Asymptotics for vectorial Allen-Cahn type problems

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The problem

We are interested in the asymptotic behavior, as $\varepsilon \to 0$ of families of solutions $(u_\varepsilon)_{\varepsilon>0}$ to gradient-type equations

$$-\Delta u_\varepsilon = -\varepsilon^{-2}\nabla V_u(u_\varepsilon), \text{ on a domain } \Omega \subset \mathbb{R}^N, \ N \geq 1 \qquad (\mathbf{Ellip}_\varepsilon)$$

$0 < \varepsilon < 1$ is a parameter, $u_\varepsilon : \mathbb{R}^N \to \mathbb{R}^k$, V is a potential $V : \mathbb{R}^k \to \mathbb{R}^+$. Solutions are critical points of the energy

$$\mathrm{E}_\varepsilon(u) = \int_\Omega e_\varepsilon(u) = \int_\Omega \varepsilon \frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon}, \text{ for } u : \Omega \mapsto \mathbb{R}^k.$$

If V is convex, then E_ε is strictly convex and positive. We impose an energy bound on solutions, namely

$$\mathrm{E}_\varepsilon(u) \leq M_0, \qquad (2)$$

for some constant $M_0 > 0$.

Remark: parabolic case

$$\frac{\partial u}{\partial t} - \Delta u_\varepsilon = -\varepsilon^{-2}\nabla V_u(u_\varepsilon). \qquad (\mathbf{Parab}_\varepsilon)$$

is also of great interest.

Assumptions on the potential V

For the potential $V : \mathbb{R}^k \to \mathbb{R}$, we will assume:

(H_1) inf $V = 0$ and the set of minimizers $\Sigma \equiv \{y \in \mathbb{R}^k, V(y) = 0\}$

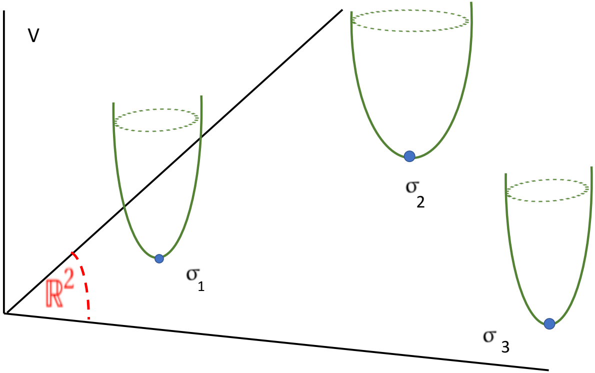
is finite and contains **at least two distinct** elements, i.e.

$$\Sigma = \{\sigma_1, ..., \sigma_q\}, q \geq 2, \ \sigma_i \in \mathbb{R}^k, \forall i = 1, ..., q.$$

(H_∞) There exists $\alpha_0 > 0$ and $R_0 > 0$ such that

$$y \cdot \nabla V(y) \geq \alpha_0 |y|^2, \text{ if } |y| > R_0.$$

The graph of V



We will assume then that

$$V(x) \underset{|x|\to+\infty}{\to} +\infty \text{ and } V \geq 0.$$

If V is strictly convex, uniqueness of solutions for given bounadry data.

We consider here instead the case where there are several mimimizers of the potential V
$\rightsquigarrow$ Transitions between minimizers

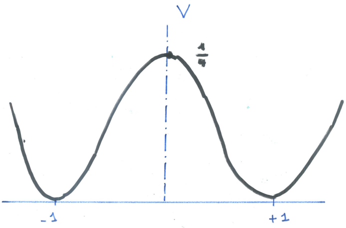
We assume hence that V has a finite number and at least two distinct minimizers.

A classical example: The Allen-Cahn potential

In the **scalar case** k=1, a classical example is given the Allen-Cahn potential:

$$V(u) = \frac{(1-u^2)^2}{4}, \qquad (AC)$$

The minimizers are $+1$ and -1.



Stationary solutions in 1D

For $\Omega = \mathbb{R}$, we may assume, by scaling that $\varepsilon = 1$, and (**Ellip**$_\varepsilon$) reduces to the ODE

$$-\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} = \nabla_w V(w) \text{ on } \mathbb{R}, \quad w(x) = u(\varepsilon x). \tag{5}$$

Finite energy solutions necessarily connect at $\pm\infty$ two minimizers σ^- and σ^+: They are called *profiles* or *heteroclinic connections*, if $\sigma^- \neq \sigma^+$.
Multiplying (5) by w_ε, we obtain the conservation law

$$\frac{d}{dx}\left(V(w) - \frac{|w'|^2}{2}\right) = 0, \tag{6}$$

For profiles one derives the identity

$$|w'| = \sqrt{2V(w)} \text{ on } \mathbb{R}.$$

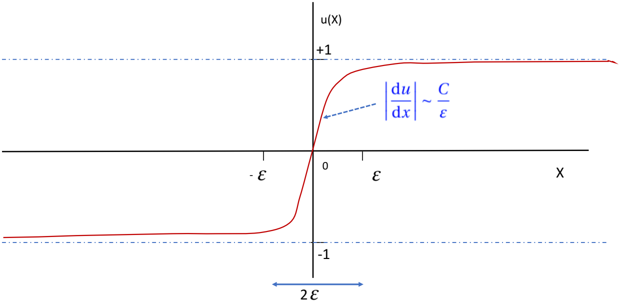
In the *scalar* case, using standard integration techniques, one shows that profils connect only nearby minimizers σ^- and σ^+, and that the solution is unique up to translations and symmetries. For the Allen-Cahn potential, it is given by

$$w(s) = \tanh\left(\frac{s}{\sqrt{2}}\right).$$

Notice the scaling law, for w_ε defined by $w_\varepsilon(s) = w\left(\frac{s}{\varepsilon}\right)$

$$\mathrm{E}_\varepsilon(w_\varepsilon) = \mathrm{E}(w).$$

1D Transition joining two minimizers



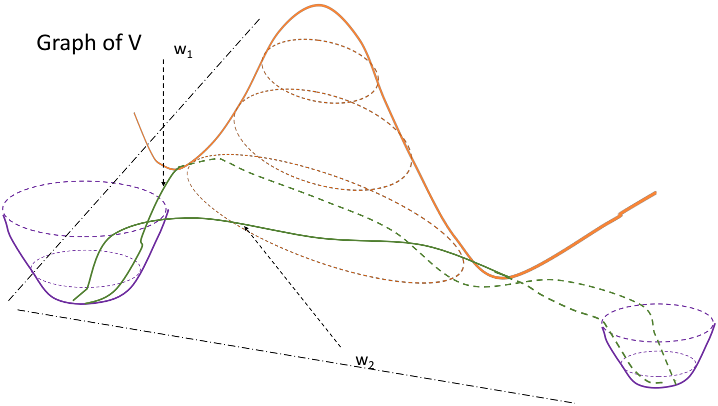
Heteroclinic solutions in the vectorial case

In contrast, in the *vectorial* case, the problem of finding profiles is a very active field of research: Several approaches have been proposed and results have been obtained recently (Alikakos, Betelu, Monteil, Santambrogio, Sternberg, Zuniga...).

No uniqueness in general!

So far, the research has mainly focuses on minimizing solutions, whereas **non-minimizing solutions** might be of interest also.

Non-uniqueness of minimizing connections

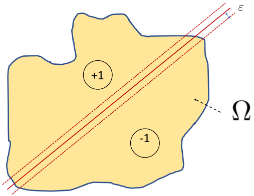


- The transition from one minimizer of V to another occurs on a region of caracteristic size ε near a point .
- In higher dimensions, one expects a similar phenomenom, replacing points points by codimension 1 hypersurfaces.

Higher Dimensional problems

An overview of results for the scalar case, for $N \geq 2$

In the scalar case, the limit $\varepsilon \to 0$ of solutions to ($\mathbf{Ellip}_\varepsilon$) or ($\mathbf{Parab}_\varepsilon$) is well understood: Ω decomposes into regions where the solution takes values either close to $+1$ or close to -1 (for the A-C potential), the regions are separated by interfaces of width $\simeq \varepsilon$.

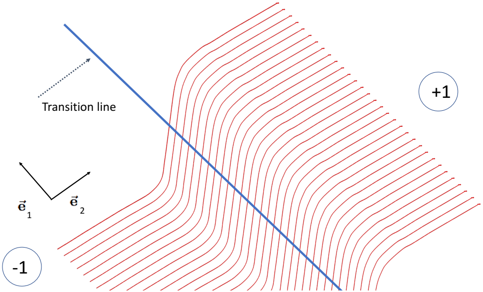


The interface converges to codimension 1 **generalized minimal surfaces**, or moved by mean curvature for ($\mathbf{Parab}_\varepsilon$).

Near a point x_0 on this interface, one has (see e.g. [Hutchinson-Tonegawa)

$$u_\varepsilon(x_1, x_2, \ldots, x_N) \underset{x_1 \to x_{0,1}}{\simeq} \tanh\left(\frac{x_1 - x_{0,1}}{\sqrt{2}\varepsilon}\right),$$

-up to multiplicities-, where the unit vector $\mathbf{e}_1$ is choosen to be orthonormal to the interface. Results due to Modica-Mortola, Schatzmann-De Mottoni, Ilmanen, Soner, Tonegawa....



General principle for the scalar case

For small ε, we have the general principle

$$\text{Solutions to } (\textbf{Ellip}_\varepsilon) \sim \text{ minimal surface} + \text{glued profiles.}$$

The vectorial case for $N \geq 2$: Do similar behaviors hold?

Given a family of $(u_\varepsilon)_{\varepsilon>0}$ of solution to $-\Delta u_\varepsilon = -\varepsilon^{-2}\nabla_u V(u_\varepsilon)$ in Ω, with $E(u_\varepsilon) \leq M_0$:

($\mathcal{Q}_1$) *Is there concentration near a codimension 1 hypersurface (or rectifiable set) $\mathfrak{S}_\star$?*

($\mathcal{Q}_2$) *It is the set $\mathfrak{S}_\star$ minimal in some suitable weak sense?*

($\mathcal{Q}_3$) *Do we have, near a point x_0 on this interface $\mathfrak{S}_\star$.*

$$u_\varepsilon(x_1, x_2, \ldots, x_N) \underset{x_1 \to x_{0,1}}{\simeq} w\left(\frac{x_1 - x_{0,1}}{\varepsilon}\right), \qquad \text{(Profile)}$$

where $w : \mathbb{R} \to \mathbb{R}^k$ is a one-dimensional profil? Uniqueness of w?

It turns out that the three questions are **linked**.

Next a few answers for N=2

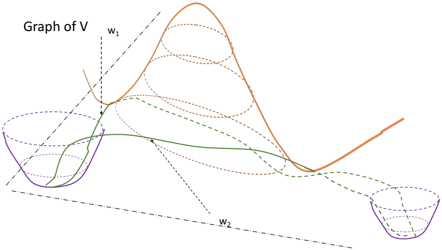
New profils or pseudo-profils : question $\mathcal{Q}_3$

The Alama Bronsard-Gui result

They construct a solution on $\mathbb{R}^2$ with a different behavior then in the scalar case. The authors make the assumption '

(H_4) *There exists two minimizers* σ^+ *and* σ^- *of* V*, and two distinct minimizing solutions* w_1 *and* w_2 *of the differential equation*

$$-w' = \nabla_w V(w) \text{ on } \mathbb{R}, \;\; w(x) \underset{x\to\pm\infty}{\rightarrow} \sigma^{\pm}. \tag{8}$$



Theorem (Alama-Bronsard-Gui, 98)

Under the above assumptions, there exists a (locally minimizing) solution $u:\mathbb{R}^2\to\mathbb{R}^k$ of $-\Delta u=\nabla_u V(u)$ such that

$$\begin{cases} u(x_1,x_2)\to w_1(x_2) \text{ as } x_1\to-\infty \\ u(x_1,x_2)\to w_2(x_2) \text{ as } x_1\to+\infty \end{cases}$$

For $\varepsilon>0$, consider the scaled map u_ε defined by $u_\varepsilon(\mathbf{x})=u\left(\frac{\mathbf{x}}{\varepsilon}\right),\mathbf{x}\in\mathbb{R}^2$

- u_ε is locally minimizing for E_ε
- concentrates on the line $\mathfrak{S}_\star=\{(x_1,0),x_1\in\mathbb{R}\}$, and

$$\begin{cases} u_\varepsilon(x_1,x_2)\simeq w_1(\frac{x_2}{\varepsilon}) \text{ for } x_1>0. \\ u_\varepsilon(x_1,x_2)\simeq w_2(\frac{x_2}{\varepsilon}) \text{ as } x_1>0. \end{cases}$$

The asymptotics (Profile) are verified on the half-lines, not on the whole line.

Periodic Pseudo-profiles

We construct solutions to $-\Delta u = \nabla_u V(u)$ on the cylinder

$$\Lambda_{\mathrm{L}} = [-\mathrm{L}, \mathrm{L}] \times \mathbb{R}, \text{ where } L > 0,$$

with periodic boundary conditions in the x_1 direction, namely such that

$$u(-\mathrm{L}, x_2) = u(\mathrm{L}, x_2) \text{ and } \frac{\partial u}{\partial x_1}(-\mathrm{L}, x_2) = \frac{\partial u}{\partial x_1}(\mathrm{L}, x_2), \text{ for any } x_2 \in \mathbb{R}. \quad \text{(Periodic)}$$

Theorem (B-Oliver, 19)

Under assumptions similar to those of the Alama-Bronsard-Gui result, if $L > L_0$, for some constant $L_0 > 0$, there exists a (smooth) solution to $-\Delta u = \nabla_u V(u)$ on Λ_L such that (Periodic) *holds and such that*

$$\frac{\partial u}{\partial x_1} \neq 0. \quad \text{(Transverse)}$$

On the proof of the previous Theorem

The proof is variational, on $W = \left\{u : \Lambda_L \to \mathbb{R}^k, \mathrm{E}(u) < +\infty, u(-\mathrm{L}, x_2) = u(\mathrm{L}, x_2)\right\}$.
It relies on a mountain-pass argument: The number

$$c_L = \inf_{p \in \mathscr{P}} \left(\sup_{s \in [0,1]} \mathrm{E}(p(s)) \right),$$

is indeed a critical value of E, where $\mathscr{P}$ denotes the set of all paths joining the maps u_1 and u_2 defined by

$$u_i(x_1, x_2) = w_i(x_2), \forall x_1 \in [-\mathrm{L}, \mathrm{L}], x_2 \in \mathbb{R}.$$

w_1 and w_2 being the two minimizing heteroclinic connections.

Scaled Solutions on $\mathbb{R}^2$

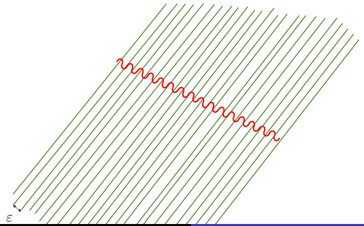
The scaled map on $\mathbb{R}^2$ defined for $\mathbf{x}=(x_1,x_2)$ by

$$u_\varepsilon(\mathbf{x}) = u\left(\frac{\mathbf{x}-N\varepsilon\mathbf{e}_1}{\varepsilon}\right), \text{ if } x_1 \in [N\varepsilon,(N+1)\varepsilon]$$

solves ($\mathbf{Ellip}_\varepsilon$) on $\mathbb{R}^2$. Notice that, for the transversal derivative

$$\varepsilon\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \rightharpoonup \mu \neq 0,$$

where $\mu = c\mathcal{H}^1(D)$, $D=\{(x_1,0), x_1\in\mathbb{R}\}$, for some constant $c>0$.



Arbitrary stationary solutions: Question $\mathscr{Q}_1$

Concentration of minimizing solutions.
In the case of minimizing solutions concentration on minimal surfaces has been established by Baldo and Fonseca-Tartar (89). Precise asymptotic behaviors might still raise some interesting questions.

Concentration of arbitrary stationary solutions
This is **a widely open subject**. Only results available for $N=2$., i.e. $\Omega \subset \mathbb{R}^2$.
Recall that we consider a family $(u_\varepsilon)_{0<\varepsilon\leq 1}$ of solutions of the equation

$$-\Delta u_\varepsilon = \nabla V_u(u_\varepsilon) \quad \text{on } \Omega.$$

satisfying the natural energy bound $\mathrm{E}_\varepsilon(v_\varepsilon) \leq \mathrm{M}_0$, where $\mathrm{M}_0 > 0$ is given.

Remark : The bound is natural in view of the energy cost of a 1D transition

$$-\ddot{u} = \varepsilon^{-2}\nabla v_u(u),$$

whose energy is uniformly bounded.

The result for $\mathscr{Q}_1$ in dimension $N=2$

Theorem (B., 18)

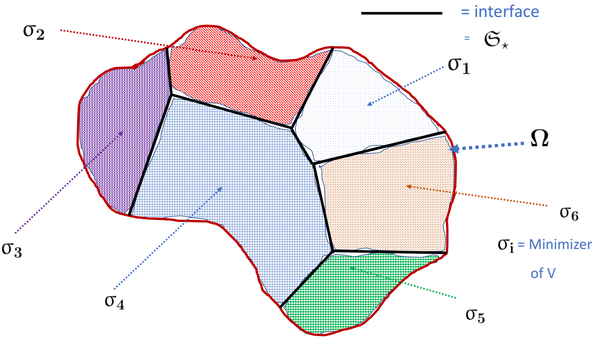
There exists a subset $\mathfrak{S}_\star$ of Ω, and a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to 0 such that the following holds:

i) $\mathfrak{S}_\star$ *is closed, rectifiable of dimension 1, with locally a finite number of connected components and such that* $\mathscr{H}^1(\mathfrak{S}) \leq \mathrm{C_H}\,\mathrm{M_0}$.

ii) *Let* $\mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$, *and* $(\mathfrak{U}^i_\star)_{i\in I}$ *be the connected components* $\mathfrak{U}_\star$. *For* $i \in I$, *There exists* $\sigma_i \in \Sigma$ *such that*

$$u_\varepsilon \to \sigma_i \text{ uniformly on every compact subset of } \mathfrak{U}^i_\star.$$

Convergence towards minimizers of V



Properties of the interfaces

The interface have a very simple structure, since they are merely an union of segments, hence locally minimizing.

Theorem

There exists a set $\mathfrak{E}_\star \subset \mathfrak{S}_\star$ *such that* $\mathscr{H}^1(\mathfrak{E}_\star) = 0$ *and such that, for* $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, *the set* $\mathfrak{S}_\star$ *is locally near* x_0, *a segment.*

More precisely, there exists a unit vector $\vec{e}_{x_0}$ *and a radius* $r_0 > 0$, *depending on* x_0, *such that*

$$\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, r_0) = (x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0}). \tag{11}$$

$\mathfrak{E}_\star$ is the set of singular points, which is nonempty in general.

Minimality and Limiting measures

Minimality is related to the presence of measures concentrating on $\mathfrak{S}_\star$. Consider the positive measure ζ_ε defined on Ω by

$$\zeta_\varepsilon \equiv \frac{V(u_\varepsilon)}{\varepsilon} \mathrm{d}x, \text{ so that } \zeta_\varepsilon(\Omega) \leq M_0. \tag{12}$$

Since the family $(\zeta_\varepsilon)_{\varepsilon>0}$ is uniformly bounded, passing possibly to a further subsequence, we have

$$\zeta_{\varepsilon_n} \equiv \frac{V(u_{\varepsilon_n})}{\varepsilon_n} \mathrm{d}x \rightharpoonup \zeta_\star, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty, \tag{13}$$

It turns out that the measure $\zeta_\star$ concentrates on $\mathfrak{S}_\star$, and that it is absolutely continuous with respect to the $\mathscr{H}^1$-measure on $\mathfrak{S}_\star$. This property implies that the measure $\zeta_\star$ is determined by the set $\mathfrak{S}_\star$ and the density $\Theta_\star$, and we have

$$\zeta_\star = \Theta_\star(\mathscr{H}^1 \llcorner \mathfrak{S}_\star) = \Theta_\star \mathrm{d}\lambda, \text{ where } \mathrm{d}\lambda = \mathscr{H}^1 \llcorner \mathfrak{S}_\star. \tag{14}$$

Global stationarity property

Theorem

The rectifiable one-varifold $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ corresponding to the measure $\zeta_\star$ is stationary.

This is equivalent to the statement: Given any smooth vector field $\vec{X} \in C_c(\Omega, \mathbb{R}^2)$ on Ω with compact support, the following identity holds

$$\int_\Omega \mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X} \mathrm{d}\zeta_\star = 0. \tag{15}$$

For $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the number $\mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X}(x)$ is defined by

$$\mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X}(x) = \left(\vec{e}_x \cdot \vec{\nabla} \vec{X}(x)\right) \cdot \vec{e}_x, \text{ for } x \in \mathfrak{S}_\star. \tag{16}$$

Singularities

Allard and Almgren showed that such one-dimensional varifolds have a network structure and are the sum of segments with densities.

The typical example of a stationary one-varifold with a singularity at 0 is given by the union of d of half-lines, intersecting at the origin, with constant densities. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d$ be d-distinct unit vectors in $\mathbb{R}^2$. Set

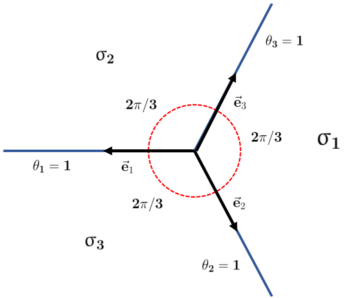
$$\mathscr{S}_\star = \bigcup_{i=1}^{d} \mathbb{H}_i, \text{ where for } i = 1, \ldots, d, \text{ we set } \mathbb{H}_i = \{t\vec{e}_i, t \geq 0\}, \tag{17}$$

and let $\theta_1, \ldots, \theta_d$ be d positive numbers. If θ_i represents the density Θ of $\mathscr{S}_\star$ on $\mathbb{H}_i$ (which is hence constant there), then $\mathbf{V}(\mathscr{S}_\star, \Theta)$ is a stationary one-dimensional rectifiable varifold iff

$$\sum_{i=1}^{d} \theta_i \vec{e}_i = 0. \tag{18}$$

Examples of singularities

The following type of singularity may occur, as constructed by Bronsard, Schatzmann and Gui (94), for a potential with three wells



Locally infinite singularities

A result of Allard and Almgren states that

$$\text{discrete set of densities} \implies \text{singularities of finite type}$$

They also constructed an example of singularity of infinite type.
singularities are hence (locally) finite in the following cases:

- Minimizing case
- Scalar case

No result in the general case.

We next discuss the method, and compare with the existing theory in the scalar case.

Comparison with the scalar case

Scalar Allen-Cahn theory: The discrepancy. Proofs heavily rely on properties of the discrepancy function

$$\xi_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon} V(u_\varepsilon) - \varepsilon \frac{|\nabla u|^2}{2}.$$

Recall that for $N=1$ solutions we have

$$\frac{1}{\varepsilon} V(u_\varepsilon) - \varepsilon \frac{|\dot{u}|^2}{2} = \xi_\varepsilon(u_\varepsilon) = 0 \text{ for 1D profiles.}$$

In higher dimensions, the positivity of ξ_ε for scalar solutions was observed first by Payne, Sperb, L. Modica,.. for entire solutions, thanks to clever use of maximum principle. Ilmanen and Hutchinson-Tonegawa proved

$$\xi_\varepsilon(u_\varepsilon) \to 0 \text{ in } L^1_{\mathrm{loc}}(\Omega).$$

The monotonicity formula in the scalar case

The monotonicity formula

$$\frac{d}{dr}\left(\frac{1}{r^{N-2}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega,$$

holds for any potential, is relevant for concentration on $N-2$ dimensional sets. For concentration on $N-1$ dimensional sets, the stronger monotonicity formula

$$\frac{d}{dr}\left(\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega,$$

is more appropriate. We have, if $N=2$, the identity

$$\frac{d}{dr}\left(\frac{\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)}{r}\right)=\frac{1}{r^2}\int_{\mathbb{D}^2(r)}\xi_\varepsilon(u_\varepsilon)\mathrm{d}x+\frac{1}{r}\int_{\mathbb{S}^1(r)}\left|\frac{\partial u_\varepsilon}{\partial r}\right|^2\mathrm{d}\ell \tag{19}$$
$$\gtrsim 0.$$

Formula (32) has been established by ilmanen in the *scalar case*.

Summary of methods in the scalar Allen-Cahn case

Thanks to the sign of the discrepancy, the chain of arguments in the scale Allen-Cahn case goes as follows

$$\text{sign of discrepancy} \implies \text{monotonicity} \implies \text{clearing} - \text{out}$$

whereas

$$\begin{cases} \text{clearing out} + \text{monotonicity} \implies \text{concentration on } N-1 \text{ dimensional sets} \\ \text{monotonicity} \implies \text{(Preiss)} \ \text{ rectifiability of concentration set} \end{cases}$$

and

$$\text{sign of discrepancy } + \text{stress} - \text{energy tensor}$$
$$\Downarrow$$
$$\text{stationary sets or motion by mean} - \text{curvature}$$

Conclusion: Sign of discrepancy is crucial !

Vectorial Allen-Cahn: main difficulty

Main observations for the vectorial case

- The **discrepancy** has **no sign**, in view of the existence of periodic pseudo-profiles

The chain of argument is broken...

$\Downarrow$

New ideas are required !

Concentration on lines in $N=2$ Vectorial Allen-Cahn

The set $\mathfrak{S}_\star$ corresponds to the concentration set for the energy. The total mass of the measures $(\nu_\varepsilon)_{0<\varepsilon\leq 1}$

$$\nu_\varepsilon \equiv e_\varepsilon(u_\varepsilon)\,dx \text{ on } \Omega. \tag{20}$$

is uniformly bounded, since $\nu_\varepsilon(\Omega)\leq \mathrm{M}_0$:by compactness, up to a subsequence

$$\nu_{\varepsilon_n} \rightharpoonup \nu_\star \text{ sur } \Omega \text{ as } n\to+\infty. \tag{21}$$

A central part of the argument is the **"clearing-out"** statement

Theorem

Let $x_0\in\Omega$ and $r>0$ be given such that $\mathbb{D}^2(x_0,r)\subset\Omega$. There exists a constant $\eta_0>0$ such that, if we have

$$\frac{\nu_\star\left(\overline{\mathbb{D}^2(x_0,r)}\right)}{r}<\eta_0, \text{ then it holds } \nu_\star\left(\overline{\mathbb{D}^2(x_0,\frac{r}{2})}\right)=0. \tag{22}$$

Rectifiability and beyond

The argument for rectifiability of $\mathfrak{S}_\star$ is specific to 1-dimensional sets: a compact connected set of dimension 1 is rectifiable.

Rectifiability implies that $\mathscr{S}_\star$ has an approximate tangent for $\mathscr{H}^1$-almost every $x_0 \in \mathscr{S}$, i.e. there exists a unit vector $\vec{e}_{x_0}$ s.t. $\forall\theta > 0$

$$\lim_{r\to 0} \frac{\mathscr{H}^1\left(\mathscr{S} \cap \left(\mathbb{D}^2(x_0,r) \setminus \mathscr{C}\mathrm{one}(x_0,\vec{e}_{x_0},\theta)\right)\right)}{r} = 0, \tag{23}$$

where, for $|\vec{e}|=1$, $\mathscr{C}\mathrm{one}(x_0,\vec{e},\theta)$ stands for the cone

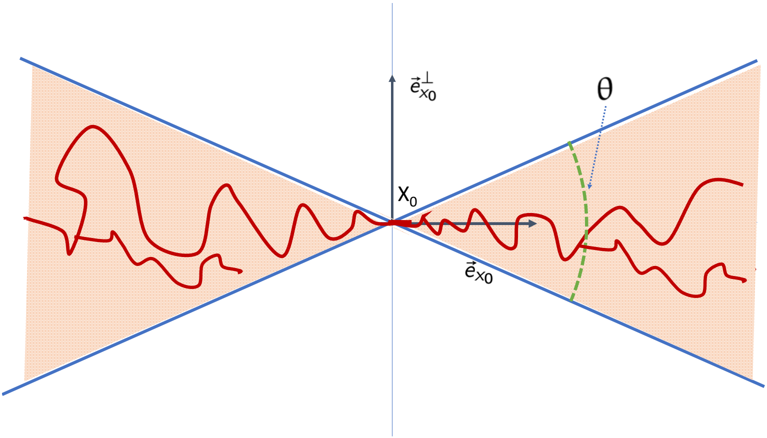
$$\mathscr{C}\mathrm{one}(x_0,\vec{e},\theta) = \left\{y \in \mathbb{R}^2, |\vec{e}^\perp\cdot(y-x_0)| \leq \tan\theta|\vec{e}\cdot(y-x_0)|\right\}, \vec{e}^\perp\cdot\vec{e}=0, |\vec{e}^\perp|=1.$$

The point x_0 is said to be regular if (23) is satisfied. We have:

Proposition

Let x_0 be a regular point of $\mathfrak{S}_\star$. $\forall\theta>0$, $\exists R_{\mathrm{cone}}(\theta,x_0)$ s.t.

$$\mathfrak{S}_\star \cap \mathbb{D}^2(x_0,r) \subset \mathscr{C}\mathrm{one}(x_0,\vec{e}_{x_0},\theta),\ \forall 0<r\leq R_{\mathrm{cone}}(\theta,x_0). \tag{24}$$


$\vec{e}_{x_0}^{\perp}$
θ
x_0
$\vec{e}_{x_0}$

Stationarity and the limiting Hopf differential

We pass to the limit the quadratic gradient terms (depends on the orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2)$!

$$\varepsilon u_{\varepsilon_n x_i} \cdot u_{\varepsilon_n x_j} \rightharpoonup \mu_{\star,i,j} \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty, \text{for } i,j = 1,2.$$

Set $\quad \omega_\star = (\mu_{\star,1,1} - \mu_{\star,2,2}) - 2i\mu_{\star,1,2}.$

The measures $\zeta_\star$ and $\omega_\star$ are strongly related in view of our next result.

Lemma

We have, in the sense of distributions

$$\frac{\partial \omega_\star}{\partial \bar{z}} = 2\frac{\partial \zeta_\star}{\partial z} \quad \text{in } \mathscr{D}'(\Omega). \tag{25}$$

Relation (25) is the two-dimensional analog of the conservation law (3) for the ordinary differential equation. It expresses the fact that the energy of the solution u_ε is stationary with respect to variations of the domain.

Taking the real and imaginary parts of this relation, we obtain, in the sense of distributions, the *modified Cauchy-Riemann relations*

$$\begin{cases} \dfrac{\partial}{\partial x_2}(2\mu_{\star,1,2}) = \dfrac{\partial}{\partial x_1}\left(2\zeta_\star - \mu_{\star,1,1} + \mu_{\star,2,2}\right) \text{ and} \\ \dfrac{\partial}{\partial x_1}(2\mu_{\star,1,2}) = \dfrac{\partial}{\partial x_2}\left(2\zeta_\star + \mu_{\star,1,1} - \mu_{\star,2,2}\right), \end{cases} \tag{26}$$

the second relation being in some sense the closest to (3).

A new discrepancy relation

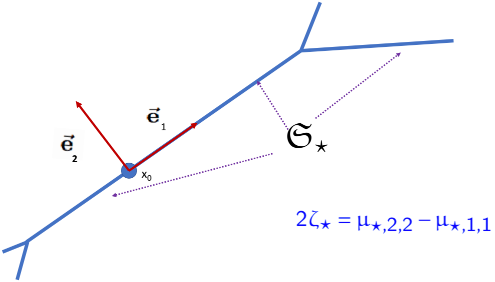
A crucial step in the proofs is to show that $\nu_\star$ and $\zeta_\star$ are absolutely continuous with respect to $d\lambda$. Then we have:

Proposition

Let x_0 be a regular point of $\in \mathfrak{S}_\star$. Assume that the orthonormal frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ is choosen so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$. We have locally near x_0 the identity

$$2\zeta_\star = \mu_{\star,2,2} - \mu_{\star,1,1} \text{ and } \mu_{\star,1,2}(x_0) = 0. \tag{27}$$

$x_0 \in \mathfrak{S}_\star$ regular point



Remark: In the scalar case, one obtains $\mu_{\star,1,1} = \mu_{\star,1,2} = 0$ so that

$$\omega_\star = \mu_{\star,2,2} = 2\zeta_\star.$$

Monotonicity for $\zeta_\star$

In order to prove the absolute continuity of the measures with respect to $\mathscr{H}^1 \llcorner \mathfrak{S}_\star$, we rely on a monotonicity formula for $\zeta_\star$.

Proposition

Let $x_0 \in \Omega$, let $\rho > 0$ such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. If $0 < 0 < r_0 \leq r_1 \leq \rho$, then we have

$$\frac{\zeta_\star(\mathbb{D}^2(x_0, r_1))}{r_1} \geq \frac{\zeta_\star(\mathbb{D}^2(x_0, r_0))}{r_0}.$$

Thank you for your attention!