A novel and simple approach to normalised solutions to Schrödinger equations

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Schrödinger-type equations model a lot of natural phenomena and their solutions have interesting and important properties. This gives rise to the search for *normalised solutions*, i.e., when the L^2-norm is prescribed. We propose a simple and novel approach, based on minimisation arguments, to find a solution $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ to the problem

$$\begin{cases} -\Delta u + \lambda u = f(u) \\ \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = \rho^2, \end{cases} \tag{1}$$

where $N \geq 1$, $\rho > 0$ is given a priori, and $f \in \mathcal{C}(\mathbb{R})$ satisfies suitable assumptions. Weak solutions of (1) can be looked for as critical points of a certain functional of class $\mathcal{C}^1$, known as the *energy functional*, restricted to the sphere $\left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \|u\|_2 = \rho \right\}$. Three cases can be distinguished: when the energy restricted to the sphere is bounded below for all, some, or no values of ρ, which are known in the literature as the *mass-subcritical*, *mass-critical*, and *mass-superctitical* cases respectively and depend on the behaviour of f with respect to the *mass-critical* exponent $2 + \frac{4}{N} \in \left]2, 2^*\right[$ (where $2^* = +\infty$ if $N \in \{1, 2\}$ and $2^* = \frac{2N}{N-2}$ otherwise).

In my first lecture, I will focus on the mass-subcritical and mass-critical cases, where a *least-energy* solution will be obtained via *direct minimisation*; here, it suffices to assume that f has suitable behaviours at the origin and infinity. In my second lecture, I will focus on the mass-supercritical case, where a least-energy solution will be obtained via *constrained minimisation*; here, we will also need global conditions that relate the quantities $f(t)t$ and $F(t)$, where $F(t) = \int_0^t f(s) \, \mathrm{d}s$.

If time allows us, I will mention systems, non-linearities with Sobolev-critical growth at infinity, and evolution equations.

References

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