} \frac{g_n(x)}{x} \int_{0}^{x} P_{2n} \left( \frac{t}{x} \right) \cos \rho t \, dt \]

\[ = \cos \rho x + \sum_{n=0}^{\infty} (-1)^n g_n(x) j_{2n} (\rho x), \]

where \( j_k (\rho x) := \sqrt{\frac{\pi}{2 \rho x}} J_{k+1/2} (\rho x) \) are spherical Bessel functions.
For computing

\[ g_0(x) = \varphi(0, x) - 1 \]

we use the SPPS method from


**Figure:** Mike Porter
Consider
\[
\varphi_N(\rho, x) = \cos \rho x + \sum_{n=0}^{N} (-1)^n g_n(x) j_{2n}(\rho x).
\]

For \( \rho \in \mathbb{R} \) we have
\[
|\varphi(\rho, x) - \varphi_N(\rho, x)| \leq \varepsilon_N(x) \quad \text{-independent of } \rho.
\]

For \( \rho \in \mathbb{C} \) belonging to the strip \( |\text{Im} \rho| \leq C, C \geq 0, \)
\[
|\varphi(\rho, x) - \varphi_N(\rho, x)| \leq \varepsilon_N(x) \frac{\sinh(Cx)}{C}.
\]

Estimates of the convergence rates depending on the smoothness of \( q \) in:

\[
\begin{aligned}
-u'' + e^x u &= \lambda u, \quad 0 \leq x \leq \pi, \\
u(0, \lambda) &= u(\pi, \lambda) = 0.
\end{aligned}
\]

*Figure:* Absolute and relative error of the first 100 eigenvalues computed in Matlab, machine precision, \( N = 29 \).
Same example, next 400 eigenvalues

Vladislav V. Kravchenko (Rostov)
In 200-digits precision arithemtics in Wolfram Mathematica the first 10000 eigenvalues and eigenfunctions are obtained with the absolute error of order $10^{-105}$.

Figure: Abs. error of the first 500 eigenvalues (blue) and eigenfunctions (violet).
The inverse Sturm-Liouville problem on a finite interval

Given two sequences of real numbers \( \{ \lambda_n \}_{n=0}^{\infty} \) and \( \{ \alpha_n \}_{n=0}^{\infty} \) (satisfying certain conditions).
Find \( q(x) \) and the numbers \( h, H \) such that \( \{ \lambda_n \}_{n=0}^{\infty} \) be the spectrum of the problem

\[
-y'' + q(x)y = \lambda y,
\]

\[
y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0,
\]

and \( \alpha_n, n = 0, 1, \ldots \) be the corresponding weight numbers.

\[
\alpha_n := \int_0^{\pi} \phi^2(\rho_n, x) \, dx.
\]
Several numerical methods for the inverse S-L problem


All of them are iterative and no one is able to recover the boundary conditions.

“As was mentioned earlier, if complete spectral data is available, then it is in theory possible to determine the boundary conditions as part of the solution of the problem. We do not believe, however, that this is numerically feasible in most cases.” W. Rundell, P. E. Sacks
We use the representation

\[ G(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n} \left( \frac{t}{x} \right), \quad 0 \leq t \leq x. \]

Since

\[ g_0(x) = \varphi(0, x) - 1, \]

if the coefficient \( g_0 \) is known the potential \( q \) can be computed by

\[ q = \frac{g_0''}{g_0 + 1} \] (1)

The number \( h \) is obtained from the equality

\[ h = g_0'(0). \] (2)

The number \( H \) is determined from

\[ H = \omega - h - \frac{1}{2} \int_{0}^{\pi} q(t) \, dt, \]

where

\[ \omega = \pi \lim_{n \to \infty} n(\rho_n - n). \]
The Gel’fand-Levitan equation

\[ F(x, t) = \sum_{n=0}^{\infty} \left( \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_0^n} \right) \]

where

\[ \alpha_0^n = \begin{cases} \pi/2, & n > 0, \\ \pi, & n = 0. \end{cases} \]

The functions \( G \) and \( F \) are related by the equation

\[ G(x, t) + F(x, t) + \int_0^x F(t, s) G(x, s) \, ds = 0, \quad 0 < t < x. \] (G-L)

\[ (G-L) \to G(x, t) \to q(x) = 2 \frac{dG(x, x)}{dx}. \]
\[ G(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n}\left(\frac{t}{x}\right) \]

\[ G(x, t) + F(x, t) + \int_0^x F(t, s) G(x, s) \, ds = 0, \quad 0 < t < x. \]

\[ \frac{g_k(x)}{4k + 1} + \sum_{m=0}^{\infty} g_m(x) A_{km}(x) = b_k(x) \quad \text{for all } k = 0, 1, \ldots, \]
Theorem The coefficients $g_n$ satisfy the infinite system of linear algebraic equations

$$\frac{g_k(x)}{4k + 1} + \sum_{m=0}^{\infty} g_m(x) A_{km}(x) = b_k(x) \quad \text{for all } k = 0, 1, \ldots, \quad (3)$$

where

$$A_{km}(x) := (-1)^{m+k} x \sum_{n=0}^{\infty} \left( \frac{j_{2m}(\rho_n x) j_{2k}(\rho_n x)}{\alpha_n} - \frac{j_{2m}(nx) j_{2k}(nx)}{\alpha_n^0} \right),$$

$$b_k(x) := (-1)^k x \sum_{n=0}^{\infty} \left( \frac{\cos nx j_{2k}(nx)}{\alpha_n^0} - \frac{\cos \rho_n x j_{2k}(\rho_n x)}{\alpha_n} \right).$$
**Figure:** Potential $q(x) = |3 - |x^2 - 3||$ recovered from 200 eigenvalues, with 6 equations.
**Figure:** Potential recovered from 200 eigenvalues, with 6 equations.

V. V. Kravchenko *Direct and inverse Sturm-Liouville problems: A method of solution*, Birkhäuser (2020)

V. V. Kravchenko, S. M. Torba *A direct method for solving inverse Sturm-Liouville problems*. Inverse Problems, **37** (2021) # 015015 (32pp)
Scattering

\[-y'' + q(x)y = \lambda y, \quad x \in (-\infty, \infty),\]

\[\int_{-\infty}^{\infty} (1 + |x|) |q(x)| \, dx < \infty.\]

Jost solutions:

\[e(\rho, x) = e^{i\rho x} \left(1 + o(1)\right), \quad x \to +\infty\]

\[g(\rho, x) = e^{-i\rho x} \left(1 + o(1)\right), \quad x \to -\infty\]

1. **Discrete spectrum:** There may exist a finite number of eigenvalues \( \lambda_n < 0 \), for which \( \exists y(x) \in L_2(-\infty, \infty) \). Then \( y(x) = c_1 e(\rho, x) = c_2 g(\rho, x) \).

Norming constants:

\[
\alpha_k := \left( \int_{-\infty}^{\infty} e^2(\rho_k, x) \, dx \right)^{-1}.
\]
Discrete spectrum: There may exist a finite number of eigenvalues \( \lambda_n < 0 \), for which \( \exists y(x) \in L_2(-\infty, \infty) \). Then \( y(x) = c_1 e(\rho, x) = c_2 g(\rho, x) \).

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\]

Reflection coefficient:
Figure: A schematic representation of the scattering model.
Direct problem: Given $q(x)$, find the scattering data

$$s(\rho), \rho \in \mathbb{R}, \quad \lambda_k, \alpha_k, k = 1, N$$
- **Direct problem:** Given $q(x)$, find the scattering data
  
  $$s(\rho), \rho \in \mathbb{R}, \quad \lambda_k, \alpha_k, k = 1, N$$

- **Inverse problem:** Given scattering data, find $q(x)$.
- **Direct problem:** Given \( q(x) \), find the scattering data

\[ s(\rho), \, \rho \in \mathbb{R}, \quad \lambda_k, \, \alpha_k, \, k = 1, N \]

- **Inverse problem:** Given scattering data, find \( q(x) \).

- **Applications:** Inverse scattering transform method; nonlinear Fourier transform.


• **Direct problem:** Given $q(x)$, find the scattering data

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• **Applications:** Inverse scattering transform method; nonlinear Fourier transform.


Transmutation operator:

\[ e(\rho, x) = e^{i\rho x} + \int_{x}^{\infty} A(x, t) e^{i\rho t} dt \]

[B. Ya. Levin (1956)]
Same approach works

Transmutation operator:

\[ e(\rho, x) = e^{i\rho x} + \int_x^\infty A(x, t)e^{i\rho t} \, dt \]

[B. Ya. Levin (1956)]

\[ A(x, t) = \sum_{n=0}^{\infty} a_n(x)L_n(t-x)e^{\frac{x-t}{2}} \quad (L_n - \text{Laguerre polynomials}) \]

A recursive integration procedure for \( a_n(x) \), beginning with

\[ a_0(x) = e^{\frac{i}{2}, x}e^{\frac{x}{2}} - 1. \]

[V. V. Kravchenko 2019; B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko 2020]
\[ e(\rho, x) = e^{i\rho x} \left( 1 + \sum_{n=0}^{\infty} a_n(x) \int_0^\infty L_n(t) e^{-\left(\frac{1}{2} + i\rho\right)t} dt \right) . \]
\[ e(\rho, x) = e^{i\rho x} \left( 1 + \sum_{n=0}^{\infty} a_n(x) \int_{0}^{\infty} L_n(t) e^{-\left(1/2-i\rho\right) t} \, dt \right). \]

Series representation for Jost solutions:

\[ e(\rho, x) = e^{i\rho x} \left( 1 + (z + 1) \sum_{n=0}^{\infty} (-1)^n z^n a_n(x) \right), \]

where

\[ z := \left( \frac{1}{2} + i\rho \right) / \left( \frac{1}{2} - i\rho \right). \]
\[ \rho \mapsto z = \frac{1/2 + i\rho}{1/2 - i\rho} \]

\[ \text{Im}(\rho) > 0 \]

\[ z = -1 \quad z = 0 \quad z = 1 \]

\[ |z| < 1 \]
Example
\[
q(x) = \begin{cases} 
q_1(x), & x < 0 \\
q_2(x), & x > 0
\end{cases}
\]

where

\[
q_1(x) = \frac{16 \left( \sqrt{2} + 1 \right)^2 e^{-2\sqrt{2}x}}{\left( \left( \sqrt{2} + 1 \right)^2 e^{-2\sqrt{2}x} - 1 \right)^2}
\]

and

\[
q_2(x) = \frac{96 e^{2x} \left( 81 e^{8x} - 144 e^{6x} + 54 e^{4x} - 9 e^{2x} + 1 \right)}{(36 e^{6x} - 27 e^{4x} + 12 e^{2x} - 1)^2}.
\]
Reflection coefficient

\[ s(\rho) = \frac{(\rho + i)(\rho + 2i)(101\rho^2 - 3i\rho - 400)}{(\rho - i)(\rho - 2i)(50\rho^4 + 280i\rho^3 - 609\rho^2 - 653i\rho + 400)}. \]

**Figure:** \( s(\rho) \) computed. Abs. error \( < 10^{-12} \).
Inverse scattering

Given the scattering data

\[ s(\rho), \rho \in \mathbb{R}, \quad \lambda_k, \alpha_k, \ k = 1, N, \]

find \( q(x) \).

[V. V. Kravchenko, On a method for solving the inverse scattering problem on the line. Math Meth Appl Sci. 42 (2019), 1321-1327]
Gelfand-Levitan-Marchenko equation

\[ F(x + y) + A(x, y) + \int_x^\infty A(x, t) F(t + y) dt = 0, \quad y > x, \]

where

\[ F(x) = \sum_{k=1}^{N} \alpha_k e^{-\tau_k x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\rho) e^{i\rho x} d\rho. \]
\[ A(x, t) = \sum_{n=0}^{\infty} a_n(x) L_n(t - x) e^{\frac{x-t}{2}} \]

\[
F(x + y) + A(x, y) + \int_{x}^{\infty} A(x, t) F(t + y) dt = 0
\]

\[
a_m(x) + \sum_{n=0}^{\infty} a_n(x) A_{mn}(x) = r_m(x), \quad m = 0, 1, \ldots
\]
\[ A_{mn}(x) : = (-1)^{m+n} \left( \sum_{k=1}^{N} \alpha_k e^{-2\tau_k x} \frac{(\frac{1}{2} - \tau_k)^{m+n}}{(\frac{1}{2} + \tau_k)^{m+n+2}} \right) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\rho) e^{2i\rho x} \frac{(\frac{1}{2} + i\rho)^{m+n}}{(\frac{1}{2} - i\rho)^{m+n+2}} d\rho, \]

\[ r_m(x) : = (-1)^{m+1} \left( \sum_{k=1}^{N} \alpha_k e^{-2\tau_k x} \frac{(\frac{1}{2} - \tau_k)^{m}}{(\frac{1}{2} + \tau_k)^{m+1}} \right) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\rho) e^{2i\rho x} \frac{(\frac{1}{2} + i\rho)^{m}}{(\frac{1}{2} - i\rho)^{m+1}} d\rho, \]

It is enough to find \( a_0(x) \)
\[ a_0(x) = e^{i \frac{x}{2}} e^{\frac{x}{2}} - 1 \]

\[ q = \frac{a_0'' - a_0'}{a_0 + 1} \]
Figure: Potential $q(x)$ recovered with 25 equations.
Inverse scattering transform method

Korteweg - de Vries equation

\[ u_t + u_{xxx} - 6uu_x = 0. \]

Diagram: schematic representation of the IST

\[ u(x,0) = q(x) \quad \rightarrow \quad -y'' + q(x)y = \lambda y \]

direct scattering problem

\[ \text{Scatt.data } (t=0) \]

\[ \downarrow \text{linear evolution law} \]

\[ u(x,t) \quad \leftarrow \quad \text{inverse scattering problem} \]

\[ \text{Scatt.data } (t) \]

\[ u(x, 0) = xe^{-x^2}. \]
Рис. 4.3. Неселитонный вклад в решение уравнения Кортенсга — де Фриза $u_t - 6u_{xx} + u_{xxxx} = 0$ для начального профиля $u(x, 0) = xe^{-x^2}$. Направление распространения — слева направо. Появление колебаний впереди импульса связано с наложением периодических граничных условий. Решение получено У. Фернандо с м.л.
Sturm-Liouville problem on the half-line


1. Sturm-Liouville problem on the half-line

2. Inverse scattering on the half-line
3. Quantum scattering for the perturbed Bessel equation

\[ Lu := -u'' + \left( \frac{\ell(\ell + 1)}{x^2} + q(x) \right) u = \rho^2 u, \quad x > 0 \]


4. Recovery of Sturm-Liouville problem from the Weyl function

THANK YOU