The logarithmic Brunn–Minkowski conjecture

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$$\left|A+B\right|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

where $|\cdot|$ denotes the Lebesgue measure and the Minkowski linear combination of sets is given by

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This inequality captures the optimal concavity of the Lebesgue measure and becomes an equality if A and B are homothetic and convex.

The Brunn–Minkowski inequality (continued)

Choosing B to be a Euclidean ball B_ε of radius ε, we get that

$$\left|A + B_\varepsilon\right|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + \varepsilon |B_1|^{\frac{1}{n}}.$$

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Therefore, the surface area of A satisfies

$$\begin{aligned} |\partial A| = \liminf_{\varepsilon \to 0^+} \frac{|A + B_\varepsilon| - |A|}{\varepsilon} &\geq \liminf_{\varepsilon \to 0^+} \frac{\left(|A|^{1/n} + \varepsilon |B_1|^{1/n}\right)^n - |A|}{\varepsilon} \\ &= n|A|^{\frac{n-1}{n}} |B_1|^{\frac{1}{n}}. \end{aligned}$$

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Thus, one easily deduces the isomerimetric inequality: along all measurable sets of fixed volume, Euclidean balls have minimal surface area.

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Combining the Brunn–Minkowski inequality with AM-GM, we get that

$$\left|\lambda A + (1-\lambda)B\right| \geq \left(\lambda|A|^{1/n} + (1-\lambda)|B|^{1/n}\right)^n \geq |A|^\lambda |B|^{1-\lambda}.$$

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Conversely, applying this dimension-free inequality to

$$A_1 = \frac{1}{|A|^{1/n}} \cdot A, \quad B_1 = \frac{1}{|B|^{1/n}} \cdot B \;\text{ and }\; \lambda = \frac{|A|^{1/n}}{|A|^{1/n} + |B|^{1/n}},$$

we get that

$$\frac{|A+B|^{1/n}}{|A|^{1/n} + |B|^{1/n}} = \big|\lambda A_1 + (1-\lambda)B_1\big|^{1/n} \geq |A_1|^\lambda |B_1|^{1-\lambda} = 1,$$

thus recovering the original Brunn–Minkowski inequality.

Brunn–Minkowski theory

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Broadly speaking, modern Brunn–Minkowski theory tries to relate the *size* of the *sum* of given sets with the *size* of the individual *summands*, where *size* and *sum* are interpreted more loosely than in the classical Brunn–Minkowski inequality. Particular attention is given to delicate inequalities which hold for *origin-symmetric* convex sets in $\mathbb{R}^n$.

Brunn–Minkowski theory (continued)

In this talk, we will be interested in the case that:

• The *size* of a subset A of $\mathbb{R}^n$ is measured by a log-concave measure, i.e. a measure μ for which

$$\mu\big(\lambda A + (1-\lambda)B\big) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$$

for every Borel sets A, B and $\lambda \in (0,1)$. By classical results of Prékopa, Leindler and Borell a full-dimensional measure is log-concave if and only if it is of the form $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$, where $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function.

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$$\gamma_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-|x|^2/2}\,\mathrm{d}x,$$

where $|x|$ is the Euclidean length of a vector x.

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The most profound Brunn–Minkowski-type inequality for the Gaussian measure is Ehrhard's inequality (1983), which asserts that for every Borel sets A, B in $\mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\Phi^{-1}\big(\gamma_n\big(\lambda A + (1-\lambda)B\big)\big) \geq \lambda \Phi^{-1}\big(\gamma_n(A)\big) + (1-\lambda)\Phi^{-1}\big(\gamma_n(B)\big),$$

where Φ^{-1} is the inverse of the distribution function $\Phi(x) = \gamma_1((-\infty, x])$.

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Ehrhard's original proof required both sets A, B to be convex. The general version stated here is due to Borell (2003).

Ehrhard's inequality also implies the Gaussian isoperimetric inequality: among all measurable sets of fixed Gaussian measure, half spaces of the form $\{x \in \mathbb{R}^n : \; x_1 < s\}$ have minimal Gaussian surface area.

Sums of symmetric convex sets

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If K is a symmetric convex set in $\mathbb{R}^n$, then its support function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ is given by

$$h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle$$

and we can write

$$K = \left\{ x \in \mathbb{R}^n : \ \langle x, \theta \rangle \leq h_K(\theta) \text{ for every } \theta \in \mathbb{S}^{n-1} \right\}.$$

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It is straightforward from the definition that if K, L are symmetric convex sets and for $\alpha, \beta > 0$,

$$h_{\alpha K + \beta L} \equiv \alpha h_K + \beta h_L,$$

which implies that

$$\lambda K + (1-\lambda)L = \left\{ x : \ \langle x, \theta \rangle \leq \lambda h_K(\theta) + (1-\lambda) h_L(\theta), \ \forall \ \theta \in \mathbb{S}^{n-1} \right\}.$$

Sums of symmetric convex sets (continued)

If $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}_+$ is a positive even function, the *Wulff shape* of φ is the symmetric convex set defined as

$$\mathbb{W}[\varphi] = \left\{ x \in \mathbb{R}^n : \ \langle x, \theta \rangle \leq \varphi(\theta) \text{ for every } \theta \in \mathbb{S}^{n-1} \right\}.$$

Notice that $\mathbb{W}[\varphi]$ is the largest symmetric convex set M for which $h_M \leq \varphi$.

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Definition

The geometric mean $K^\lambda L^{1-\lambda}$ of two symmetric convex sets K, L in $\mathbb{R}^n$ is the Wulff shape of the function $h_K^\lambda \cdot h_L^{1-\lambda}$. More generally, for $p \in (0, \infty)$ the L^p-average of K and L is defined as

$$\lambda K +_p (1-\lambda) L = \mathbb{W}\big[(\lambda h_K^p + (1-\lambda) h_L)^{\frac{1}{p}}\big].$$

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$$\lambda K +_p (1-\lambda)L = \mathbb{W}\big[(\lambda h_K^p + (1-\lambda)h_L)^{\frac{1}{p}}\big].$$

Notice that $\lambda K +_p (1-\lambda)L \subseteq \lambda K +_q (1-\lambda)L$ for $0 \leq p \leq q$.

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Conjecture (Log-Brunn–Minkowski inequality)

For every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$,

$$\big|K^{\lambda}L^{1-\lambda}\big| \geq |K|^{\lambda}|L|^{1-\lambda}.$$

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In their original paper, Böröczky, Lutwak, Yang and Zhang confirmed the conjecture on the plane.

The log-Brunn–Minkowski inequality self-improves

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Theorem (Saroglou, 2015)

If the log-Brunn–Minkowski conjecture is true in dimension n, then for every even log-concave measure μ on $\mathbb{R}^n$, every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$,

$$\mu\big(K^\lambda L^{1-\lambda}\big) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}. \tag{$*$}$$

The log-Brunn–Minkowski inequality self-improves

Moreover, the log-Brunn–Minkowski inequality for the measure μ implies all L^p Brunn–Minkowski inequalities for μ.

Proposition (Livshyts, Marsiglietti, Nayar and Zvavitch, 2017)

If a symmetric log-concave measure μ satisfies $(*)$, then for every $p \in (0, \infty)$, every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0, 1)$,

$$(**) \qquad \mu\big(\lambda K +_p (1-\lambda)L\big)^{\frac{p}{n}} \geq \lambda \mu(K)^{\frac{p}{n}} + (1-\lambda)\mu(L)^{\frac{p}{n}}.$$

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Remark. Taking $p = 1$, $K = B(0,1)$ and $L = \{x\}$ and (for instance) $\mu = \gamma_n$ we see that, as $x \to \infty$, $(**)$ cannot hold without the assumption that the convex sets are symmetric.

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Using ideas of Cordero-Erausquin, Fradelizi and Maurey (2004), it is not hard to show the following special case of the conjecture.

Proposition

For every unconditional convex sets K, L in $\mathbb{R}^n$, every unconditional measure μ on $\mathbb{R}^n$ and every $\lambda, p \in (0,1)$, we have

$$\mu\big(\lambda K +_p (1-\lambda)L\big)^{\frac{p}{n}} \geq \lambda\mu(K)^{\frac{p}{n}} + (1-\lambda)\mu(L)^{\frac{p}{n}}.$$

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Böröczky and Kalantzopoulos (2020) relaxed the unconditionality assumption to the weaker property that K and L are symmetric with respect to any (not necessarily pairwise orthogonal) n hyperplanes.

A special case: the B-theorem

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Choosing K and L to be dilates of each other in $(*)$, the log-BM conjecture implies that for every symmetric convex set K and $a, b > 0$,

$$(\dagger) \qquad \mu\big(a^\lambda b^{1-\lambda} K\big) \geq \mu(aK)^\lambda \mu(bK)^{1-\lambda}$$

for every symmetric log-concave measure μ. In the case of the standard Gaussian measure γ_n, inequality $(\dagger)$ was postulated by Banaszczyk in the 90's and became known as the B-conjecture.

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Theorem (Cordero-Erausquin, Fradelizi and Maurey, 2004)

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Theorem (Cordero-Erausquin, Fradelizi and Maurey, 2004)

Inequality $(\dagger)$ *holds true for the standard Gaussian measure* γ_n.

Moreover, $(\dagger)$ has been confirmed for a family of *Gaussian mixtures* (E., Nayar and Tkocz, 2018) which includes the symmetric exponential measure, i.e. the measure $\mathrm{d}\nu_n(x) = \frac{1}{2^n} e^{-\sum_{i=1}^n |x_i|} \mathrm{d}x$.

A special case: the Gardner–Zvavitch problem

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The $p = 1$ case of $(**)$ asserts that for every symmetric convex sets K, L and every $\lambda \in (0, 1)$,

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Theorem (E. and Moschidis, 2020)

For every symmetric convex sets K, L and every $\lambda \in (0,1)$,

$$\gamma_n\big(\lambda K + (1-\lambda)L\big)^{\frac{1}{n}} \geq \lambda \gamma_n(K)^{\frac{1}{n}} + (1-\lambda)\gamma_n(L)^{\frac{1}{n}}.$$

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Theorem (Kolesnikov–E. Milman, 2017; Chen–Huang–Li–Liu, 2018)

There exists a universal constant $c \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and every $p \geq 1 - \frac{c}{n^{1+o(1)}}$ the following holds. For every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$, we have

$$\left|\lambda K +_p (1-\lambda)L\right|^{\frac{p}{n}} \geq \lambda |K|^{\frac{p}{n}} + (1-\lambda)|L|^{\frac{p}{n}}.$$

The local approach to Brunn–Minkowski inequalities

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Let μ be a symmetric log-concave measure on $\mathbb{R}^n$. We would like to prove that for every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$,

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This is equivalent (...) to the fact that for every K, L, the function

$$(***) \qquad [0,1] \ni \lambda \longmapsto \mu\big(\lambda K +_p (1-\lambda)L\big)^{\frac{p}{n}}$$

is concave on $[0,1]$.

The local approach to BM inequalities (continued)

We can and will assume that both K and L have smooth boundaries with strictly positive principal curvatures. Then, given a C^2 even function $h : \mathbb{S}^{n-1} \to \mathbb{R}$ and $\varepsilon \in (-1, 1)$ small enough, we will write $K +_p \varepsilon \cdot h$ for the Wulff shape of $(h_K^p + \varepsilon h^p)^{1/p}$.

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Problem (Local L^p Brunn–Minkowski inequality)

Let μ be a symmetric log-concave measure on $\mathbb{R}^n$. Is it true that for every strictly smooth symmetric convex set K in $\mathbb{R}^n$ and every C^2 even function $h : \mathbb{S}^{n-1} \to \mathbb{R}$,

$$(\diamond) \qquad \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0} \mu\big(K +_p \varepsilon \cdot h\big)^{\frac{p}{n}} \leq 0?$$

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Let $M(\varepsilon) = \mu(K +_p \varepsilon \cdot h)$. Then, inequality $(\diamond)$ is equivalent to

$$(\diamond\diamond) \qquad M(0)M''(0) \leq \frac{n-p}{n} M'(0)^2.$$

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• Cheng, Huang, Li and Liu (2018) used Schauder estimates to prove uniqueness in the L^p Minkowski problem which by previous work of Böröczky, Lutwak, Yang and Zhang (2012) shows that the local L^p Brunn–Minkowski inequality implies its global counterpart.

The local approach to BM inequalities (continued)

The contributions in the aforementioned main theorem are as follows:

• Kolesnikov and E. Milman (2017) confirmed the local L^{p_n} Brunn–Minkowski inequality for the Lebesgue measure, where $p_n = 1 - \frac{c}{n^{1+o(1)}}$.

• Cheng, Huang, Li and Liu (2018) used Schauder estimates to prove uniqueness in the L^p Minkowski problem which by previous work of Böröczky, Lutwak, Yang and Zhang (2012) shows that the local L^p Brunn–Minkowski inequality implies its global counterpart.

• A different proof of this implication was given by Putterman (2019).

The local approach to BM inequalities (continued)

The local approach to BM inequalities (continued)

Proposition (Kolesnikov and E. Milman, 2017)

Let $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$ be a log-concave measure. For $x \in \partial K$, let n_x be the unit normal of ∂K at x and define $f : \partial K \to \mathbb{R}$ by $f(x) = \frac{h^p(n_x)}{p h_K^{p-1}(n_x)}$. Then

$$M'(0) = \int_{\partial K} f(x)\ \mathrm{d}\mu_{\partial K}(x);$$

$$M''(0) = \int_{\partial K} H_x f(x)^2 - \langle II^{-1}(x)\nabla_{\partial K} f(x), \nabla_{\partial K} f(x)\rangle\ \mathrm{d}\mu_{\partial K}(x) + (1-p)\int_{\partial K} \frac{f(x)^2}{\langle x, n_x\rangle}\ \mathrm{d}\mu_{\partial K}(x),$$

where $\mu_{\partial K}$ is the restriction of μ on ∂K, II is the second fundamental form of ∂K and H_x is the weighted mean curvature at x, i.e.

$$H_x = \mathrm{tr}(II(x)) - \langle \nabla V(x), n_x\rangle.$$

The local approach to BM inequalities (continued)

So, we have to show that for every even $f : \partial K \to \mathbb{R}$,

$$\int_{\partial K} \underbrace{Hf^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle}_{\Phi(\partial K, V, f, \nabla f)} \, \mathrm{d}\mu_{\partial K} + (1-p) \int_{\partial K} \frac{f(x)^2}{\langle x, n_x \rangle} \, \mathrm{d}\mu_{\partial K}(x)$$

$$\leq \frac{n-p}{n\mu(K)} \left(\int_{\partial K} f(x) \, \mathrm{d}\mu_{\partial K}(x) \right)^2 .$$

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$$\leq \frac{n-p}{n\mu(K)} \left(\int_{\partial K} f(x) \,\mathrm{d}\mu_{\partial K}(x) \right)^2 .$$

Remark. This inequality with $p = 1$ and μ being the Lebesgue measure, first appeared in work of Colesanti (2008).

The Reilly formula

Denote by $\mathscr{L}_\mu$ the elliptic operator associated to μ, whose action on a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{L}_\mu u = \Delta u - \langle \nabla V, \nabla u \rangle$.

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Theorem (Reilly formula)

For every smooth $u : K \to \mathbb{R}$,

$$\int_K (\mathscr{L}_\mu u)^2 \, \mathrm{d}\mu = \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \, \mathrm{d}\mu + \int_{\partial K} \Psi \, \mathrm{d}\mu_{\partial K},$$

for some explicit $\Psi = \Psi(\partial K, V, u, \nabla u)$.

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for some explicit $\Psi = \Psi(\partial K, V, u, \nabla u)$.

Crucial observation! If $f : \partial K \to \mathbb{R}$ is the Neumann boundary data of u, i.e. $f(x) = \langle \nabla u(x), n_x \rangle$ for $x \in \partial K$, then

$$\Phi(\partial K, V, f, \nabla f) \leq \Psi(\partial K, V, u, \nabla u).$$

The Reilly formula (continued)

Conclusion. To derive an L^p Brunn–Minkowski inequality for μ it suffices to show that for every symmetric K and *for every* even $f : \partial K \to \mathbb{R}$ *there exists* a $u : K \to \mathbb{R}$ with Neumann boundary data f, such that

$$\int_K (\mathscr{L}_\mu u)^2 - \|\nabla^2 u\|_{\mathrm{HS}}^2 - \langle \nabla^2 V \nabla u, \nabla u\rangle \, \mathrm{d}\mu + (1-p)\int_{\partial K} \frac{f(x)^2}{\langle x, n_x\rangle} \, \mathrm{d}\mu_{\partial K}(x) \leq \frac{n-p}{n\mu(K)} \left(\int_{\partial K} f(x) \, \mathrm{d}\mu_{\partial K}(x) \right)^2.$$

Back to the Lebesgue measure

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Remark. The local L^p Brunn–Minkowski inequality

$$(\diamondsuit) \qquad \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0} m\big(K +_p \varepsilon \cdot h\big)^{\frac{p}{n}} \le 0?$$

for the Lebesgue measure m is invariant under transformations of the form $K \mapsto sK$ where $s \in \mathbb{R}_+$. At the level of the main inequality above, this means that in the case of the Lebesgue measure, the desired conclusion is invariant under transformations of the form $f \mapsto f(x) + t\langle x, n_x\rangle$.

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Therefore, we can assume without loss of generality that $\int_{\partial K} f \, \mathrm{d}m_{\partial K} = 0$. Since for every such f there exists $u : K \to \mathbb{R}$ such that $\Delta u = 0$ and $\langle \nabla u(x), n_x\rangle = f(x)$ on ∂K we have the following sufficient condition.

Back to the Lebesgue measure (continued)

Corollary

Suppose that there exists $p \in [0,1)$ such that for any symmetric convex set K in $\mathbb{R}^n$, any even harmonic function $u : K \to \mathbb{R}$ satisfies

$$(\square) \qquad \int_{\partial K} \frac{\langle \nabla u(x), n_x \rangle^2}{\langle x, n_x \rangle} \,\mathrm{d}m_{\partial K}(x) \leq \frac{1}{1-p} \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 \,\mathrm{d}m.$$

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Back to the Lebesgue measure (continued)

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Remark. Taking $p = 1$, we deduce the classical Brunn–Minkowski inequality.

Back to the Lebesgue measure (continued)

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Unfortunately even for $n = 2$, there does not exist a uniform $p < 1$ such that ($\Box$) is satisfied. The reason for that is that, unlike the local L^p Brunn–Minkowski inequality ($\diamond$), the stronger inequality ($\Box$) is not invariant under linear transformations of the convex set K.

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Theorem (Kolesnikov and E. Milman)

There exists a universal constant $C \in (0, \infty)$ satisfying the following. For any symmetric convex set K there exists an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that any even function $u : TK \to \mathbb{R}$,

$$\int_{\partial TK} \frac{|\nabla u(x)|^2}{\langle x, n_x \rangle} \, \mathrm{d}m_{\partial TK}(x) \le C n^{1+o(1)} \int_{TK} \|\nabla^2 u\|_{\mathrm{HS}}^2 \, \mathrm{d}m.$$

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Observe that they control the gradient $|\nabla u(x)|^2$ instead of $\langle \nabla u(x), n_x \rangle^2$ and they do not assume that u is harmonic.

The solution of the Gardner–Zvavitch problem

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The lack of homogeneity of γ_n does not allow us to assume wlog that $\int_{\partial K} f \, \mathrm{d}\gamma_{\partial K} = 0$. In fact, this case is easily treatable (...) and thus, by rescaling, we can assume that $\int_{\partial K} f \, \mathrm{d}\gamma_{\partial K} = \gamma_n(K)$.

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The solution of the Gardner–Zvavitch problem (continued)

Theorem (E. and Moschidis, 2020)

For every $n \in \mathbb{N}$ and every symmetric convex set K in $\mathbb{R}^n$, every smooth symmetric function $u : K \to \mathbb{R}$ with $\mathscr{L} u = 1$ on K, satisfies

$$(\heartsuit\heartsuit) \qquad \int \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \, \mathrm{d}\gamma_K \geq \frac{1}{n},$$

where γ_K is the normalized Gaussian probability measure on K.

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where γ_K is the normalized Gaussian probability measure on K.

For a matrix A, denote by $\widehat{A}$ its traceless part, $\widehat{A} = A - \frac{\mathrm{tr}(A)}{n}\mathrm{Id}$. Then,

$$\|A\|_{\mathrm{HS}}^2 = \|\widehat{A}\|_{\mathrm{HS}}^2 + \frac{(\mathrm{tr}A)^2}{n}.$$

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$$\|A\|_{\mathrm{HS}}^2 = \|\widehat{A}\|_{\mathrm{HS}}^2 + \frac{(\mathrm{tr} A)^2}{n}.$$

In particular, if $\widehat{\nabla}^2 u$ is the traceless part of $\nabla^2 u$, we have

$$\|\nabla^2 u\|_{\mathrm{HS}}^2 = \|\widehat{\nabla}^2 u\|_{\mathrm{HS}}^2 + \frac{(\Delta u)^2}{n}.$$

The solution of the Gardner–Zvavitch problem (continued)

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Notice that

$$\|\widehat{\nabla}^2 u\|_{\mathrm{HS}}^2 = \|\widehat{\nabla}^2 (u - r)\|_{\mathrm{HS}}^2,$$

for every $r \in \mathrm{Ker}(\widehat{\nabla}^2)$, in particular $r(x) = \frac{|x|^2}{2n}$.

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for every $r \in \mathrm{Ker}(\widehat{\nabla}^2)$, in particular $r(x) = \frac{|x|^2}{2n}$. Then, as before

$$\|\widehat{\nabla}^2(u-r)\|_{\mathrm{HS}}^2 = \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 - \frac{(\Delta(u-r))^2}{n} = \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 - \frac{(\Delta u - 1)^2}{n}.$$

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Combining these identities and using the equation $\mathscr{L} u = 1$,

$$\begin{aligned} \|\nabla^2 u(x)\|_{\mathrm{HS}}^2 &= \|\nabla^2 (u-r)(x)\|_{\mathrm{HS}}^2 + \frac{2}{n}\Delta u(x) - \frac{1}{n} \\ &= \|\nabla^2 (u-r)(x)\|_{\mathrm{HS}}^2 + \frac{2}{n}\sum_{i=1}^n x_i \partial_i u(x) + \frac{1}{n}. \end{aligned}$$

The Brascamp–Lieb inequality

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Theorem (Brascamp–Lieb, 1976)

Let $\beta \in (0,\infty)$ and $V : \mathbb{R}^n \to \mathbb{R}$ be such that $\nabla^2 V \geq \beta \mathsf{Id}$. Then, if $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$, every smooth function $h : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\mathrm{Var}_\mu h := \int h^2 \,\mathrm{d}\mu - \left(\int h \,\mathrm{d}\mu\right)^2 \leq \frac{1}{\beta}\int |\nabla h|^2 \,\mathrm{d}\mu.$$

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In particular, since each $\partial_i(u-r)$ is odd and K is symmetric, we have

$$\sum_{j=1}^n \int \left(\partial_i\partial_j(u-r)\right)^2 \mathrm{d}\gamma_K \geq \mathrm{Var}_{\gamma_K}\left(\partial_i(u-r)\right) = \int \left(\partial_i(u-r)\right)^2 \mathrm{d}\gamma_K.$$

The solution of the Gardner–Zvavitch problem (continued)

Adding up, we get

$$\begin{aligned}\int \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 \,\mathrm{d}\gamma_K &\geq \int_K |\nabla(u-r)|^2 \,\mathrm{d}\gamma_K \\ &= \int_K |\nabla u(x)|^2 - \frac{2}{n}\sum_{i=1}^n x_i \partial_i u(x) + \frac{|x|^2}{n^2} \,\mathrm{d}\gamma_K(x).\end{aligned}$$

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$$= \int_K |\nabla u(x)|^2 - \frac{2}{n}\sum_{i=1}^n x_i \partial_i u(x) + \frac{|x|^2}{n^2} \,\mathrm{d}\gamma_K(x).$$

Putting everything together,

$$\int \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \,\mathrm{d}\gamma_K \geq \int 2|\nabla u(x)|^2 + \frac{|x|^2}{n^2} + \frac{1}{n} \,\mathrm{d}\gamma_K(x)$$

and the proof is complete. $\square$

Thank you!