On an interior Calderón operator and a related Steklov eigenproblem for Maxwell's equations

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The presentation is based on joint work with
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This publication has been dedicated to
Professor Nicholas Alikakos
on the occasion of his retirement.

Synopsis

We consider (i) a Steklov-type problem for time-harmonic Maxwell's equations in a homogeneous isotropic nonconducting medium, along with (ii) the related interior Calderón operator, and (iii) an appropriate Dirichlet-to-Neumann type map.

The corresponding Neumann-to-Dirichlet map turns out to be compact, and this provides a Fourier basis of Steklov eigenfunctions for the associated energy spaces.

We provide natural spectral representations for

- ★ the appropriate trace spaces,
- ★ Calderón's operator,
- ★ the solutions of the corresponding BVPs, subject to either
    - electric,

  or
    - magnetic,

  boundary conditions on a cavity.

But a long introduction about trace issues and the Steklov eigenproblem, comes before discussing our results.

Recall that if $\mathcal{O}$ is a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary, there exists a linear and continuous operator Γ from $W^{k,p}(\mathcal{O})$ to $(L^p(\partial\mathcal{O}))^k$ defined by

$$\Gamma(u) = (\gamma_0(u), \cdots, \gamma_{k-1}(u)),$$

where

- $\gamma_0(u)$ is the trace of u
- $\gamma_j(u)$ is the j-th normal derivative of u, for $j = 1, 2, \cdots, k-1$.

In particular, for $u \in C^k(\overline{\mathcal{O}})$, we have

- $\gamma_0(u) = u|_{\partial\mathcal{O}}$
- $\gamma_j(u) = \frac{\partial^j u}{\partial \nu^j}$, for all $j = 1, 2, \cdots, k-1$,

where ν denotes the outer unit normal to $\partial\mathcal{O}$.
The vector $\Gamma(u)$ is called the total trace of u.

The description of the trace spaces $\gamma_j(W^{k,p}(\mathcal{O}))$ for $j = 1, 2, \cdots, k-1$, and of the total trace space $\Gamma(W^{k,p}(\mathcal{O}))$ are important problems in the theory of Sobolev spaces.

From a historical point of view, this problem dates at least back to 1906, when Hadamard provided his famous counterexample which pointed out the need to clarify which conditions on the datum g guarantee that the solution v to the Dirichlet problem

$$\Delta v = 0, \text{ in } \mathcal{O} \ ; \ \ v = g, \text{ on } \partial\mathcal{O},$$

has square summable gradient.
In the framework of Sobolev spaces, this problem can be reformulated as: find necessary and sufficient conditions on g, such that $g = \gamma_0(u)$, for some $u \in H^1(\mathcal{O}) \equiv W^{1,2}(\mathcal{O})$.

A short discussion on trace theorems

Conclusive results are available for smooth domains, and are nowadays classical.
The standard approach consists in flattening the boundary of $\mathcal{O}$ by means of suitable local diffeomorphisms. Hence the problem is recast to that of describing the trace spaces of $W^{k,p}(\mathbb{R}^N)$ on $(N-1)$-dimensional hyperplanes which can be identified with $\mathbb{R}^{N-1}$.
A classical method for describing the trace spaces of $W^{k,p}(\mathbb{R}^N)$ on $\mathbb{R}^{N-1}$, in the case p $= 2$, is via Fourier Transform.

If $p \neq 2$ this approach is no more applicable. In this case, the description of the trace spaces relies on a method originally developed by Gagliardo in 1957[1], for the case $k = 1$, that involves the use of a particular case of the so-called Besov spaces.

If the domain is sufficiently smooth, the definition of Besov spaces can be "transplanted" from $\mathbb{R}^{N-1}$ to $\partial\mathcal{O}$, providing well-defined function spaces at the boundary of $\mathcal{O}$.

[1] Besov introduced the general spaces $B^s_{p,q}$, bearing now his name, in 1961.

However, when $\mathcal{O}$ is an arbitrary bounded open set with a Lipschitz boundary, there is no such simple description and not many results are available in the literature.
A complete description of the traces of all derivatives up to the order $k-1$ of a function $u \in W^{k,p}(\mathcal{O})$ is due to Besov, who provided (in the early 1970s) an explicit, but quite technical, representation theorem.

Simpler descriptions are not in general available, with the exception of a few special cases (e.g., the important work by Grisvard when $\mathcal{O}$ is a curvilinear polygon in $\mathbb{R}^2$).

Another approach to trace spaces has been developed for $k = 1$, $p = 2$, by G. Auchmuty[2] who gave an alternative equivalent description of the trace space $\gamma_0(H^1(\mathcal{O}))$ in terms of Fourier series associated with the eigenfunctions of the classical Steklov problem for the Laplace operator.

This method has been employed (for a Lipschitz domain $\mathcal{O}$) very recently by P. D. Lamberti and L. Provenzano[3] for the case $k = 2$, where new families of multi-parameter biharmonic Steklov problems have been introduced in order to describibe the traces of functions in $H^2(\mathcal{O})$; the same authors generalized their results[4] for $k \geq 2$.

So, what is a Steklov problem ??

[2] SIAM Journal on Mathematical Analysis **38** 2006, 894-905.
[3] Revista Matemática Complutense 2021, in press, 36 pp.
[4] Le Matematiche **75** 2020, 137-165.

The classical Steklov eigenproblem

The classical Steklov eigenproblem, for a bounded smooth domain Ω in $\mathbb{R}^n$ with boundary Γ, reads

$$\Delta v = 0 \text{ in } \Omega \; ; \; D_\nu v = \lambda v \text{ on } \Gamma = \partial\Omega,$$

where ν denotes the unit outer normal to Γ, $D_\nu v$ the normal derivative of v, and the unknown v is a real- or complex-valued function called Steklov eigenfunction, while the unknown λ is a non-negative real number called Steklov eigenvalue.

The Steklov eigenvalues can be equivalently defined as the eigenvalues of the celebrated Dirichlet-to-Neumann map defined from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$ by $g \mapsto D_\nu v$, where v is the solution to the Dirichlet problem

$$\Delta v = 0 \text{ in } \Omega \; ; \; v = g \text{ on } \Gamma.$$

It turns out that the non-zero eigenvalues of the D-t-N operator are the reciprocals of the eigenvalues of the corresponding N-t-D (Neumann - to - Dirichlet) map, which can be considered as a compact self-adjoint map from $L^2(\Gamma)$ to itself. In particular, the eigenvalues have finite multiplicity and can be represented as a non-decreasing divergent sequence.

The classical Steklov eigenproblem

Boundary spectral parameters were introduced in 1902 by Steklov. Steklov eigenproblems are models for linear surface water waves, mechanical oscillators immersed in a viscous fluid, vibration modes of a structure in contact with an incompressible fluid, heat diffusion, optimal design, medical and geophysical imaging, inverse scattering theory to reconstruct the index of refraction of inhomogeneous media, etc.

We refer, e.g., to the paper by Lamberti and Provenzano[5] for an introduction to Steklov-type problems.

Further, many studies have been carried out on numerical methods for Steklov eigenvalue problems[6].

An interesting application of the Steklov eigenproblem appears in the recent work by S. Heyden and M. Ortiz[7], on the functional optimality of the sulcus pattern of the human brain!

[5] Current Trends in Analysis and its Applications, 171-178, Trends Math., Birkhäuser, 2015, and references therein.

[6] Zhang Y., Bi H., Yang Y., Open Mathematics **18**, 2020, 216-236, and references therein.

[7] Mathematical Medicine and Biology: A Journal of the IMA **36**, 2019, 207-221.

Vladimir Andreevich Steklov (1864 - 1926)

Steklov was not only an outstanding mathematician, who made many important contributions to Applied Mathematics, but also had an unusually bright personality.
The Mathematical Institute of the Russian Academy of Sciences in Moscow bears his name.

On his life and work it is worth reading the very interesting paper:
N. Kuznetsov, T. Kulczycki, M. Kwasnicki, A. Nazarov, S. Poborchi, I. Polterovich, B. Siudeja, "*The Legacy of Vladimir Andreevich Steklov*"[8].

Some comments about Steklov's work:

[8]Notices of the AMS **61**, 2014, 9-22.

Vladimir Andreevich Steklov (1864 - 1926)

For his Master's thesis, Steklov worked on the equations of a solid body moving in an ideal non-viscous fluid; there were four cases to be considered. Two of these cases had been solved by Clebsch in 1871. Steklov solved the third case in his thesis (1893). The final case was solved by Lyapunov (Steklov's supervisor) in 1893.
For his doctoral dissertation he worked on problems that arose in potential theory, electrostatics and hydromechanics, using rigorous mathematical analysis.
After obtaining his doctorate, he studied BVPs of Dirichlet type, where Laplace's equation must be solved on a surface.
He wrote "General Theory of Fundamental Functions"[9] in which he examined expansions of functions as series in an infinite system of orthogonal eigenfunctions[10].
He also worked on hydrodynamics and the theory of elasticity.
Additionally, he wrote a number of works on the history of science.
He was an Invited Speaker of the ICM in 1924 in Toronto.

[9] In fact the term "Fundamental Functions", which is due to Poincaré, means eigenfunctions in today's terminology.

[10] Steklov was not the first to study this problem; Fourier had examined a special case of this situation about 60 years before.

Let Ω be a compact Riemannian manifold of dimension $n \geq 2$ with (possibly non-smooth) boundary $M := \partial\Omega$.
Consider the Steklov problem

$$\Delta u = 0\,,\ \text{in}\ \ \Omega\,;\quad D_\nu u = \lambda u\,,\ \text{on}\ \ M\,,$$

where Δ is the Laplace-Beltrami operator acting on functions on Ω, and D_ν is the outward normal derivative along the boundary M.
It is known that the spectrum of the Steklov problem is discrete, as long as the trace operator $H^1(\Omega) \to L^2(M)$ is compact.
In this case, the eigenvalues form a sequence

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty\,,$$

and the corresponding eigenfunctions form an orthonormal basis of $L^2(M)$.

For example, in the case where Ω is the unit ball in $\mathbb{R}^3$ (so $M = S^2$), the eigenvalues are

$$\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2, \cdots, \lambda_j = j, \cdots.$$

The multiplicity of λ_j is equal to $2j + 1$.
The corresponding to λ_j eigenfunction ϕ_j is the solution of

$$\Delta_{S^2}\phi_j + j(j+1)\phi_j = 0.$$

The Steklov eigenvalues can be interpreted as the eigenvalues of the Dirichlet-to-Neumann operator $\mathcal{D} : H^{\frac{1}{2}}(M) \to H^{-\frac{1}{2}}(M)$ which maps a function $f \in H^{\frac{1}{2}}(M)$ to $\mathcal{D}f = D_\nu(\mathcal{H}f)$, where $\mathcal{H}f$ is the harmonic extension of f to Ω.

Moreover, the Steklov problem provides a new playground for exciting interactions[11] between geometry and spectral theory, exhibiting phenomena that could not be observed in other eigenvalue problems, and leading to the field, of high current interest and activity, referred to as "geometric spectral theory".

[11] Girouard A., Polterovich I., Journal of Spectral Theory **7**, 2017, 321-359.

Steklov spectrum and shape optimization

The Steklov spectrum holds some surprises in shape optimization.
For instance:

- The classical Faber - Krahn inequality states that among Euclidean domains with fixed measure, the first Dirichlet eigenvalue is minimized by a ball.
- The Szegö - Weinberger inequality states that the first nonzero Neumann eigenvalue is maximized by a ball.

In both the above cases, no topological assumptions are made.

- The analogous result for Steklov eigenvalues is the Weinstock inequality, which states that among planar domains with fixed perimeter, $\lambda_1(\Omega)$ is maximized by a disk provided that Ω is simply connected.

In contrast with the Dirichlet and Neumann case, this assumption cannot be removed; indeed the result fails for appropriate annuli.

- The asymptotic distribution of Steklov eigenvalues has an unusual (compared to the Dirichlet and Neumann cases) high sensitivity to the regularity of the boundary.

Mayer and Krechetnikov[12]: dynamics of liquid sloshing

The velocity potential $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies

$$\begin{aligned} \Delta\Phi &= 0\,, && \text{inside the mug} \\ D_t^2\Phi + g\,D_\nu\Phi &= 0\,, && \text{on the free surface} \\ D_\nu\Phi &= 0\,, && \text{on the sides and the bottom.} \end{aligned}$$

Separation of variables leads to

$$\begin{aligned} \Delta f &= 0\,, && \text{inside the mug} \\ D_\nu f &= \frac{\lambda}{g} f\,, && \text{on the free surface} \\ D_\nu f &= 0\,, && \text{on the sides and the bottom.} \end{aligned}$$

[12] Physical Review E **85**, 2012, 046117.

Consider the Laplace equation in a bounded domain Ω of $\mathbb{R}^3$, with $\Gamma = \partial\Omega$,

$$\Delta u(x) = 0,$$

and its fundamental solution

$$\Phi(x, y) = \frac{1}{4\pi} \frac{1}{|x - y|} \,.$$

The well-known representation formula for the solution reads

$$u(x) = \int_\Gamma \Phi(x, y) \underbrace{\frac{\partial u(y)}{\partial \nu_y}}_{\gamma_1^{\text{int}} u(y)} ds_y - \int_\Gamma \underbrace{\frac{\partial \Phi(x, y)}{\partial \nu_y}}_{\gamma_1^{\text{int}} \Phi(x,y)} \overbrace{u(y)}^{\gamma_0^{\text{int}} u(y)} ds_y \,, x \in \Omega.$$

In the Dirichlet problem the datum $\gamma_0^{\text{int}} u$ is given, while in the Neumann problem it is the datum $\gamma_1^{\text{int}} u$ that is known.

Single and double layer potential BIOs: non-rigorous presentation

For a given density function υ and for $x \in \mathbb{R}^3 \setminus \Gamma$, we define in the usual way the "single layer" and the "double layer" potential, which determine bounded linear maps from suitable spaces on Γ to corresponding ones on Ω.

The application of trace operators on these maps, generates

- The Single Layer Boundary Integral Operator
  $(\mathcal{V}\upsilon)(x) = \int_\Gamma \Phi(x,y)\,\upsilon(y)\,ds_y\,, \;\; x \in \Gamma$
- The Double Layer Boundary Integral Operator
  $(\mathcal{W}\upsilon)(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_y}\,\upsilon(y)\,ds_y\,, \;\; x \in \Gamma$
- The Adjoint Double Layer Boundary Integral Operator
  $(\mathcal{W}'\upsilon)(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_x}\,\upsilon(y)\,ds_y\,, \;\; x \in \Gamma$
- The Hypersingular Boundary Integral Operator
  $(\mathcal{D}\upsilon)(x) = -\lim_{\Omega \ni \xi \to x \in \Gamma} \nu_x \cdot \mathrm{grad}_\xi (\mathcal{W}\upsilon)(\xi)\,, \;\; x \in \Gamma$

The Calderón projector

In terms of the above BIOs, the representation formula for the solution of $\Delta u(x) = 0$ reduces to a system of two boundary integral equations which can be written, for $x \in \Gamma$, as

$$\mathfrak{g} = \mathfrak{C}\,\mathfrak{g},$$

where

$$\mathfrak{g} := \begin{pmatrix} \gamma_0^{\mathrm{int}} u \\ \gamma_1^{\mathrm{int}} u \end{pmatrix},$$

and

$$\mathfrak{C} := \begin{pmatrix} (1-\sigma)\mathcal{I} - \mathcal{W} & \mathcal{V} \\ \mathcal{D} & \sigma\mathcal{I} + \mathcal{W}' \end{pmatrix}$$

is the so-called Calderón projector.
$\mathcal{I}$ denotes the identity operator, while the coefficient σ is given by

$$\sigma(x) = \lim_{\epsilon \to 0} \frac{1}{4\pi\epsilon^2} \int_{y \in \Omega : |y-x| = \epsilon} ds_y \,,\ x \in \Gamma,$$

and, if Γ is at least differentiable within a vicinity of $x \in \Gamma$, we have

$$\sigma(x) = \frac{1}{2}\,, \quad \text{for almost all } x \in \Gamma.$$

The D-t-N map & the Calderón operator

From the above vector BIE, we get that the representation of the Dirichlet-to-Neumann map is given by

$$\gamma_1^{\mathrm{int}} u(x) = \mathcal{C}\, \gamma_0^{\mathrm{int}} u(x)\,, \;\; x \in \Gamma,$$

where $\mathcal{C}$ is the Calderón operator (often called Steklov-Poincaré operator), that is given by

$$\mathcal{C} = \mathcal{V}^{-1}\,(\sigma\mathcal{I} + \mathcal{W}),$$

or, more frequently, by its symmetric representation

$$\mathcal{C} = \mathcal{D} + (\sigma\mathcal{I} + \mathcal{W}')\,\mathcal{V}^{-1}\,(\sigma\mathcal{I} + \mathcal{W})\,.$$

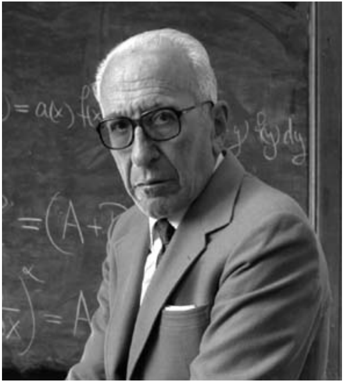
Therefore, we have described the D-t-N map

$$\gamma_1^{\mathrm{int}} u(x) = (\mathcal{C}\,\gamma_0^{\mathrm{int}})u(x)\,, \;\; x \in \Gamma,$$

which maps some given Dirichlet datum $\gamma_0^{\mathrm{int}} u \in H^{\frac{1}{2}}(\Gamma)$ to the corresponding Neumann datum $\gamma_1^{\mathrm{int}} u \in H^{-\frac{1}{2}}(\Gamma)$ of the harmonic function $u \in H^1(\Omega)$.

★ The construction of the D-t-N map (i.e., of the Calderón operator), even for the Laplace equation, is not so simple....

Alberto Pedro Calderón (1920 - 1998)



Calderón was one of the 20^{th} century leading mathematicians.
His fundamental influence is felt strongly in abstract fields, such as Harmonic Analysis, Partial Differential Equations, Complex Analysis, and Geometry, as well as in more concrete areas, such as Signal Processing, Geophysics, and Tomography.

In a homogeneous isotropic nonconducting medium filling a domain Ω in $\mathbb{R}^3$, time-harmonic Maxwell's equations read

$$\mathrm{curl} E - \mathrm{i}\,\omega\,\mu\,H = 0\,,\ \ \mathrm{curl} H + \mathrm{i}\,\omega\,\varepsilon\,E = 0\,, \tag{1.1}$$

where E, H are, respectively, the spatial parts of the electric and the magnetic field, ε and μ are the electric permittivity and the magnetic permeability of the medium, and $\omega > 0$ is the angular frequency (we have adopted the time convention $\mathrm{e}^{-\,\mathrm{i}\,\omega\,t}$).
In the considered case of homogeneous isotropic media, ε and μ are real constants, therefore E and H are automatically divergence-free.

Of *sine qua non* importance in electromagnetics is the following BVP, involving the so-called perfect conductor condition on the boundary Γ of Ω:

$$\begin{cases} \mathrm{curl} E - \mathrm{i}\,\omega\,\mu\,H = 0\,,\ \ \mathrm{curl} H + \mathrm{i}\,\omega\,\varepsilon\,E = 0, & \text{in } \Omega, \\ \nu \times E = m, & \text{on } \Gamma, \end{cases} \tag{1.2}$$

i.e., the tangential trace of the electric field on Γ is given by a fixed vector m.

The interior Calderón operator

The interior Calderón operator is defined as the mapping of the tangential component of the electric field to the tangential component of the magnetic field on Γ, i.e., $m \mapsto \nu \times H$.
This is the origin of another term used for this operator, namely the "*electric to magnetic boundary component map*"[13].

Operating by curl on (1.2) and setting $\widetilde{m} := -\mathrm{i}\,\omega\,\varepsilon\, m$ we obtain

$$\begin{cases} \mathrm{curl}\mathrm{curl} H - \omega^2 \varepsilon\,\mu\, H = 0, & \text{in } \Omega, \\ \nu \times \mathrm{curl} H = \widetilde{m}, & \text{on } \Gamma. \end{cases} \tag{1.3}$$

The corresponding interior Calderón operator for (1.3), maps $\widetilde{m}$ to $\nu \times H$.

In view of the vector identity $\mathrm{curl}\,\mathrm{curl} w = \mathrm{grad}\,\mathrm{div} w - \Delta w$ and the fact that H is divergence-free, (1.3) can also be written as

$$\begin{cases} \Delta H + \omega^2\,\varepsilon\,\mu\, H = 0, & \text{in } \Omega, \\ \nu \times \mathrm{curl} H = \widetilde{m}, & \text{on } \Gamma. \end{cases}$$

[13] Calderón operators are also known as *capacity*, or *impedance*, or admittance, or *Steklov-Poincaré* operators.

$\operatorname{curl}\operatorname{curl}$: superconductors and MHD

The study of the operators "$\operatorname{curl}\operatorname{curl}$" and "$\operatorname{curl}\operatorname{curl} - \varrho^2 \boldsymbol{I}$" is essential not only in the mathematical theory of classical electromagnetics, but in other related important applications areas, such as, e.g., the theory of superconductors, magnetohydrodynamics (MHD) (where the equations - consisting of an elegant and subtle coupling of the Navier-Stokes and Maxwell's equations - govern the motion of electrically conducting viscous incompressible fluids in a magnetic field, and in particular in the ideal linear MHD equations that describe the stability properties of a "tokamak", i.e., a device which uses a powerful magnetic field to confine a hot plasma in the shape of a torus), etc.

The Steklov problem in Electromagnetics

Although Steklov boundary conditions have been considered for many classes of operators, in the literature there are not so many results concerning Steklov-type eigenvalues for Maxwell's equations; interesting exceptions are the recent papers by S. Cogar, D. Colton and P. Monk[14], and by J. Camanõ, C. Lackner and P. Monk[15].

In the first of these papers, the use of Steklov eigenvalues for Maxwell's equations is suggested to detect changes in a scatterer using remote measurements of the scattered wave, i.e., as a novel "target signature" for nondestructive testing via inverse scattering.

Because the Steklov eigenvalue problem for Maxwell's equations is not a standard eigenvalue problem for a compact operator, a modified Steklov problem is proposed, that restores compactness.

In particular it is shown that it is possible to measure Steklov eigenvalues for a bounded inhomogeneous scatterer by solving a sequence of modified far field equations.

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[14] in "Maxwell's Equations: Analysis and Numerics" (Langer U., Pauly D., Repin S., eds.), Radon Series on Computational and Applied Mathematics **24**, De Gruyter, 2019, 145-169.

[15] SIAM Journal on Mathematical Analysis **49**, 2017, 4376-4401.

In order to measure the modified Steklov eigenvalues of a domain from far field measurements, in the second mentioned work, Camanõ, Lackner and Monk perturb the usual far-field equation of the Linear Sampling Method[16] by using the far-field pattern of an auxiliary impedance problem related to the modified Steklov problem.
The boundary condition appearing in the Steklov problem discussed by Camanõ, Lackner and Monk, is[17] $\nu \times \mathrm{curl} u = \lambda u_{\mathrm{T}}$, on $\partial\Omega$; accordingly the energy space they use is $\{u \in H(\mathrm{curl}, \Omega) \colon u_{\mathrm{T}} \in (L^2(\partial\Omega))^3\}$, and the corresponding eigenvectors turn out to be divergence free.

Our boundary condition is $\nu \times \mathrm{curl} u = \lambda u$, on $\partial\Omega$; clearly it is stronger, in the sense that it implies that our eigenfunctions are automatically tangential: this allows us to discard the part of the spectrum associated with possible non tangential eigenvectors, which are responsible for the appearance of an accumulation point in the spectrum of the operator discussed in the counterexample of the paper by Camanõ, Lackner and Monk. Our energy space is $\left\{u \in (H^1(\Omega))^3 : \ u \cdot \nu = 0\right\}$.

[16] According to the LSM, the norm of the approximate solution of the far field equation is an indicator for the obstacle in a scattering problem. Using the Singular Value Decomposition of the far field operator, the LSM is very easy to implement, and (assuming that the necessary data is available) the obstacle reconstruction (inverse) problem can be solved very quickly.

[17] By $u_{\mathrm{T}} := -\nu \times (\nu \times u) = \pi_{\mathrm{T}} u$ is denoted the tangential component of u.

In a similar setting to the work by by Camanõ, Lackner and Monk, M. Halla[18] studies, in a precise, technical way, both the original Steklov eigenvalue problem for Maxwell's equations, and the aforementioned modified Steklov problem, and analyzes their Fredholmness and approximation. Among other properties, he shows that the essential spectrum consists of only the point 0, and that the eigenvalues are discrete in $\mathbb{C} \setminus \{0\}$. He also shows that infinitely many eigenvalues exist, when all of the coefficients are real-valued.

Very interesting work on the existence and stability of electromagnetic Steklov eigenvalues has recently been done by S. Cogar[19]. With an interesting trace class modification of the appearing operator, and assuming that the magnetic permeability $\mu = 1$, he surpasses the fact that[20] there are no existence results for Steklov eigenvalues, in the case of non-smooth, or non-real, electric permittivity ε.

[18] ◇ arXiv:1909.01983v1 [math.SP].
◇ ESAIM M^{2}AN **55** 2021, 57-76.

[19] ◇ SIAM Journal on Applied Mathematics **80** 2020, 881-905.
◇ Inverse Problems and Imaging 2021, doi: 10.3934/ipi.2021011.
◇ SIAM Journal on Mathematical Analysis **52** 2020, 6412-6441 (with P. Monk).

[20] contrary to the corresponding scattering problem for the Helmholtz equation, where the case of existence of Steklov eigenvalues can be treated even for complex coefficients, in view of the Agmon theory for non-selfadjoint opereators

The Calderón operator in Electromagnetics

The Calderón operator and its variants have been and are studied extensively in Electromagnetics, see, e.g., the classical book by M. Cessenat[21].

Another approach to the representation of an exterior Calderón operator associated with a scattering problem for not necessarily spherical domains is proposed by G. Kristensson, $\mathbb{S}$, N. Wellander and A. Yannacopoulos[22]:
The appropriate series expansions are performed with respect to generalized harmonics (the set of eigenfunctions to the Laplace-Beltrami operator for the domain's boundary).
Further, the norm in an appropriate trace space of the exterior Calderón operator is obtained in view of an eigenproblem for a suitable quadratic form.

Note that, in the case of a sphere, the eigenfunctions of the Laplace-Beltrami operator are the spherical harmonics, hence the classical Steklov eigenfunctions.

[21] *Mathematical Methods in Electromagnetics, Linear Theory and Applications*, World Scientific, 1996.
[22] SN Partial Differential Equations and Applications **1**:6, 2020.

Back to the interior Calderón operator

We consider the boundary value problem (cf. frame 23):

$$\begin{cases} \mathrm{curl}\,\mathrm{curl} H - \omega^2 \varepsilon\, \mu\, H = 0, & \text{in } \Omega, \\ \nu \times \mathrm{curl} H = \widetilde{m}, & \text{on } \Gamma, \end{cases} \tag{2.1}$$

that introduces the associated interior Calderón operator as the map

$$\widetilde{m} \mapsto \nu \times H,$$

where H is the magnetic field, ε and μ are, respectively, the electric permittivity and the magnetic permeability of the medium, and $\omega > 0$ is the angular frequency.
The fixed vector m is the tangential trace of the electric field E (i.e., $\nu \times E = m$) on Γ, and $\widetilde{m} := -\mathrm{i}\,\omega\,\varepsilon\, m$.

Some spaces

Let Ω be a bounded open set in $\mathbb{R}^3$ with sufficiently smooth boundary Γ.
By $L^2(\Omega)$, $L^2(\Gamma)$ and $H^1(\Omega)$, $H^1_0(\Omega)$ we denote the standard Lebesgue and Sobolev spaces.
The fractional order Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, are defined as

- $\boxed{0 < s < 1}$ $H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{3}{2}}} \in L^2(\Omega \times \Omega) \right\}$
- $\boxed{s > 1}$ $H^s(\Omega) = \left\{ u \in H^{[s]}(\Omega) : D^\alpha u \in H^{s-[s]}(\Omega), \forall \alpha \in \mathbb{N}_0^3 : |\alpha| = [s] \right\}$

with norm: $$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{H^{[s]}(\Omega)} + \sum_{|\alpha|=[s]} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{3+2(s-[s])}} dx\, dy \right)^{\frac{1}{2}}$$

- $H^s_0(\Omega)$ = the closure of C_0^∞ in the $H^s(\Omega)$-norm
- $H^{-s}(\Omega)$ is the dual space of $H^s_0(\Omega)$
- $H^{\frac{1}{2}}(\Gamma)$ = the space of traces of $H^1(\Omega)$ on Γ
- $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)$
- $H^s(\Gamma)$ = the space of traces of $H^{s+\frac{1}{2}}(\Omega)$ on Γ
- $H^{-s}(\Gamma)$ is the dual space of $H^s(\Gamma)$

Some spaces for Electromagnetics

- $TL^2(\Gamma) = \{u \in (L^2(\Gamma))^3 : \nu \cdot u = 0 \text{ on } \Gamma\}$
- $TH^{\frac{1}{2}}(\Gamma) = \{u \in (H^{\frac{1}{2}}(\Gamma))^3 : \nu \cdot u = 0 \text{ on } \Gamma\}$
- $TH^{-\frac{1}{2}}(\Gamma) = (TH^{\frac{1}{2}}(\Gamma))'$

- $H(\mathrm{curl}, \Omega) = \{u \in (L^2(\Omega))^3 : \mathrm{curl} u \in (L^2(\Omega))^3\}$,
  with norm: $||u||_{H(\mathrm{curl},\Omega)} = \left(||u||^2_{(L^2(\Omega))^3} + ||\mathrm{curl} u||^2_{(L^2(\Omega))^3}\right)^{\frac{1}{2}}$
- $H(\mathrm{div}, \Omega) = \{u \in (L^2(\Omega))^3 : \mathrm{div} u \in L^2(\Omega)\}$,
  with norm: $||u||_{H(\mathrm{div},\Omega)} = \left(||u||^2_{(L^2(\Omega))^3} + ||\mathrm{div} u||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$
- $H_0(\mathrm{div}, \Omega) = \{u \in H(\mathrm{div}, \Omega) : \nu \cdot u = 0 \text{ on } \Gamma\}$
- $X_{\mathrm{T}}(\Omega) = H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega)$,
  with norm:
  $||u||_{H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)} = \left(||u||^2_{(L^2(\Omega))^3} + ||\mathrm{curl} u||^2_{(L^2(\Omega))^3} + ||\mathrm{div} u||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$
- $X_{\mathrm{T}}(\mathrm{div}\, 0, \Omega) = \{u \in X_{\mathrm{T}}(\Omega) : \mathrm{div} u = 0 \text{ in } \Omega\}$

The main BVP

In the sequel, we will always assume that

Ω is a bounded, connected, open set in $\mathbb{R}^3$, with a $C^{1,1}$ boundary $\Gamma = \partial\Omega$

We focus on a Calderón operator associated with the following BVP:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{u} - \alpha\boldsymbol{u} - \theta\operatorname{grad}\operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl}\boldsymbol{u} = \boldsymbol{f}, & \text{on } \Gamma, \end{cases} \tag{2.2}$$

where $\boldsymbol{u}$ is the unknown vector field.

The "penalty term"[23] $\theta\operatorname{grad}\operatorname{div}\boldsymbol{u}$ (with $\theta > 0$) is introduced in order to guarantee the coercivity of the quadratic form associated with the Calderón operator.

Note that the boundary operator $\nu \times \operatorname{curl}\boldsymbol{u}$ in (2.2) can be considered as the "electromagnetic version" of the operator $D_\nu u$ on Γ, usually associated with the scalar Laplace operator.

[23] cf. Costabel M., Dauge M., Mathematical Methods in the Applied Sciences **22** 1999, 243-258.

The interior Calderón operator for the main BVP

The interior Calderón operator $\mathcal{C}$ is then defined by

$$\mathcal{C}(f) = \nu \times \boldsymbol{u}\,, \tag{2.3}$$

where $\boldsymbol{u}$ is the solution of (2.2).
Thus, the Calderón operator establishes a correspondence between the electric and magnetic fields on Γ:

$$\nu \times \operatorname{curl} \boldsymbol{u} \longmapsto \nu \times \boldsymbol{u}\,.$$

Now, in our problem, the corresponding to the N-t-D map is

$$\nu \times \operatorname{curl} \boldsymbol{u} \longmapsto \boldsymbol{u}\,,$$

the eigenvalues of which are the reciprocals of the eigenvalues of the following Steklov-type problem for Maxwell's equations

$$\begin{cases} \operatorname{curl}\operatorname{curl} \boldsymbol{u} - \alpha \boldsymbol{u} - \theta \operatorname{grad}\operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} \boldsymbol{u} = \lambda \boldsymbol{u}, & \text{on } \Gamma. \end{cases} \tag{2.4}$$

The problem (2.4) has to be interpreted in the weak sense as:
find $u \in X_{\mathrm{T}}(\Omega)$ such that

$$\int_\Omega \mathrm{curl} u \cdot \mathrm{curl}\varphi \, dx - \alpha \int_\Omega u \cdot \varphi \, dx + \theta \int_\Omega \mathrm{div}\, u \, \mathrm{div}\varphi \, dx = -\lambda \int_\Gamma u \cdot \varphi \, d\sigma, \quad (2.5)$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

In particular, any $u \in X_{\mathrm{T}}(\Omega)$ satisfies the condition $u \cdot \nu = 0$ on Γ.

It is known that[24] $X_{\mathrm{T}}(\Omega)$ is continuously embedded in $(H^1(\Omega))^3$, and compactly embedded in $(L^2(\Omega))^3$, where Ω is a bounded, connected, open set in $\mathbb{R}^3$, with a $C^{1,1}$ boundary Γ.

Thus, in our problem,

$$X_{\mathrm{T}}(\Omega) = \left\{ u \in (H^1(\Omega))^3 : \; u \cdot \nu = 0 \right\}.$$

[24] This follows by the Gaffney inequality:

$$\|u\|_{(H^1(\Omega))^3} \leq c \left(\|u\|_{L^2(\Omega)} + \|\mathrm{curl} u\|_{L^2(\Omega)} + \|\mathrm{div} u\|_{L^2(\Omega)} \right),$$

where $c > 0$, for all $u \in X_{\mathrm{T}}(\Omega)$.

The tangential components trace and its basic properties

Let $w : \overline{\Omega} \to \mathbb{R}^3$ be a smooth vector field; we have that

$$w|_\Gamma = \underbrace{(\nu \cdot w|_\Gamma)\,\nu}_{\text{normal component}} + \underbrace{(\nu \times w|_\Gamma) \times \nu}_{\text{tangential component}} .$$

We denote by

$$\pi_{\mathrm{T}} w := \nu \times (w \times \nu)$$

the *tangential components trace*[25] of w on Γ.

Some of its main properties are:

- $\pi_{\mathrm{T}}\big((H^1(\Omega))^3\big) = TH^{\frac{1}{2}}(\Gamma)$,
- $\pi_{\mathrm{T}}\Big((H^{\frac{1}{2}}(\Gamma))^3\Big) = TH^{\frac{1}{2}}(\Gamma)$,
- $\Big(\pi_{\mathrm{T}}\Big((H^{\frac{1}{2}}(\Gamma))^3\Big)\Big)' = TH^{-\frac{1}{2}}(\Gamma)$,
- π_{T} is a compact operator from $(H^1(\Omega))^3$ to $TL^2(\Gamma)$.

We shall often use the same symbol for a function and its trace.

[25] The *tangential trace* of w on Γ is defined as $\gamma_{\mathrm{T}} w := \nu \times w$.

Solvability of the main BVP

Let $\alpha \in \mathbb{R}$ and $\theta > 0$ be fixed.
We consider the solvability of problems (2.2) and (2.4).

We begin by establishing that for any fixed $\eta \geq 0$ sufficiently large, the problem

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\boldsymbol{u} - \alpha\ \boldsymbol{u} - \theta\,\mathrm{grad}\,\mathrm{div}\boldsymbol{u} = 0, & \text{in } \Omega, \\ \nu \times \mathrm{curl}\boldsymbol{u} - \eta\,\boldsymbol{u} = \boldsymbol{f}, & \text{on } \Gamma \end{cases} \tag{2.6}$$

has a (unique) solution for every datum $\boldsymbol{f} \in TL^2(\Gamma)$.

The weak formulation of problem (2.6) reads: find $\boldsymbol{u} \in X_{\mathrm{T}}(\Omega)$ such that

$$\int_\Omega \mathrm{curl}\boldsymbol{u}\cdot\mathrm{curl}\varphi\, dx - \alpha\int_\Omega \boldsymbol{u}\cdot\varphi\, dx + \theta\int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\varphi\, dx + \eta\int_\Gamma \boldsymbol{u}\cdot\varphi\, d\sigma = -\int_\Gamma \boldsymbol{f}\cdot\varphi\, d\sigma, \tag{2.7}$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

Solvability of the main BVP

When $\alpha > 0$, the quadratic form associated with the LHS of (2.7) is not necessarily positive definite, and this complicates the analysis of the problem.

- We first consider the (natural) assumption $\alpha < A_1$,

where

$$A_1 := \inf_{\substack{u \in (H_0^1(\Omega))^3 \\ u \neq 0}} \frac{\int_\Omega |\mathrm{curl} \boldsymbol{u}|^2 \, dx + \theta \int_\Omega |\mathrm{div} \boldsymbol{u}|^2 \, dx}{\int_\Omega |u|^2 dx} \,. \tag{2.8}$$

By the Gaffney inequality it follows that $A_1 > 0$, and that A_1 is the first eigenvalue of the problem

$$\int_\Omega \mathrm{curl} \boldsymbol{u} \cdot \mathrm{curl} \varphi \, dx + \theta \int_\Omega \mathrm{div} \boldsymbol{u} \, \mathrm{div} \varphi \, dx = A \int_\Omega u \cdot \varphi \, dx \,, \forall \, \varphi \in (H_0^1(\Omega))^3, \tag{2.9}$$

with unknowns $u \in (H_0^1(\Omega))^3$ (the Dirichlet eigenfunction), and $A \in \mathbb{R}$ (the Dirichlet eigenvalue).

- The case $\alpha > A_1$ will be discussed at the end of this talk, cf. frames 63 and 64.

We can now state

Solvability of the main BVP: the case $\alpha < A_1$

Theorem (2.10)

Let $\alpha < A_1$, and $\theta > 0$. Then there exists $c_{\alpha,\theta} \geq 0$ such that for any $\eta \geq c_{\alpha,\theta}$ the quadratic form defined by the LHS of (2.7) is coercive in $X_{\mathrm{T}}(\Omega)$, and problem (2.7) has a unique solution $u \in X_{\mathrm{T}}(\Omega)$, for all $f \in TL^2(\Gamma)$.

Moreover, for $\alpha \leq 0$ one can take $c_{\alpha,\theta} = 0$.

Finally, if $\alpha \leq 0$ and $\eta = 0$, and if, in addition, f satisfies the condition $\mathrm{div}_\Gamma f = 0$ on Γ, then $\mathrm{div}\, u = 0$ in Ω.

Remark (2.11)

In the last case above, the problem (2.7) can be formulated directly in the energy space $X_{\mathrm{T}}(\mathrm{div}\, 0, \Omega)$.

This weak formulation can be stated as:
find $u \in X_{\mathrm{T}}(\mathrm{div}\, 0, \Omega)$ such that

$$\int_\Omega \mathrm{curl} u \cdot \mathrm{curl} \varphi \, dx - \alpha \int_\Omega u \cdot \varphi \, dx = - \int_\Gamma f \cdot \varphi \, d\sigma \,, \tag{2.12}$$

for all $\varphi \in X_{\mathrm{T}}(\mathrm{div}\, 0, \Omega)$.

For any $\alpha < A_1, \theta > 0$ and $\eta \geq c_{\alpha,\theta}$ as in Theorem 2.10, we consider the operator $\mathcal{L}^{\eta}_{\alpha,\theta}$ from $X_{\mathrm{T}}(\Omega)$ to its dual $(X_{\mathrm{T}}(\Omega))'$ defined by the pairing

$$\langle \mathcal{L}^{\eta}_{\alpha,\theta}(u), \varphi \rangle = \int_{\Omega} \mathrm{curl}\, u \cdot \mathrm{curl}\, \varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \mathrm{div}\, u \, \mathrm{div}\, \varphi \, dx + \eta \int_{\Gamma} u \cdot \varphi \, d\sigma,$$

for all $u, \varphi \in X_{\mathrm{T}}(\Omega)$.

Consider the operator $\mathcal{J}$ from $TL^2(\Gamma)$ to $(X_{\mathrm{T}}(\Omega))'$ defined by the pairing

$$\langle \mathcal{J}(f), \varphi \rangle = \int_{\Gamma} f \cdot \varphi \, d\sigma,$$

for all $f \in TL^2(\Gamma)$, $\varphi \in X_{\mathrm{T}}(\Omega)$.

Recall that $\pi_{\mathrm{T}}(u)$ coincides with the trace of u on Γ, for any $u \in X_{\mathrm{T}}(\Omega)$.

By Theorem 2.10, $\mathcal{L}^{\eta}_{\alpha,\theta}$ is invertible, hence we can introduce the operator $\mathcal{A}^{\Gamma}_{\eta}$ from $TL^2(\Gamma)$ to itself, defined by

$$\mathcal{A}^{\Gamma}_{\eta} := -\pi_{\mathrm{T}} \circ \left(\mathcal{L}^{\eta}_{\alpha,\theta}\right)^{-1} \circ \mathcal{J}. \tag{2.13}$$

Theorem (2.14)

The operator $\mathcal{A}^{\Gamma}_{\eta}$ is compact and self-adjoint in $TL^2(\Gamma)$.

For $\eta \geq 0$, we define the sesquilinear form[26]

$$\langle u, v\rangle^{\eta}_{\alpha,\theta} := \int_{\Omega} \mathrm{curl}\, u \cdot \mathrm{curl}\, \bar{v}\, dx - \alpha \int_{\Omega} u \cdot \bar{v}\, dx + \theta \int_{\Omega} \mathrm{div}\, u\, \mathrm{div}\, \bar{v}\, dx + \eta \int_{\Gamma} u \cdot \bar{v}\, d\sigma, \tag{2.15}$$

for all $u, v \in X_{\mathrm{T}}(\Omega)$.

We observe that, if $\eta \geq c_{\alpha,\theta}$, this form defines[27] a scalar product in $X_{\mathrm{T}}(\Omega)$.

In view of this, problem (2.7) can be written as[28]

$$\langle u, \varphi\rangle^{\eta}_{\alpha,\theta} = -\langle f, \varphi\rangle_{(L^2(\Gamma))^3} ,$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

[26] Note that $\langle \cdot, \cdot\rangle^{0}_{\alpha,\theta}$ is the sesquilinear form appearing in the LHS of (2.5).

[27] by Theorem 2.10; see also the Gaffney inequality.

[28] $\langle \cdot, \cdot\rangle_{(L^2(\Gamma))^3}$ denotes the standard scalar product in $(L^2(\Gamma))^3$: $\langle p, q\rangle_{(L^2(\Gamma))^3} := \int_{\Gamma} p \cdot \bar{q}\, d\sigma$.

Resolvent operators

We also consider the operator $\mathcal{A}_\eta^\Omega$ from $X_{\mathrm{T}}(\Omega)$ to itself, defined by

$$\mathcal{A}_\eta^\Omega(u) = -\left(\mathcal{L}_{\alpha,\theta}^\eta\right)^{-1} \circ \mathcal{J} \circ \pi_{\mathrm{T}} \,. \tag{2.16}$$

Theorem (2.17)

The operator $\mathcal{A}_\eta^\Omega$ is compact and self-adjoint with respect to (2.15).

It is evident that

$$\mathcal{A}_\eta^\Gamma \circ \pi_{\mathrm{T}} = \pi_{\mathrm{T}} \circ \mathcal{A}_\eta^\Omega \,. \tag{2.18}$$

The eigenvalue problem

We consider again the eigenvalue problem (2.4):

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{u} - \alpha\boldsymbol{u} - \theta\operatorname{grad}\operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl}\boldsymbol{u} = \lambda\boldsymbol{u}, & \text{on } \Gamma. \end{cases}$$

Recall that the weak formulation of (2.4) is given in (2.5):

$$\int_\Omega \operatorname{curl}\boldsymbol{u}\cdot\operatorname{curl}\varphi\, dx - \alpha\int_\Omega \boldsymbol{u}\cdot\varphi\, dx + \theta\int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\varphi\, dx = -\lambda\int_\Gamma \boldsymbol{u}\cdot\varphi\, d\sigma, \forall\varphi \in X_{\mathrm{T}}(\Omega).$$

It turns out that this eigenvalue problem can be recast as an eigenvalue problem for the operator $\mathcal{A}_\eta^\Gamma$ (or, for the operator $\mathcal{A}_\eta^\Omega$).

Since these operators are compact and self-adjoint, their spectra can be easily described.

In particular, we have the following result.

Theorem (2.19)

Let $\alpha < A_1$ and $\theta > 0$. The spectrum of the operator $\mathcal{A}_\eta^\Omega$ can be represented as $\{0\} \cup \{\gamma_n : n \in \mathbb{N}\}$, where γ_n, $n \in \mathbb{N}$, are negative eigenvalues of finite multiplicity, $\gamma_n \to 0$ as $n \to \infty$, and 0 is an eigenvalue of infinite multiplicity with eigenspace given by $(H_0^1(\Omega))^3$.

Moreover, the point spectrum of the operator $\mathcal{A}_\eta^\Gamma$ is given by $\{\gamma_n : n \in \mathbb{N}\}$.

Furthermore, if $\mathcal{A}_\eta^\Omega u = \gamma_n u$, for some $u \in X_{\mathrm{T}}(\Omega)$, then $\mathcal{A}_\eta^\Gamma \pi_{\mathrm{T}} u = \gamma_n \pi_{\mathrm{T}} u$.

The eigenvalue problem

By Theorem 2.19 and using the Min-Max Principle for the compact, self-adjoint operator $\mathcal{A}_\eta^\Omega$, we have

Theorem (2.20)

Let $\alpha < A_1$ and $\theta > 0$. The eigenvalues of problem (2.4) form a sequence λ_n, $n \in \mathbb{N}$, in $\mathbb{R}$, given by $\lambda_n = \gamma_n^{-1} + \eta$, for all $n \in \mathbb{N}$, and the eigenfunctions coincide with those of the operator $\mathcal{A}_\eta^\Omega$ associated with γ_n.

Moreover, $\lambda_n \to -\infty$, as $n \to \infty$, and can be represented as

$$\lambda_n = - \min_{\substack{V \subset X_{\mathrm{T}}(\Omega) \\ \dim V = n}} \max_{u \in V \setminus (H_0^1(\Omega))^3} \frac{\int_\Omega \left(|\mathrm{curl}\, u|^2 - \alpha |u|^2 + \theta |\mathrm{div}\, u|^2 \right) dx}{\int_\Gamma |\pi_{\mathrm{T}} u|^2 \, dx}, \qquad (2.21)$$

where, as usual, each eigenvalue is repeated as many times as its multiplicity.

Note that the eigenvalue problem for the operator $\mathcal{A}_\eta^\Omega$ can be written, in terms of the unknowns $u \in X_{\mathrm{T}}(\Omega)$ (the eigenvector), and λ (the eigenvalue), as

$$\int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, \varphi \, dx - \alpha \int_\Omega u \cdot \varphi \, dx + \theta \int_\Omega \mathrm{div}\, u \, \mathrm{div}\, \varphi \, dx + \eta \int_\Gamma u \cdot \varphi \, d\sigma = -\lambda \int_\Gamma u \cdot \varphi \, d\sigma,$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

It follows by the previous results, that $X_{\mathrm{T}}(\Omega)$ can be decomposed as an orthogonal sum, with respect to the scalar product $\langle \cdot, \cdot \rangle_{\alpha,\theta}^\eta$, as

$$X_{\mathrm{T}}(\Omega) = \mathrm{Ker} \mathcal{A}_\eta^\Omega \oplus \left(\mathrm{Ker} \mathcal{A}_\eta^\Omega\right)^\perp = (H_0^1(\Omega))^3 \oplus \left(\mathrm{Ker} \mathcal{A}_\eta^\Omega\right)^\perp .$$

The eigenvalue problem

We note that $u \in \left(\mathrm{Ker}\mathcal{A}_\eta^\Omega\right)^\perp$ if and only if

$$\int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\varphi \, dx - \alpha \int_\Omega u \cdot \varphi \, dx + \theta \int_\Omega \mathrm{div}\, u \, \mathrm{div}\varphi \, dx = 0, \tag{2.22}$$

for all $\varphi \in (H_0^1(\Omega))^3$ or, equivalently, for all $\varphi \in (C_c^\infty(\Omega))^3$.

Thus, $u \in \left(\mathrm{Ker}\mathcal{A}_\eta^\Omega\right)^\perp$ *if and only if* u is a weak solution in $(H^1(\Omega))^3$ of

$$\begin{cases} \mathrm{curl}\, \mathrm{curl}\, u - \alpha u - \theta\, \mathrm{grad}\, \mathrm{div}\, u = 0, & \text{in } \Omega, \\ \nu \cdot u = 0, & \text{on } \Gamma. \end{cases} \tag{2.23}$$

Setting[29]

$$\mathcal{H}(\Omega) := \left(\mathrm{Ker}\mathcal{A}_\eta^\Omega\right)^\perp,$$

we can write

$$X_{\mathrm{T}}(\Omega) = (H_0^1(\Omega))^3 \oplus \mathcal{H}(\Omega). \tag{2.24}$$

Corollary (2.25)

$$\mathrm{Ker}\mathcal{A}_\eta^\Gamma = \{0\}$$

[29] These functions are the analogues of the harmonic functions considered by Auchmuty.

We introduce a Calderón operator associated with the interior problem (2.7). In order to identify the appropriate condition under which our Calderón operator is well-defined we need the following result.

We denote by $\Sigma = \{\lambda_n : n \in \mathbb{N}\}$ the set of Steklov eigenvalues of problem (2.5).

Theorem (2.26)

Let $\alpha < A_1$ and $\theta > 0$. The problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} u - \alpha u - \theta \operatorname{grad}\operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = \lambda\, u + f, & \text{on } \Gamma, \end{cases} \tag{2.27}$$

is uniquely solvable in $X_{\mathrm{T}}(\Omega)$, for all $f \in TL^2(\Gamma)$, if and only if $\lambda \notin \Sigma$.

Then we can give the following definition.

Definition (2.28)

Assume that $\alpha < A_1$ and $\theta > 0$ are such that $0 \notin \Sigma$.
The interior Calderón operator is the operator defined as

$$\mathcal{C}: TL^2(\Gamma) \to TL^2(\Gamma),$$
$$f \mapsto \mathcal{C}(f) := \nu \times u,$$

where $u \in X_{\mathrm{T}}(\Omega)$ is the solution of the problem

$$\begin{cases} \mathrm{curl}\,\mathrm{curl} u - \alpha u - \theta\, \mathrm{grad}\,\mathrm{div} u = 0, & \text{in } \Omega, \\ \nu \times \mathrm{curl} u = f, & \text{on } \Gamma. \end{cases}$$

We conclude this section by discussing the condition $0 \notin \Sigma$.

We consider two auxiliary eigenvalue problems:

(1) The classical eigenvalue problem for the Neumann Laplacian

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } \Omega, \\ D_\nu\phi = 0, & \text{on } \Gamma, \end{cases} \tag{2.29}$$

for $\phi \in H^1(\Omega)$, which is well-known to admit a divergent sequence $\lambda_n^{\mathcal{N}}$, $n \in \mathbb{N}$, of non-negative eigenvalues of finite multiplicity, with $\lambda_1^{\mathcal{N}} = 0$.

(2) The eigenvalue problem for the $\operatorname{curl}\operatorname{curl}$ operator with the "magnetic" boundary condition

$$\begin{cases} \operatorname{curl}\operatorname{curl} u = \lambda u, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = 0, & \text{on } \Gamma, \end{cases} \tag{2.30}$$

for $u \in X_{\mathrm{T}}(\operatorname{div} 0, \Omega)$, which also admits a divergent sequence $\lambda_n^{\mathcal{M}}$, $n \in \mathbb{N}$, of non-negative eigenvalues of finite multiplicity, with $\lambda_1^{\mathcal{M}} = 0$.

For the relation between problem (2.30) and the eigenvalue problem for Maxwell's system, one can see the paper by Z. Zhang[30].

[30] Zeitschrift für Angewandte Mathematik und Physik **69**:104, 2018.

Consider now problem (2.4) with $\lambda = 0$, namely

$$\begin{cases} \operatorname{curl}\operatorname{curl} u - \alpha u - \theta \operatorname{grad}\operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = 0, & \text{on } \Gamma. \end{cases} \tag{2.31}$$

For the values of α and θ for which Σ is well-defined, the following result gives a necessary and sufficient condition for the validity of the hypothesis $0 \notin \Sigma$.

Theorem (2.32)

Problem (2.31) has a non-trivial solution $u \in X_T(\Omega)$ if and only if $\alpha \in \{\theta\lambda_n^{\mathcal{N}} : n \in \mathbb{N}\} \cup \{\lambda_n^{\mathcal{M}} : n \in \mathbb{N}\}$.

Note that Theorem 2.32 may be considered as the "magnetic version" of Theorem 1.1 in the paper by M. Costabel and M. Dauge[31], already mentioned in relation to the penalty term regarding the coercivity of the Calderón operator (frame 32).

[31] Mathematical Methods in the Applied Sciences **22** 1999, 243-258.

Throughout this section we again assume that $\alpha < A_1$ and $\theta > 0$.

Recall that the space $X_{\mathrm{T}}(\Omega)$ can be considered as a Hilbert space with respect to the scalar product $\langle \cdot, \cdot \rangle^{\eta}_{\alpha,\theta}$.

Moreover, the space $\mathcal{H}(\Omega)$ of solutions to problem (2.23):

$$\begin{cases} \mathrm{curl}\,\mathrm{curl} u - \alpha u - \theta\, \mathrm{grad}\, \mathrm{div} u = 0, & \text{in } \Omega, \\ \nu \cdot u = 0, & \text{on } \Gamma, \end{cases}$$

is a closed subspace, and it admits a Hilbert basis of Steklov eigenfunctions u_n^{Ω}, $n \in \mathbb{N}$, which satisfy the equation

$$\int_{\Omega} \mathrm{curl} u_n^{\Omega} \cdot \mathrm{curl} \varphi \, dx - \alpha \int_{\Omega} u_n^{\Omega} \cdot \varphi \, dx + \theta \int_{\Omega} \mathrm{div} u_n^{\Omega} \, \mathrm{div} \varphi \, dx = -\lambda_n \int_{\Gamma} u_n^{\Omega} \cdot \varphi \, d\sigma, \quad (2.33)$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

Note that (2.33) can be equivalently written as

$$\langle u_n^{\Omega}, \varphi \rangle^{\eta}_{\alpha,\theta} = -(\lambda_n - \eta) \langle u_n^{\Omega}, \varphi \rangle_{(L^2(\Gamma))^3} , \quad (2.34)$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

In the sequel, we consider normalized eigenfunctions u_n^Ω, $n \in \mathbb{N}$, i.e., we assume that

$$\left\langle u_n^\Omega, u_m^\Omega \right\rangle_{\alpha,\theta}^{\eta} = \delta_{nm} ,$$

where δ_{nm} is the Kronecker symbol.
Taking into account Theorem 2.19 and the fact that the traces of the eigenfunctions u_n^Ω provide also a basis of the space $TL^2(\Gamma)$, we set

$$u_n^\Gamma := \sqrt{|\lambda_n - \eta|}\, \pi_{\mathrm{T}} u_n^\Omega \tag{2.35}$$

and, in view of (2.34), we observe that u_n^Γ, $n \in \mathbb{N}$, is an orthonormal basis of $TL^2(\Gamma)$.

We proceed by proving the following theorem which provides a spectral representation for the solutions of problem (2.2):

$$\begin{cases} \operatorname{curl}\operatorname{curl} u - \alpha u - \theta \operatorname{grad}\operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = f, & \text{on } \Gamma, \end{cases}$$

for data $f \in TL^2(\Gamma)$ and a corresponding spectral representation for the associated Calderón operator.

Recall that (by Theorem 2.26), if $0 \notin \Sigma$, then problem (2.2) is uniquely solvable.

Theorem (2.36)

Assume that $0 \notin \Sigma$.
Let $f \in TL^2(\Gamma)$ *be represented as*

$$f = \sum_{n=1}^{\infty} c_n u_n^{\Gamma} ,$$

where $\{c_n\}_{n\in\mathbb{N}} \in \ell^2$.
Then the solution $u \in X_{\mathrm{T}}(\Omega)$ *of problem* (2.2), *is given by*

$$u = \sum_{n=1}^{\infty} \left(\frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} \, c_n \right) u_n^{\Omega} \,. \tag{2.37}$$

Moreover, the corresponding interior Calderón operator can be represented as

$$\mathcal{C}(f) = \nu \times \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} \, u_n^{\Gamma} \,. \tag{2.38}$$

In Theorem 2.53 we shall prove the corresponding result for data $f \in TH^{-\frac{1}{2}}(\Gamma)$, and this will allow to extend the Calderón operator and define it as an operator from $TH^{-\frac{1}{2}}(\Gamma)$ to $TH^{\frac{1}{2}}(\Gamma)$, as one would expect.
For $s > 0$, we define

$$T\mathcal{H}^s(\Gamma) := \left\{ f = \sum_{n=1}^{\infty} c_n u_n^\Gamma : \|f\|_{s,\Gamma} := \left(\sum_{n=1}^{\infty} |c_n|^2 |\, \lambda_n - \eta|^{2s} \right)^{\frac{1}{2}} < \infty \right\}, \quad (2.39)$$

endowed with the norm $\| \cdot \|_{s,\Gamma}$.
Further, we define

$$T\mathcal{H}^{-s}(\Gamma) := \left(T\mathcal{H}^s(\Gamma) \right)' . \quad (2.40)$$

Theorem 2.45(i) (in frame 58 below), shows that, for $s = \frac{1}{2}$ and $s = -\frac{1}{2}$, this definition is equivalent to any of the classical definitions of the trace spaces:

$$T\mathcal{H}^{\frac{1}{2}}(\Gamma) = TH^{\frac{1}{2}}(\Gamma) \text{ and } T\mathcal{H}^{-\frac{1}{2}}(\Gamma) = TH^{-\frac{1}{2}}(\Gamma).$$

It is easy to see that the space $T\mathcal{H}^{-s}(\Gamma)$ can be identified with a space of sequences, namely

$$\left\{ F = \{c_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \ \|F\|_{-s,\Gamma} := \left(\sum_{n=1}^{\infty} |c_n|^2 \, |\lambda_n - \eta|^{-2s} \right)^{\frac{1}{2}} < \infty \right\}, \quad (2.41)$$

with the understanding that the action of an element $F = \{c_n\}_{n\in\mathbb{N}} \in T\mathcal{H}^{-s}(\Gamma)$ on an element $f = \sum_{n=1}^{\infty} d_n u_n^{\Gamma} \in T\mathcal{H}^{s}(\Gamma)$ is given by the pairing

$$\langle F, f \rangle = \sum_{n=1}^{\infty} c_n d_n \, , \quad (2.42)$$

which means that

$$\langle F, u_n^{\Gamma} \rangle = c_n, \ \forall n \in \mathbb{N}.$$

We note that any function $F \in TL^2(\Gamma)$ defines an element of $T\mathcal{H}^{-s}(\Gamma)$ by means of the formula

$$\langle F, f \rangle = \int_{\Gamma} F \cdot f \, d\sigma \ , \ \forall f \in T\mathcal{H}^{s}(\Gamma) \, .$$

Spectral representations

The following result allows in particular to characterize the space $T\mathcal{H}^{\frac{1}{2}}(\Gamma)$ as the trace space of $X_{\mathrm{T}}(\Omega)$.
It also provides spectral representations for the solutions of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} u - \alpha u - \theta \operatorname{grad}\operatorname{div} u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma, \end{cases} \tag{2.43}$$

and of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} u - \alpha u - \theta \operatorname{grad}\operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times u = f, & \text{on } \Gamma, \\ \nu \cdot u = 0, & \text{on } \Gamma. \end{cases} \tag{2.44}$$

We understand the solutions to problem (2.43) as functions $u \in \mathcal{H}(\Omega)$ satisfying the condition $\pi_{\mathrm{T}} u = u = f$.

Note that by (2.24), if $f = 0$, the unique solution is $u = 0$, since u would have to belong also to the space $(H_0^1(\Omega))^3$.

Thus, the solution to problem (2.43), for all admissible data f as below, will be unique.

Theorem (2.45)

(i) *The image of the trace operator* π_{T} *is given by*

$$\pi_{\mathrm{T}}\left(X_{\mathrm{T}}(\Omega)\right) = T\mathcal{H}^{\frac{1}{2}}(\Gamma)\,, \tag{2.46}$$

hence $T\mathcal{H}^{\frac{1}{2}}(\Gamma)$ *coincides with the usual Sobolev space* $TH^{\frac{1}{2}}(\Gamma)$*.*

(ii) *Let* $f \in T\mathcal{H}^{\frac{1}{2}}(\Gamma)$ *be represented as*

$$f = \sum_{n=1}^{\infty} c_n u_n^{\Gamma}\,, \tag{2.47}$$

where $\left\{c_n\sqrt{|\lambda_n - \eta|}\right\}_{n\in\mathbb{N}} \in \ell^2$*.*
Then the solution $u \in X_{\mathrm{T}}(\Omega)$ *of problem* (2.43) *is given by*

$$u = \sum_{n=1}^{\infty} c_n\sqrt{|\lambda_n - \eta|}\, u_n^{\Omega}\,. \tag{2.48}$$

Theorem (2.49) (Thm 2.45 continued)

(iii) *Let $f \in T\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and let $f \times \nu$ be represented as*

$$f \times \nu = \sum_{n=1}^{\infty} c_{n,\nu}\, u_n^{\Gamma}\,, \tag{2.50}$$

where $\left\{ c_{n,\nu} \sqrt{|\lambda_n - \eta|} \right\}_{n \in \mathbb{N}} \in \ell^2$.
Then the solution $u \in X_{\mathrm{T}}(\Omega)$ of problem (2.44) is given by

$$u = \sum_{n=1}^{\infty} c_{n,\nu}\, \sqrt{|\lambda_n - \eta|}\, u_n^{\Omega}\,. \tag{2.51}$$

Spectral representations

Theorem 2.45 allows to consider equation (2.2) also with a datum f replaced by an element $F \in T\mathcal{H}^{-\frac{1}{2}}(\Gamma)$.
In this case the formulation reads: find $u \in X_{\mathrm{T}}(\Omega)$ such that

$$\langle u, \varphi \rangle^{0}_{\alpha,\theta} = -\langle F, \bar{\varphi} \rangle \,, \tag{2.52}$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

Indeed, the trace of φ on Γ belongs to $T\mathcal{H}^{\frac{1}{2}}(\Gamma)$, and hence the RHS of (2.52) is well-defined.

Theorem (2.53)

Assume that $0 \notin \Sigma$. *Let* $F \in T\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ *be represented as*

$$F = \sum_{n=1}^{\infty} c_n \bar{u}_n^{\Gamma}, \tag{2.54}$$

where $\left\{ c_n |\lambda_n - \eta|^{-\frac{1}{2}} \right\}_{n\in\mathbb{N}} \in \ell^2$.
Then the solution $u \in X_{\mathrm{T}}(\Omega)$ *of problem* (2.52) *is given by*

$$u = \sum_{n=1}^{\infty} \left(\frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} c_n \right) \bar{u}_n^{\Omega}. \tag{2.55}$$

We have employed the notation $\bar{u}_n^{\Gamma} = \overline{u_n^{\Gamma}}$, and $\bar{u}_n^{\Omega} := \overline{u_n^{\Omega}}$. Although the orthonormal bases in the theorem, consist of eigenfunctions which are not necessarily real, it is always possible to select orthonormal bases of real eigenfunctions[32].

[32] Since the coefficients of our operator are real, it follows that if u is an eigenfunction then also $\bar{u}$ is an eigenfunction; hence one can choose a basis of real eigenfunctions, without worrying about passing from u_n^{Γ} to its complex conjugate.

Remark (2.56)

By formula (2.55), it follows that the interior Calderón operator defined in Definition 2.28 can be extended from $TL^2(\Gamma)$ *to* $T\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ *by setting*

$$\mathcal{C}(F) = \nu \times \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} \, \vec{u}_n^{\Gamma} \, , \tag{2.57}$$

for all $F \in T\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ *represented as in (2.54),* $\mathcal{C}(F)$ *being an element of* $T\mathcal{H}^{\frac{1}{2}}(\Gamma)$.

The case $\alpha > A_1$

- We conclude with the case $\alpha > A_1$.

First, note that by standard spectral theory, the problem

$$\int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\varphi \, dx + \theta \int_\Omega \mathrm{div}\, u \, \mathrm{div}\varphi \, dx = A \int_\Omega u \cdot \varphi \, dx \,, \forall \; \varphi \in (H_0^1(\Omega))^3,$$

has a divergent sequence of positive eigenvalues A_n, $n \in \mathbb{N}$ with finite multiplicity.

Suppose that $\alpha \in \mathbb{R}$ is such that $A_n < \alpha < A_{n+1}$, for some $n \in \mathbb{N}$.

Let $\mathcal{V}_n$ be the subspace of $(H_0^1(\Omega))^3$ generated by all eigenfunctions associated with all eigenvalues A_k with $k \leq n$, and let

$$\mathcal{V}_n^\perp = \{ v \in X_{\mathrm{T}}(\Omega) : \quad \langle v, u \rangle^0_{\alpha,\theta} = 0, \; \forall \; u \in \mathcal{V}_n \} \, .$$

Clearly, $\mathcal{V}_n^\perp$ is a closed subspace of $X_{\mathrm{T}}(\Omega)$.

Solvability of the main BVP: the case $\alpha > A_1$

Our final result settles the case $\alpha > A_1$:

Theorem (2.58)

Assume that $A_n < \alpha < A_{n+1}$ for some $n \in \mathbb{N}$, and let $\theta > 0$. Then

$$X_{\mathrm{T}}(\Omega) = \mathcal{V}_n \oplus \mathcal{V}_n^{\perp}\,,$$

and there exists a constant $c_{\alpha,\theta} \geq 0$, such that, for any $\eta \geq c_{\alpha,\theta}$, the quadratic form

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx \, - \, \alpha \int_\Omega u \cdot v \, dx \, + \, \theta \int_\Omega \operatorname{div} u \operatorname{div} v \, dx \, + \, \eta \int_\Gamma u \cdot v \, d\sigma,$$

is coercive in $\mathcal{V}_n^{\perp}$.

THANK YOU

FOR YOUR PATIENCE
AND YOUR ATTENTION !