Direct and inverse Sturm-Liouville problems: A method of solution

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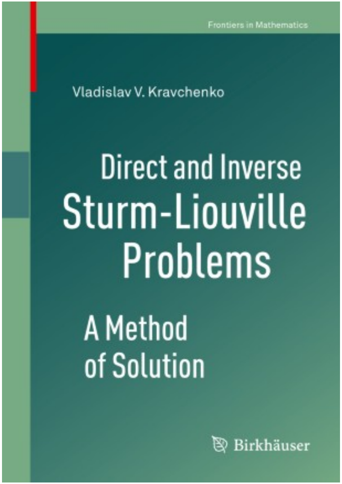
May, 2021, Athens



Frontiers in Mathematics
Vladislav V. Kravchenko
Direct and Inverse
Sturm-Liouville
Problems
A Method
of Solution
Birkhäuser

1. Series representations for transmutation (transformation) operator kernels;

2. Series representations for solutions of Sturm-Liouville equations;

3. Efficient methods for solving direct spectral problems on finite and infinite intervals;

4. Direct methods for solving inverse spectral problems on finite and infinite intervals.

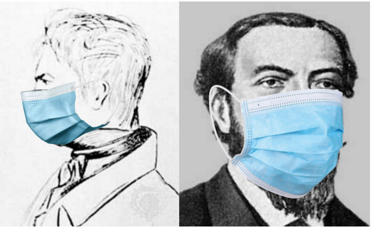
$$-y'' + q(x)y = \lambda y,$$

$$\Uparrow$$

Liouville transformation

$$-\left(P(x)Y'(x)\right)' + Q(x)Y(x) = \lambda R(x)Y(x)$$

Sturm-Liouville equation



Jacques Charles Francois Sturm (1803-1855) y Joseph Liouville (1809-1882)

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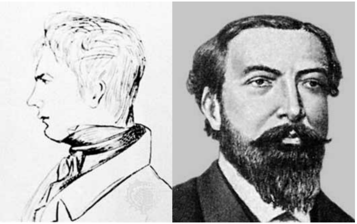


Jacques Charles Francois Sturm (1803-1855) y Joseph Liouville (1809-1882)

Transmutation operators (J. Delsarte 1938)

Let $\varphi(\rho, x)$ denote a solution of the Cauchy problem

$$-\varphi'' + q(x)\varphi = \lambda\varphi, \quad x \in (0, b),$$

$$\varphi(\rho, 0) = 1, \quad \varphi'(\rho, 0) = h \in \mathbb{C},$$

$\rho := \sqrt{\lambda} \in \mathbb{C}$, $\mathrm{Im}\,\rho \geq 0$, $q \in L_2\,(0, b)$. Then there exists a continuous function $G(x, t)$ such that

$$\varphi(\rho, x) = \cos\rho x + \int_0^x G(x, t)\cos\rho t\,dt, \quad \forall\rho \in \mathbb{C}.$$

A. Ya. Povzner 1948; B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987; V. A. Marchenko, *Sturm-Liouville Operators and Applications*, AMS Chelsea Publishing, 2011; V. A. Yurko, *Introduction to the theory of inverse spectral problems.* Fizmatlit, 2007, 384pp. (Russian); S. M. Sitnik, E. L. Shishkina, *Transmutations, singular and fractional differential equations with applications to mathematical physics*, Elsevier, 2020.

Construction of the kernel G

Theorem

$$G(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right),\quad 0\le t\le x$$

where P_k - Legendre polynomials. The series converges w.r.t t in L_2-norm (if $q\in C[0,b]$ it converges uniformly). The coefficients g_n are constructed by a simple recurrent integration procedure

$$g_n(x)=\int_0^x c_n(s)g_{n-1}(s)ds,$$

where c_n are known, starting with

$$g_0(x)=\varphi(0,x)-1.$$

[VK, L. J. Navarro and S. M. Torba, Appl. Math. & Comp. **314** (2017)]

Neumann series of Bessel functions

$$G(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right)$$

$$\searrow$$

$$\varphi(\rho,x)=\cos\rho x+\int_0^x G(x,t)\cos\rho t\,dt.$$

$$\Downarrow$$

$$\begin{aligned}\varphi(\rho,x) &= \cos\rho x+\sum_{n=0}^{\infty}\frac{g_n(x)}{x}\int_0^x P_{2n}\left(\frac{t}{x}\right)\cos\rho t\,dt\\ &= \cos\rho x+\sum_{n=0}^{\infty}(-1)^n g_n(x)\, j_{2n}(\rho x),\end{aligned}$$

where $j_k(\rho x) := \sqrt{\frac{\pi}{2\rho x}}J_{k+1/2}(\rho x)$ are spherical Bessel functions.

For computing

$$g_0(x) = \varphi(0, x) - 1$$

we use the SPPS method from

- V. V. Kravchenko, R. M. Porter, Spectral parameter power series for Sturm-Liouville problems. *Mathematical Methods in the Applied Sciences* **33** (2010), 459–468.

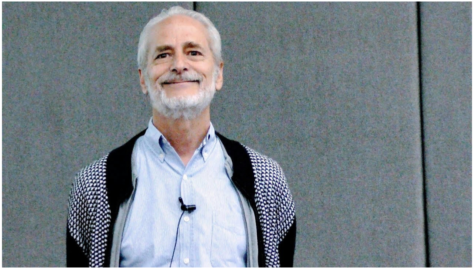


Figure: Mike Porter

Truncation accuracy estimate

Consider

$$\varphi_N(\rho,x)=\cos\rho x+\sum_{n=0}^{N}(-1)^n g_n(x)\,j_{2n}(\rho x).$$

For $\rho\in\mathbb{R}$ we have

$$|\varphi(\rho,x)-\varphi_N(\rho,x)|\le\varepsilon_N(x)\quad\text{-independent of }\rho.$$

For $\rho\in\mathbb{C}$ belonging to the strip $|\operatorname{Im}\rho|\le C$, $C\ge 0$,

$$|\varphi(\rho,x)-\varphi_N(\rho,x)|\le\varepsilon_N(x)\frac{\sinh(Cx)}{C}.$$

Estimates of the convergence rates depending on the smoothness of q in: [VK, L. J. Navarro and S. M. Torba, *Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions*, Appl. Math. and Comp., v. 314 (2017), 173-192].

Example First Paine problem (J. D. Pryce *Numerical solution of Sturm-Liouville problems*, Oxford: Clarendon Press, 1993)

$$\begin{cases} -u'' + e^x u = \lambda u, \quad 0 \le x \le \pi, \\ u(0,\lambda) = u(\pi,\lambda) = 0. \end{cases}$$

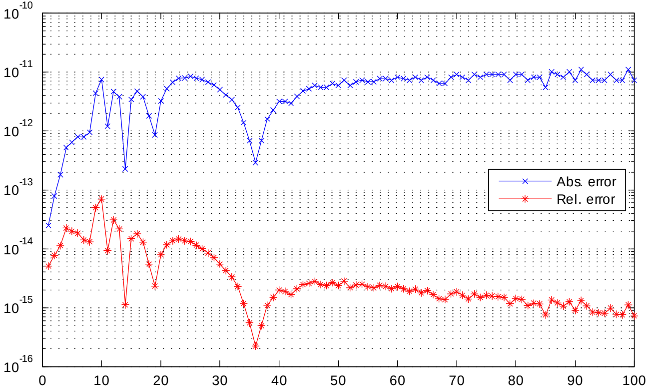
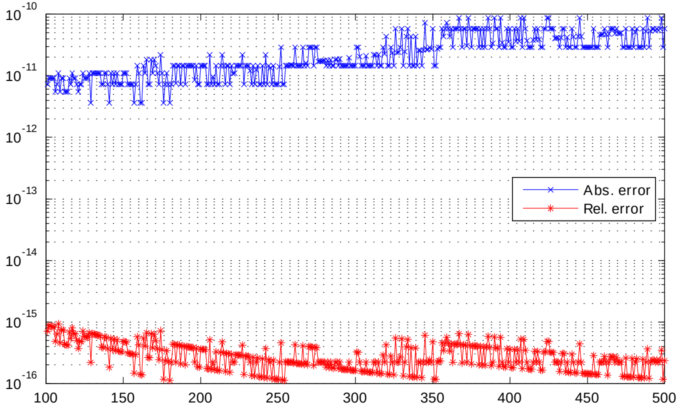


Figure: Absolute and relative error of the first 100 eigenvalues computed in Matlab, machine precision, $N = 29$.

Same example, next 400 eigenvalues



In 200-digits precision arithemtics in Wolfram Mathematica the first 10000 eigenvalues and eigenfunctions are obtained with the absolute error of order 10^{-105}.

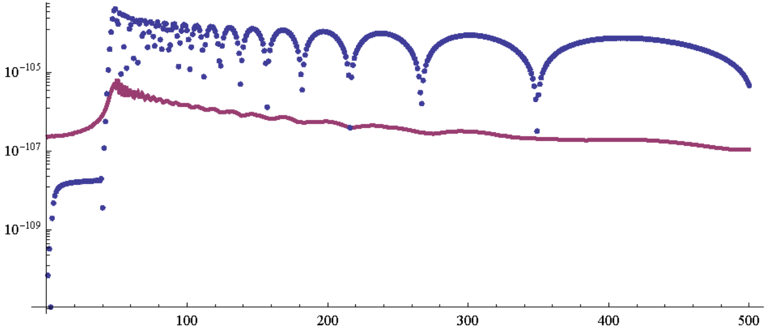


Figure: Abs. error of the first 500 eigenvalues (blue) and eigenfunctions (violet).

The inverse Sturm-Liouville problem on a finite interval

Given two sequences of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ (satisfying certain conditions).
Find $q(x)$ and the numbers h, H such that $\{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of the problem

$$-y'' + q(x)y = \lambda y,$$

$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0,$$

and α_n, $n = 0, 1, \ldots$ be the corresponding weight numbers.

$$\alpha_n := \int_0^{\pi} \varphi^2(\rho_n, x) dx.$$

Several numerical methods for the inverse S-L problem

W. Rundell, P. E. Sacks, *Reconstruction techniques for classical inverse Sturm–Liouville problems.* Math. Comput. 58 (1992), 161–83.

M. Ignatiev, V. Yurko, *Numerical methods for solving inverse Sturm-Liouville problems.* Results Math. 52 (2008), no. 1-2, 63–74.

A. Kammanee, C. Böckmann, *Boundary value method for inverse Sturm-Liouville problems.* Applied Mathematics and Computation, 214 (2009) 342-352.

B. M. Brown, V. S. Samko, I. W. Knowles, M. Marletta, *Inverse spectral problem for the Sturm–Liouville equation*. Inverse Problems 19 (2003) 235–252.

B. D. Lowe, M. Pilant, W. Rundell, The recovery of potentials from finite spectral data. SIAM J. Math. Anal. 23 (1992), no. 2, 482–504.

All of them are iterative and no one is able to recover the boundary conditions.

> *"As was mentioned earlier, if complete spectral data is available, then it is in theory possible to determine the boundary conditions as part of the solution of the problem. We do not believe, however, that this is numerically feasible in most cases." W. Rundell, P. E. Sacks*

We use the representation

$$G(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right), \quad 0\le t\le x.$$

Since

$$g_0(x)=\varphi(0,x)-1,$$

if the coefficient g_0 is known the potential q can be computed by

$$q=\frac{g_0''}{g_0+1} \tag{1}$$

The number h is obtained from the equality

$$h=g_0'(0). \tag{2}$$

The number H is determined from

$$H=\omega-h-\frac{1}{2}\int_0^{\pi}q(t)\,dt,$$

where

$$\omega=\pi\lim_{n\to\infty}n(\rho_n-n).$$

The Gel'fand-Levitan equation

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{\cos \rho_n x \, \cos \rho_n t}{\alpha_n} - \frac{\cos nx \, \cos nt}{\alpha_n^0} \right)$$

where

$$\alpha_n^0 = \begin{cases} \pi/2, & n > 0, \\ \pi, & n = 0. \end{cases}$$

The functions G and F are related by the equation

$$G(x,t) + F(x,t) + \int_0^x F(t,s) G(x,s)\, ds = 0, \quad 0 < t < x. \qquad \text{(G-L)}$$

$$\text{(G-L)} \to G(x,t) \to q(x) = 2 \frac{dG(x,x)}{dx}.$$

$$G(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right)$$

$$\swarrow \searrow$$

$$G(x,t)+F(x,t)+\int_0^x F(t,s)G(x,s)\,ds=0,\quad 0<t<x.$$

$$\Downarrow$$

$$\frac{g_k(x)}{4k+1}+\sum_{m=0}^{\infty}g_m(x)A_{km}(x)=b_k(x)\qquad \text{for all } k=0,1,\ldots,$$

Theorem The coefficients g_n satisfy the infinite system of linear algebraic equations

$$\frac{g_k(x)}{4k+1} + \sum_{m=0}^{\infty} g_m(x) A_{km}(x) = b_k(x) \qquad \text{for all } k = 0, 1, \ldots, \qquad (3)$$

where

$$A_{km}(x) := (-1)^{m+k} x \sum_{n=0}^{\infty} \left(\frac{j_{2m}(\rho_n x)\, j_{2k}(\rho_n x)}{\alpha_n} - \frac{j_{2m}(nx)\, j_{2k}(nx)}{\alpha_n^0} \right),$$

$$b_k(x) := (-1)^k x \sum_{n=0}^{\infty} \left(\frac{\cos nx \; j_{2k}(nx)}{\alpha_n^0} - \frac{\cos \rho_n x \; j_{2k}(\rho_n x)}{\alpha_n} \right).$$

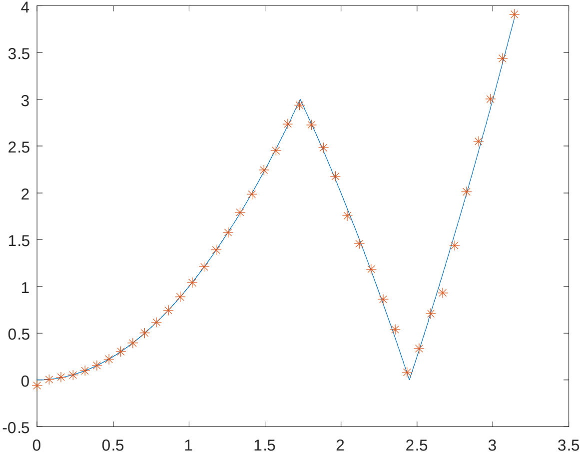


Figure: Potential $q(x) = \left|3 - \left|x^2 - 3\right|\right|$ recovered from 200 eigenvalues, with 6 equations.

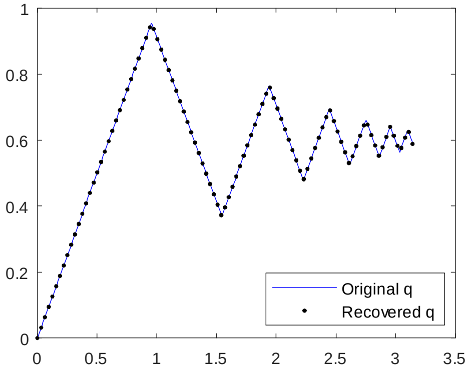


Figure: Potential recovered from 200 eigenvalues, with 6 equations.

V. V. Kravchenko *Direct and inverse Sturm-Liouville problems: A method of solution*, Birkhäuser (2020)

V. V. Kravchenko, S. M. Torba *A direct method for solving inverse Sturm-Liouville problems*. Inverse Problems, **37** (2021) # 015015 (32pp)

Scattering

$$-y'' + q(x)y = \lambda y, \quad x \in (-\infty, \infty),$$

$$\int_{-\infty}^{\infty} (1 + |x|)\, |q(x)|\, dx < \infty.$$

Jost solutions:

$$e(\rho, x) = e^{i\rho x} \left(1 + o(1)\right), \quad x \to +\infty$$

$$g(\rho, x) = e^{-i\rho x} \left(1 + o(1)\right), \quad x \to -\infty$$

[K. Chadan, P. C. Sabatier *Inverse problems in quantum scattering theory*. Springer, NY, 1989]

1. **Discrete spectrum:** There may exist a finite number of eigenvalues $\lambda_n < 0$, for which $\exists\ y(x) \in L_2(-\infty, \infty)$. Then $y(x) = c_1 e(\rho, x) = c_2 g(\rho, x)$.
   Norming constants:

$$\alpha_k := \left(\int_{-\infty}^{\infty} e^2(\rho_k, x)\, dx \right)^{-1}.$$

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2. **Reflection coefficient:**

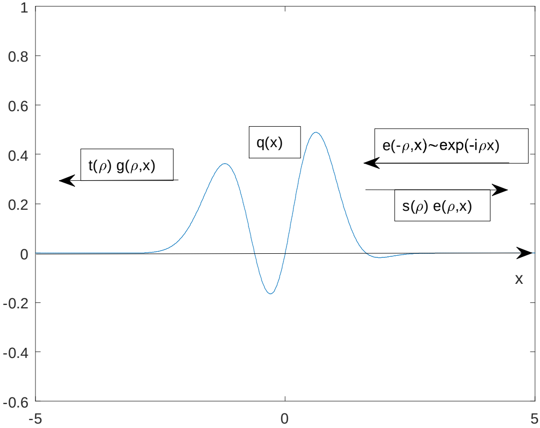

Figure: A schematic representation of the scattering model.

- **Direct problem:** Given $q(x)$, find the scattering data

$$s(\rho),\ \rho \in \mathbf{R}, \quad \lambda_k,\ \alpha_k,\ k = \overline{1, N}$$

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- **Inverse problem:** Given scattering data, find $q(x)$.
- **Applications:** Inverse scattering transform method; nonlinear Fourier transform.

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- K. Chadan, P. C. Sabatier *Inverse problems in quantum scattering theory*. Springer, 1989
  V. A. Marchenko, *Sturm-Liouville operators and applications: revised edition*. AMS Chelsea Publishing, 2011
  M. J. Ablowitz, *Nonlinear dispersive waves: asymptotic analysis and solitons.* Cambridge University Press, 2011.

Same approach works

- Transmutation operator:

$$e(\rho, x) = e^{i\rho x} + \int_x^\infty A(x,t) e^{i\rho t} dt$$

[B. Ya. Levin (1956)]

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$$e(\rho, x) = e^{i\rho x} + \int_x^\infty A(x,t) e^{i\rho t} dt$$

  [B. Ya. Levin (1956)]

- $$A(x,t) = \sum_{n=0}^{\infty} a_n(x) L_n(t-x) e^{\frac{x-t}{2}} \quad (L_n \text{ - Laguerre polynomials}).$$

  A recursive integration procedure for $a_n(x)$, beginning with

$$a_0(x) = e(\frac{i}{2}, x) e^{\frac{x}{2}} - 1.$$

  [V. V. Kravchenko 2019; B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko 2020]

-

$$\Downarrow$$

$$e(\rho, x) = e^{i\rho x}\left(1 + \sum_{n=0}^{\infty} a_n(x) \int_0^{\infty} L_n(t) e^{-\left(\frac{1}{2} - i\rho\right)t} dt\right).$$

$$\Downarrow$$

-

$$\Downarrow$$

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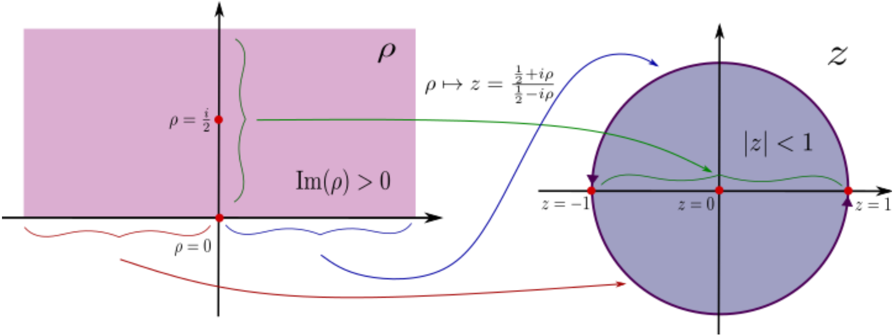
$$\Downarrow$$

- Series representation for Jost solutions:

$$e(\rho, x) = e^{i\rho x}\left(1 + (z+1)\sum_{n=0}^{\infty} (-1)^n z^n a_n(x)\right),$$

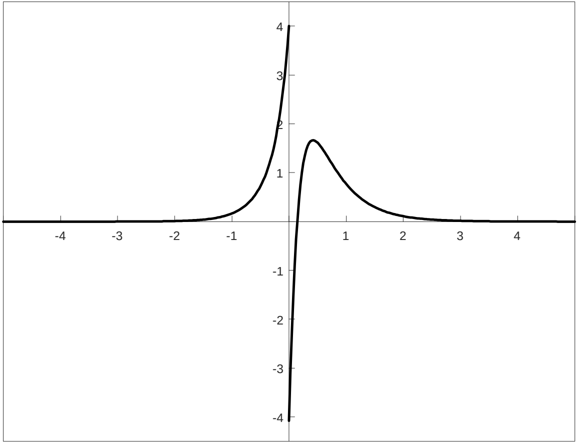
where

$$z := \left(\frac{1}{2} + i\rho\right) / \left(\frac{1}{2} - i\rho\right).$$


ρ
$\rho = \frac{i}{2}$
$\mathrm{Im}(\rho) > 0$
$\rho = 0$
$\rho \mapsto z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$
z
$|z| < 1$
$z = -1$
$z = 0$
$z = 1$

Example

[T. Aktosun, P. Sacks, Mathematical Methods in the Applied Sciences 25, (2002), 347-355.]



$$q(x) = \begin{cases} q_1(x), & x < 0 \\ q_2(x), & x > 0 \end{cases}$$

where

$$q_1(x) = \frac{16\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x}}{\left(\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x} - 1\right)^2}$$

and

$$q_2(x) = \frac{96e^{2x}\left(81e^{8x} - 144e^{6x} + 54e^{4x} - 9e^{2x} + 1\right)}{\left(36e^{6x} - 27e^{4x} + 12e^{2x} - 1\right)^2}.$$

Reflection coefficient

$$s(\rho) = \frac{(\rho + i)(\rho + 2i)(101\rho^2 - 3i\rho - 400)}{(\rho - i)(\rho - 2i)(50\rho^4 + 280i\rho^3 - 609\rho^2 - 653i\rho + 400)}.$$

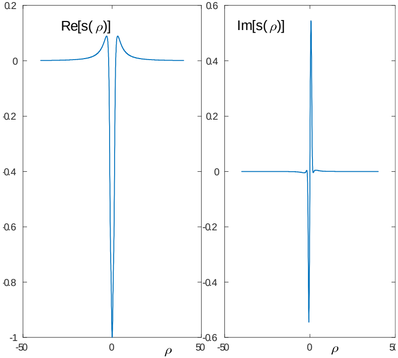


Figure: $s(\rho)$ computed. Abs. error $< 10^{-12}$.

Inverse scattering

Given the scattering data

$$s(\rho),\ \rho \in \mathbf{R}, \quad \lambda_k,\ \alpha_k,\ k = \overline{1, N},$$

find $q(x)$.
[V. V. Kravchenko, On a method for solving the inverse scattering problem on the line. Math Meth Appl Sci. 42 (2019), 1321-1327]

Gelfand-Levitan-Marchenko equation

$$F(x+y)+A(x,y)+\int_x^{\infty} A(x,t)F(t+y)dt=0, \quad y>x,$$

where

$$F(x)=\sum_{k=1}^{N}\alpha_k e^{-\tau_k x}+\frac{1}{2\pi}\int_{-\infty}^{\infty} s(\rho)\, e^{i\rho x}d\rho.$$

$$A(x,t)=\sum_{n=0}^{\infty}a_n(x)L_n(t-x)e^{\frac{x-t}{2}}$$

$$\swarrow \searrow$$

$$F(x+y)+A(x,y)+\int_x^{\infty}A(x,t)F(t+y)dt=0$$

$$\Downarrow$$

$$a_m(x)+\sum_{n=0}^{\infty}a_n(x)A_{mn}(x)=r_m(x),\quad m=0,1,\ldots$$

$$A_{mn}(x) := (-1)^{m+n}\Big(\sum_{k=1}^{N}\alpha_k e^{-2\tau_k x}\frac{\left(\frac{1}{2}-\tau_k\right)^{m+n}}{\left(\frac{1}{2}+\tau_k\right)^{m+n+2}} + \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\rho)\, e^{2i\rho x}\frac{\left(\frac{1}{2}+i\rho\right)^{m+n}}{\left(\frac{1}{2}-i\rho\right)^{m+n+2}}d\rho\Big),$$

$$r_m(x) := (-1)^{m+1}\Big(\sum_{k=1}^{N}\alpha_k e^{-2\tau_k x}\frac{\left(\frac{1}{2}-\tau_k\right)^{m}}{\left(\frac{1}{2}+\tau_k\right)^{m+1}} + \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\rho)\, e^{2i\rho x}\frac{\left(\frac{1}{2}+i\rho\right)^{m}}{\left(\frac{1}{2}-i\rho\right)^{m+1}}d\rho\Big),$$

It is enough to find $a_0(x)$

$$a_0(x) = e(\frac{i}{2}, x)e^{\frac{x}{2}} - 1$$

$$\Downarrow$$

$$q = \frac{a_0'' - a_0'}{a_0 + 1}$$

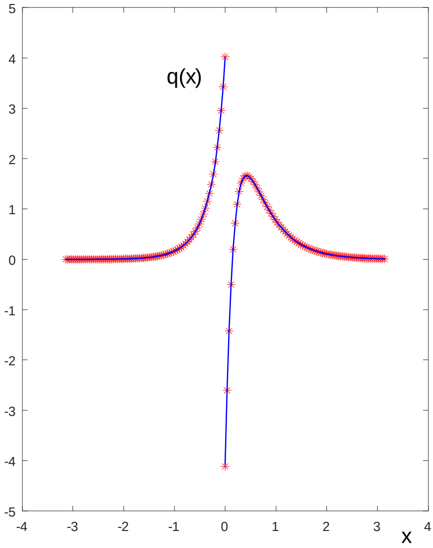


Figure: Potential $q(x)$ recovered with 25 equations.

Inverse scattering transform method

Korteweg - de Vries equation

$$u_t + u_{xxx} - 6uu_x = 0.$$

Diagram: schematic representation of the IST

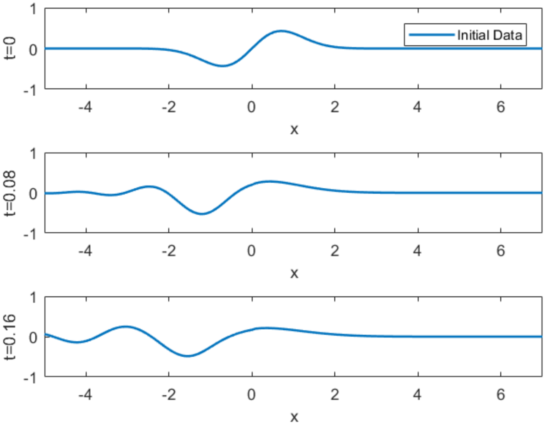
$$u(x,0) = q(x) \quad \overset{-y''+q(x)y=\lambda y}{\underset{\text{direct scattering problem}}{-\,-\,-\,-\,--\longrightarrow}} \quad \text{Scatt.data } (t=0)$$

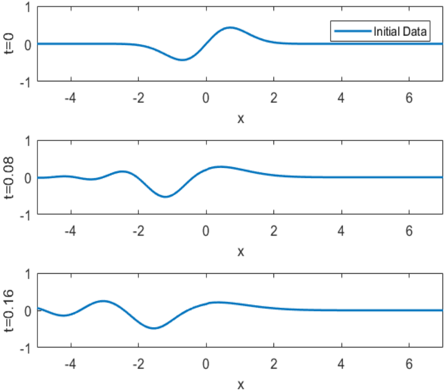
$$\Big\downarrow \ \text{linear evolution law}$$

$$u(x,t) \quad \overset{\text{inverse scattering problem}}{\longleftarrow -\,-\,-\,-\,--} \quad \text{Scatt.data } (t)$$

Example [G. L. Lamb Elements of soliton theory. John Wiley & Sons, 1980]

$$u(x,0) = xe^{-x^2}.$$





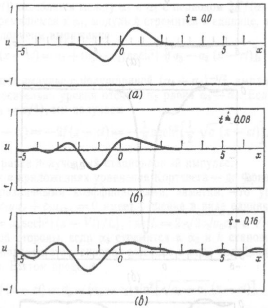


Рис. 4.3. Несолитонный вклад в решение уравнения Кортевега — де Фриза $u_t - 6uu_x + u_{xxx} = 0$ для начального профиля $u(x, 0) = xe^{-x^2}$. Направление распространения — слева направо. Появление колебаний впереди импульса связано с наложением периодических граничных условий. Решение получено У. Фергюсоном мл.

1. Sturm-Liouville problem on the half-line
   [B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, *The transmutation operator method for efficient solution of the inverse Sturm-Liouville problem on a half-line.* Mathematical Methods in the Applied Sciences, 42, (2019), 7359–7366.]
   [B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, *A representation for Jost solutions and an efficient method for solving the spectral problem on the half line.* Mathematical Methods in the Applied Sciences, 43, (2020), 9304-9319.]

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   [B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, *A representation for Jost solutions and an efficient method for solving the spectral problem on the half line.* Mathematical Methods in the Applied Sciences, 43, (2020), 9304-9319.]

2. Inverse scattering on the half-line

   [A. N. Karapetyants, K. V. Khmelnytskaya, V. V. Kravchenko, *A practical method for solving the inverse quantum scattering problem on a half line*, J. Phys.: Conf. Ser. 1540 (2020), 012007]

3. Quantum scattering for the perturbed Bessel equation

$$Lu := -u'' + \left(\frac{\ell(\ell+1)}{x^2} + q(x)\right) u = \rho^2 u, \quad x > 0$$

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4. Recovery of Sturm-Liouville problem from the Weyl function
[V. V. Kravchenko, S. M. Torba, *A practical method for recovering Sturm-Liouville problems from the Weyl function.* Inverse Problems in press https://doi.org/10.1088/1361-6420/abff06]

- THANK YOU