ENTIRE SOLUTIONS TO NONCONVEX VARIATIONAL ELLIPTIC SYSTEMS IN THE PRESENCE OF A FINITE SYMMETRY GROUP

NICHOLAS D. ALIKAKOS AND GIORGIO FUSCO

Introduction

This paper is partly based on a lecture delivered by one of the authors at the workshop "Singularities in nonlinear evolution phenomena and applications" held at the Centro di Ricerca Matematica Ennio De Giorgi on May 26–29, 2008, organized by Sisto Baldo, Matteo Novaga, and Giandomenico Orlandi. The purpose of that lecture was to describe the results in Alikakos and Fusco [5]. The first part of the present paper is an expanded version of that lecture while the second part contains new results, together with proofs, that relate to or complement the paper [5].

Part I

In [5] the object of study is the system

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

where $W : \mathbb{R}^n \to \mathbb{R}$ and the gradient $W_u := \left(\frac{\partial W}{\partial u_1}, \cdots, \frac{\partial W}{\partial u_n}\right)^\top$; the system above is the Euler–Lagrange equation corresponding to the *free energy functional*

$$J(u) = \int_{\mathbb{R}^n} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x. \tag{2}$$

One of the obstructions in the study of (1) is that the free energy is infinite for the solutions we are interested in, for dimensions $n \geq 2$, due to a simple geometric reason that we will explain later. Before going any further, we introduce the hypotheses on the potential W, along with explanations.

Hypothesis 1 (N nondegenerate global minima)**.** *The potential W is of class C^2, satisfying $W = 0$ on $A = \{a_1, \ldots, a_N\}$ and $W > 0$ in $\mathbb{R}^n \setminus A$. Furthermore, $\partial^2 W(u) \geq c^2 \mathrm{Id}$ for $|u - a_i| \leq r_0$, with $r_0 > 0$ fixed, and for $i = 1, \ldots, N$.*

The essence of Hypothesis 1 is that W is nonconvex and that we allow for several *global* minima.

In Figure 1 we show examples of potentials for which (1) has been studied in the past. In the first graph we see a double-well potential defined over $\mathbb{R}$, hence $n = 1$. The kind of solution we construct, in this case, reduces to the well-known heteroclinic connection: the solution to the ordinary differential equation $u'' - W_u(u) = 0$ that connects the phases, that is,

$$\lim_{u(x)\to-\infty} = a_1 \text{ and } \lim_{u(x)\to+\infty} = a_2.$$

This is really textbook material (see, for example, [6, Ch. 2, §12.8]) as in this case (1) becomes a simple Hamiltonian system. In the second we show a double-well

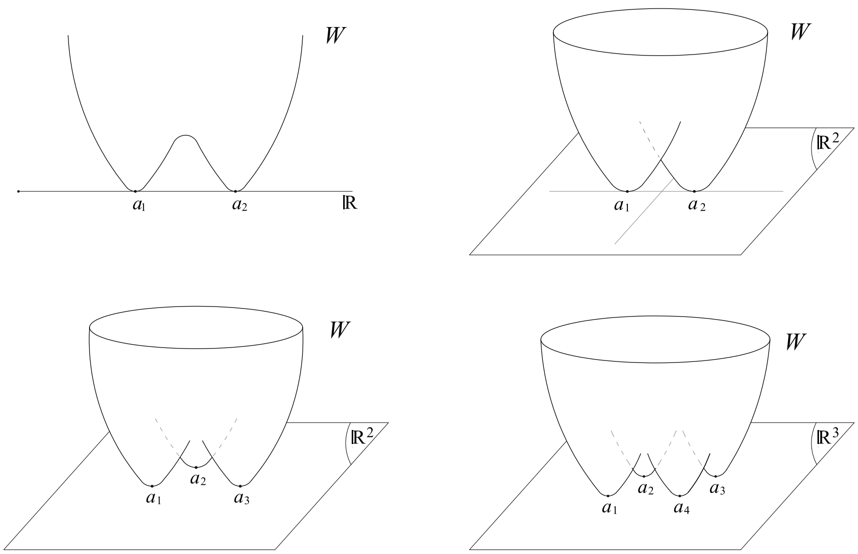

FIGURE 1.

potential over $\mathbb{R}^2$ ($n = 2$). Dang, Fife, and Peletier [11] constructed a saddle *scalar* solution for such a potential. Later, Alama, Bronsard, and Gui [1] gave a genuine vector extension of this. Saddle solutions, although related, are not included in the class of solutions we are discussing. In the third we show a triple-well potential over $\mathbb{R}^2$ ($n = 2$). Bronsard, Gui, and Schatzman [9] constructed entire solutions to (1) for a class of triple-well potentials, known as triple-junction solutions. This is a significant example of the type of solution that concerns us here. Finally in the fourth we show a quadruple-well potential over $\mathbb{R}^3$ ($n = 3$). For a class of such potentials, Gui and Schatzman [22] constructed an entire solution to (1) known as the quadruple-junction solution. This is a three-dimensional analog to the triple-junction solution and provides another significant example of the type of solution we study. Triple-junction and quadruple-junction solutions have additional significance of their own and we will comment on them later.

In all these works (for $n \geq 2$) the potentials W have been assumed to have certain symmetries. This takes us to the next hypothesis.

Hypothesis 2 (Symmetry)**.** *The potential W is invariant under a finite reflection group G acting on $\mathbb{R}^n$ (Coxeter group), that is,*

$$W(gu) = W(u), \text{ for all } g \in G \text{ and } u \in \mathbb{R}^n. \tag{3}$$

The symmetry of W allows for *equivariant* solutions to (1), that is, solutions satisfying

$$u(gx) = gu(x), \text{ for all } g \in G \text{ and } x \in \mathbb{R}^n. \tag{4}$$

The simplest reflection group over $\mathbb{R}^2$ is $\mathcal{H}_2^2$. It contains four elements: the reflections with respect to the u_1 and u_2 axes, the rotation by π, and the identity. These are exactly the symmetries of a rectangle. An $\mathcal{H}_2^2$-equivariant solution by (4) satisfies

$$\begin{aligned}\big(u_1(-x_1,x_2),u_2(-x_1,x_2)\big) &= \big(-u_1(x_1,x_2),u_2(x_1,x_2)\big)\\ \big(u_1(x_1,-x_2),u_2(x_1,-x_2)\big) &= \big(u_1(x_1,x_2),-u_2(x_1,x_2)\big)\\ \big(u_1(-x_1,-x_2),u_2(-x_1,-x_2)\big) &= \big(-u_1(x_1,x_2),-u_2(x_1,x_2)\big)\end{aligned}$$

The triple-well potential mentioned above is symmetric under $\mathcal{H}_2^3$, the group of symmetries of the equilateral triangle that contains six elements.

Finite reflection (point) groups acting on $\mathbb{R}^n$ are subgroups of linear transformations that are isometries, that is, subgroups of $\mathcal{O}(\mathbb{R}^n)$ which are generated by reflections. The relevance of the orthogonal group is due to the Laplacian in (1). In the present work we focus on *point groups*, that is, groups of transformations having a fixed point. The translation invariance of the Laplacian leads naturally to a different class of groups, the *discrete groups*, which are relevant in the study of (1) but are not considered here. If $T \in \mathcal{O}(\mathbb{R}^n)$, then $\det T = \pm 1$. If $\det T = 1$, then T is called a *rotation*. The reflection with respect to the hyperplane $\{u \in \mathbb{R}^n \mid \langle u, r\rangle = 0\}$, where $\langle \cdot,\cdot\rangle$ is the Euclidean inner product, is defined by setting

$$S_r u = u - \frac{2\langle u, r\rangle}{\langle r, r\rangle}. \tag{5}$$

It is well known that a finite reflection group over $\mathbb{R}^n$ is generated by n reflections $S_{r_1}, \ldots, S_{r_n}$, where $\{r_1, \ldots, r_n\}$ is a set of linearly independent vectors called *fundamental roots* (see [21]). Needless to say, reflection groups also contain rotations.

The quadruple-well potential mentioned above is assumed to be invariant under the symmetries of the regular tetrahedron. The regular tetrahedron is one of the five Platonic solids in $\mathbb{R}^3$ (regular convex polyhedra) all of which correspond to finite reflection groups. Algebraically, some of these groups coincide; it turns out that there are three algebraically distinct convex polyhedra: the tetrahedron, the cube, and the icosahedron, with orders as follows,

$$|\mathcal{T}^*| = 24, \quad |\mathcal{W}^*| = 48, \quad |\mathcal{J}^*| = 120.$$

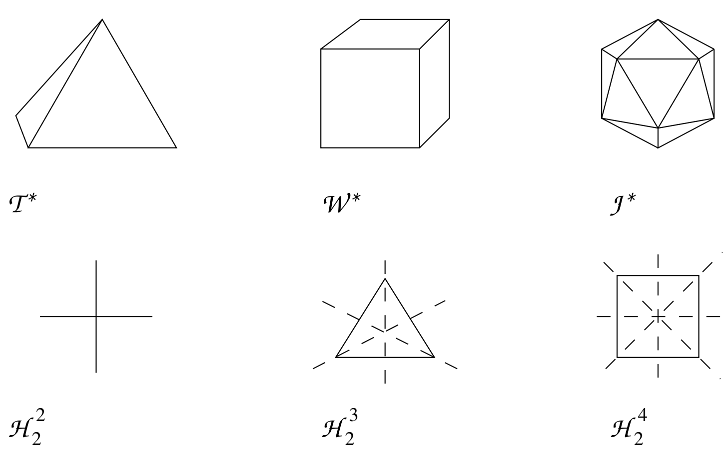

FIGURE 2.

At this point we would like to open a big parenthesis and comment on the relevance of the solutions we discuss to *singularities*, the theme of this workshop. First, concerning phase transitions, we note that for describing coexistence of three of more phases ($N \geq 3$), it is easy to see that a vector-order parameter is necesary.

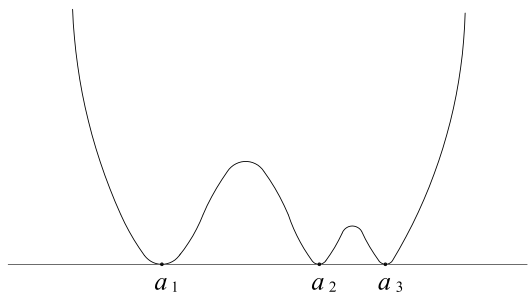

FIGURE 3.

In Figure 3 we show a triple well over $\mathbb{R}$; clearly, in this case, the problem

$$u'' - W_u(u) = 0, \text{ where } u(-\infty) = a_1, \; u(\infty) = a_3, \tag{6}$$

has no solution, that is, there is no connection between a_1 and a_3. Therefore, for coexistence of more than two phases, it is more appropriate for the triple-well potential to be defined over $\mathbb{R}^2$, with minima a_1, a_2, a_3, at the vertices of a triangle, representing the three phases (cf. [32, §1.7]). Baldo [7] has studied the minimization problem

$$\min E_\varepsilon(u) = \int_\Omega \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right\} \mathrm{d}x, \text{ with } \int_\Omega u(x)\,\mathrm{d}x = \boldsymbol{m}, \tag{7}$$

for W as in Hypothesis 1 and $\Omega \subset \mathbb{R}^n$, and has linked the minimizers u_ε of (7), as $\varepsilon \to 0$, to the partitioning of the domain Ω into sets $S_1, \ldots, S_N$, whose boundaries satisfy the minimality condition

$$\min \sum_{i,j=1}^{N} d(a_i, a_j) \mathcal{H}_{n-1}(\partial^* S_i \cap \partial^* S_j), \tag{8}$$

and the constraint

$$\sum_{j=1}^{N} |S_j| a_j = \boldsymbol{m}\,. \tag{9}$$

Here, $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure of the (reduced) boundary of each S_i. The numbers $d(a_i, a_j)$ represent the 'energy' of the transition between the phases a_i and a_j. In case there is a connection U_{ij} between a_i and a_j, then $d(a_i, a_j)$ is explicitly given in terms of its *action*,

$$d(a_i, a_j) = \int_{\mathbb{R}} \frac{1}{2} |\dot{U}_{ij}(s)|^2 + W(U_{ij}(s))\,\mathrm{d}s, \tag{10}$$

where

$$\ddot{U}_{ij} - W_u(U_{ij}) = 0, \text{ with } U_{ij}(-\infty) = a_i, \; U_{ij}(\infty) = a_j. \tag{11}$$

As it is easily seen, the minimizer u_ε of (7) satisfies the Euler–Lagrange equation

$$\begin{cases} \varepsilon^2 \Delta u_\varepsilon - W_u(u_\varepsilon) = \sigma_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \boldsymbol{n}} = 0, \text{ on } \partial\Omega. \end{cases} \tag{12}$$

Here, σ_ε is a Lagrange multiplier which turns out to converge to zero as $\varepsilon \to 0$ (see [26], [3]). On the basis of Baldo's work plus a lot of well-known related evidence, both formal and rigorous, one expects that at the ε-level, $0 < \varepsilon \ll 1$, the boundaries of the sets S_i are replaced by thin zones of thickness ε and that in each S_i and away from these zones, the solution U_ε is approximately constant and equal to some a_j. If now we rescale space and set $u(y) = u_\varepsilon(x_0 + \varepsilon y)$, it is natural to expect that in the limit, u converges to an entire solution of (1). Depending on the choice of x_0, the entire solution has different complexity. For example, if x_0 is taken in the interior of S_j, then $u \equiv a_j$. If x_0 is taken on the boundary of S_j separating S_j from S_i and away from the junctions, then u is expected to depend just on a single variable s that measures the distance from the interface and connects a_j to a_i, that is, u is the connection U_{ij} (cf. (11)). If x_0 is taken to be at the junction of three or more sets ∂S_j, then u should connect three or more different phases and should depend on two or more variables $s_i = \lambda \boldsymbol{n}_i$, with $\boldsymbol{n}_i$ a unit vector such that

$$\lim_{\lambda \to \infty} u(\lambda \boldsymbol{n}_i) = a_i, \text{ for } i = 1, 2, \ldots \tag{13}$$

These are the entire solutions considered in this paper. In case all transition energies are equal, then the functional in (8) simplifies and the boundaries of the partition form a system of surfaces of constant mean curvature. Minimal surfaces are a special case where the mean curvature is zero.

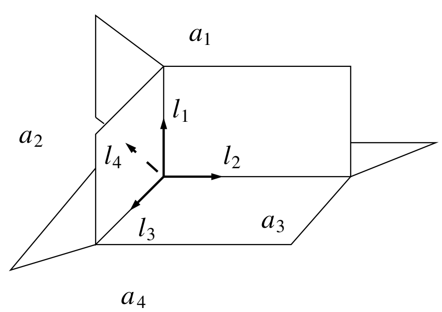

FIGURE 4.

In Figure 4 we show a quadruple-junction solution. In the context of minimal surfaces, such systems of surfaces meet each other along free-boundary curves calles 'liquid edges' and, in turn, liquid edges meet at 'supersingular' points. Each such point is the meeting oint of exactly four liquid edges, bringing together six surfaces. Any two adjacent edges form an angle ϕ, with $\cos\phi = -\frac{1}{3}$ (*Maraldi angle*). We refer to Dierkes *et al.* [12, §4.10.7] and Taylor [39]. We note that $x = 0$ in our solutions corresponds to this supersingular point.

Triple-junction solutions over the plane and quadruple-junction solutions over space have the following special significance. If the minimizers of (7) are not required to lie in a certain symmetry class, then one expects that for potentials defined over the plane and with three or more global minima, generically, there will be only triple-junction solutions. Similarly in space, for potentials with four or more global minima, one expects that, generically, the minimizers will form triple-junction configurations coming together and forming quadruple junctions.

The gradient flow in H^{-1} associated to (7) is the *vector Cahn–Hilliard equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta\Big(\varepsilon\Delta u - \dfrac{1}{\varepsilon}W_u(u)\Big), \text{ for } u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \dfrac{\partial \Delta u}{\partial \boldsymbol{n}} = 0, \text{ on } \partial\Omega. \end{cases} \tag{14}$$

Its sharp-interface limit as $\varepsilon \to 0$ is the *vector Mullins–Sekerka* free-boundary problem, a prominent gradient flow associated to (8), (9); its scalar counterpart was studied in [31] and [2]. The function

$$\mu_\varepsilon = \varepsilon\Delta u - \frac{1}{\varepsilon}W_u(u)$$

tends to a limit μ_0 as $\varepsilon \to 0$ which together with the free boundary $\Gamma(t)$ satisfies the following quasi-static problem

$$\begin{cases} \Delta\mu_0 = 0, \text{ for } x \in \Gamma_{ij}(t) := \partial S_i \cap S_j, \\ \mu_0 = H_{ij}\Big(\displaystyle\int_{-\infty}^{\infty} |\dot{U}_{ij}(s)|^2\,\mathrm{d}s\Big)\dfrac{a_i - a_j}{|a_i - a_j|}, \text{ on } \Gamma_{ij}(t), \\ (a_i - a_j)d_t^{ij}(t) = \Big[\!\!\Big[\dfrac{\partial \mu_0}{\partial \nu}\Big]\!\!\Big], \text{ on } \Gamma_{ij}(t) \subset \Omega \subset \mathbb{R}^n, \\ \dfrac{\partial \mu_0}{\partial \boldsymbol{n}} = 0, \text{ on } \partial\Omega. \end{cases} \tag{15}$$

Here, $\mu_0 = \mu_0(x,t)$ is a vector, $H_{ij}(x,t)$ is the mean curvature of $\Gamma_{ij}(t)$, $d_t^{ij}(t)$ is the distance between x and Γij, $[\![\cdot]\!]$ is the jump of the derivative of μ_0 in the normal direction to $\Gamma_{ij}(t)$, and the Laplacian in x variables.

Problem (15) has to be accompanied with *Plateau angle conditions* along any liquid edge where three surfaces intersect (see Figure 4):

$$\frac{\sin\theta_1}{d(a_2,a_3)} = \frac{\sin\theta_2}{d(a_1,a_3)} = \frac{\sin\theta_3}{d(a_1,a_2)}, \tag{16}$$

that is, the angles are fixed along the evolution. In the simplest case when all transition energies are equal (the case of a symmetric potential, for example), the angles are all equal to $\frac{2\pi}{3}$.

Formally, it can be checked that the weighted perimeter

$$\sum_{i,j=1}^{N} d(a_i,a_j)\mathcal{H}(\partial^* S_i \cap \partial^* S_j)$$

is decreasing and that the volume $|S_j(t)|$ of each phase is conserved along the evolution (15), (16).

We also note that by the Γ-convergence results in [7],

$$\begin{aligned}\frac{1}{\varepsilon}E_\varepsilon(u_\varepsilon) &= \int_\Omega \Big\{\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big\}\,\mathrm{d}x \\ &\approx \sum_{i,j=1}^{N} d(a_i,a_j)\mathcal{H}_{n-1}(\partial^* S_i \cap \partial^* S_j), \text{ as } \varepsilon \to 0, \\ &=: \operatorname{Per} u_0. \end{aligned} \tag{17}$$

On the other hand, for $y = x/\varepsilon$,

$$\frac{1}{\varepsilon}\int_\Omega \Big\{\frac{\varepsilon^2}{2}|\nabla u_\varepsilon|^2 + W(u_\varepsilon)\Big\}\,\mathrm{d}x = \int_{\Omega_\varepsilon} \Big\{\frac{1}{2}|\nabla_y u_\varepsilon|^2 + W(u_\varepsilon)\Big\}\varepsilon^{n-1}\,\mathrm{d}y.$$

Thus,

$$\begin{aligned}\int_{\Omega_\varepsilon} \Big\{\frac{1}{2}|\nabla_y u_\varepsilon|^2 + W(u_\varepsilon)\Big\}\,\mathrm{d}y &\approx \frac{1}{\varepsilon^{n-1}}\operatorname{Per} u_0 \quad (\text{as } \varepsilon \to 0) \\ &\approx \frac{1}{|\Omega_\varepsilon|^{\frac{n-1}{n}}}\operatorname{Per} u_0, \end{aligned} \tag{18}$$

where $|\Omega_\varepsilon|$ is the Lebesgue measure of Ω_ε. This computation explains the infinite free energy for $n \geq 2$ mentioned earlier. The analogous sharp-interface problem to (15), (16) for the Allen–Cahn system in two dimensions was introduced and studied in Bronsard and Reitich [10]. Mantegazza, Novaga, and Tortorelli [27] initiated a program for the global (in time) study of networks on the plane. The recent work of Freire [18] addresses the difficulties of the parametric method in dimensions greater or equal than three. Problem (15), (16) is more difficult and corresponding results are known at a formal level (cf. Bronsard, Garcke, and Stoth [8]).

After this long detour we are ready for the next hypothesis that relates the number and location of minima of the potential W to the group G.

Hypothesis 3 (Location and number of global minima)**.** *Let $F \subset \mathbb{R}^n$ be a fundamental region of G. We assume that $\bar{F}$ (the closure of F) contains a single global minimum of W, say a_1, and let* $\operatorname{Stab}(a_1)$ *be the subgroup of G that leaves a_1 fixed. Then,*

$$N := \frac{|G|}{|\operatorname{Stab}(a_1)|}. \tag{19}$$

First we recall a few basic facts about the fundamental region. For $\mathcal{H}_2^2$, a fundamental region is the first quadrant of $\mathbb{R}^2$. Its orbit under the elements of the group consists of disjoint sets whose union, after closure, is $\mathbb{R}^2$. The walls of the fundamental region are the u_1 and u_2 axes which define the fundamental reflections that generate the group. Similarly for $\mathcal{H}_2^3$, a fundamental region is the $\frac{\pi}{3}$ sector in $\mathbb{R}^2$, the walls of which define the fundamental reflections. In Figure 5 we also show a fundamental region for a group $\mathcal{W}^*$ of symmetries of the cube in $\mathbb{R}^3$.

In two dimensions the minimum a_1 can be placed in the interior of $\bar{F}$, on an edge, or on its vertex. For example, for $\mathcal{H}_2^3$ this process will give $N = 6$, $N = 3$, and $N = 1$ respectively. In higher dimensions, we have more options. We can place a_1 in the interior of $\bar{F}$, in the interior of a face, in the interior of an edge, and so on. We have calculated that in the case of the cube $(\pm 1, \pm 1, \pm 1)$ with F the simplex generated by $s_1 = e_1 + e_2 + e_3$, $s_2 = e_2 + e_3$, and $s_3 = e_3$, we have $N = 6$ on the

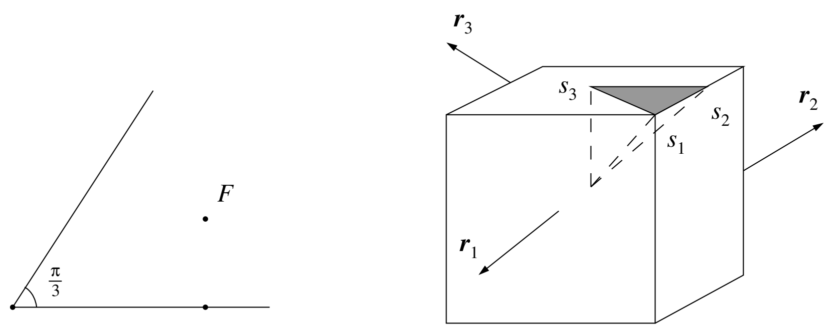

FIGURE 5.

edge s_3, $N = 8$ on the edge s_1, $N = 12$ on the edge s_2, $N = 24$ in the interior of a face, $N = 48$ in the interior of $\bar{F}$, and $N = 1$ on the vertex of F.

The hypotheses so far have been purely geometrical. Our final hypothesis is analytic and is introduced for carrying out the analysis of the problem.

Hypothesis 4 (Q-monotonicity)**.** *Let*

$$D := \mathrm{Int}\big(\textstyle\bigcup_{g\in \mathrm{Stab}(a_1)} g\bar{F}\big). \tag{20}$$

We restrict ourselves to potentials W for which there is a continuous function $Q : \bar{D} \to \mathbb{R}$ with the following properties:

$$Q \textit{ is convex}, \tag{21a}$$

$$Q(u) > 0 \textit{ and } Q_u(u) \neq 0, \ \textit{ on } \bar{D} \setminus \{a_1\}, \tag{21b}$$

$$Q(u + a_1) = |u| + \mathrm{o}(|u|) \ \textit{ as } |u| \to 0, \tag{21c}$$

$$\big\langle Q_u(u), W_u(u)\big\rangle \geq 0, \ \textit{ on } \bar{D} \setminus \{a_1\}. \tag{21d}$$

Before discussing the limitations that such a hypothesis imposes on W, let us explain right away how such a Q helps in the analysis. First, it is clear that understanding the geometry of a vector solution is much harder than for its scalar counterpart. A possible route in handling a vector field is by writing it in polar form and then attempting to control its radial part, which of course is a scalar function. From this point of view, one would write

$$u(x) = a_1 + |u(x) - a_1| \frac{u(x) - a_1}{|u(x) - a_1|}$$

with

$$Q(u(x)) = |u(x) - a_1|, \ \text{for } x \in D. \tag{22}$$

Note that the distance from a_1 is the relevant quantity since we are seeking solutions that connect the minima of W. Next, by computation and utilizing (1),

$$\begin{aligned}\Delta Q(u(x)) &= \mathrm{tr}\left\{(\partial^2 Q)(\nabla u)(\nabla u)^\top\right\} + \big\langle Q_u(u(x)), \Delta u(x)\big\rangle \\ &= \mathrm{tr}\left\{(\partial^2 Q)(\nabla u)(\nabla u)^\top\right\} + \big\langle Q_u(u(x)), W_u(u(x))\big\rangle.\end{aligned}$$

If now it happens that

$$u : D \to D, \quad \text{(positivity)} \tag{23}$$

we can continue the calculation using convexity to get

$$\Delta Q(u(x)) \geq \big\langle Q_u(u(x)), W_u(u(x))\big\rangle \geq 0, \text{ for } x \in D. \tag{24}$$

From this we can deduce global bounds on $Q\big(u(x)\big)$ and so, ultimately on $u(x)$. Insisting on $Q(u) = |u - a_1|$ is unnecessary and very restrictive since all we need by the computation above is convexity. The price, however, for this more general option is that we have to develop a global coordinate system in $\mathbb{R}^n$ in terms of the level sets of Q (see Proposition 2 in Part II). Another crucial element in the calculation above is the positivity of u and we will come back to this point later.

Now we address the restrictions that Q imposes on W. First, for $n = 1$ and odd symmetry, for a double-well potential W with $D = \{u > 0\}$, it is easy to see that Q-monotonicity implies that W is monotone in D along the ray emanating from a_1 and thus, only the graph on the right in Figure 6 satisfies the condition.

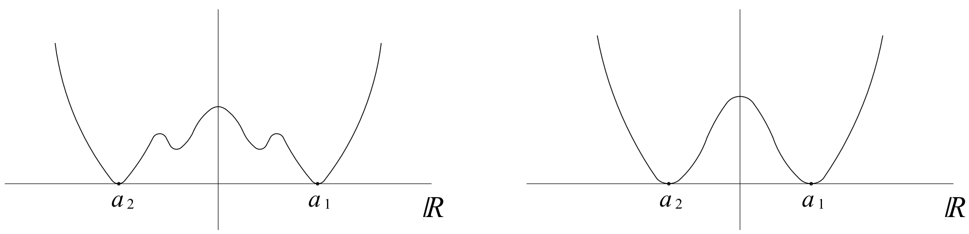

FIGURE 6.

It turns out that in higher space dimensions, Q-monotonicity is less restrictive. In Figure 7 we consider $n = 2$ with $\mathcal{H}_2^2$ symmetry, which is the next simplest case; hence, $D = \{(u_1, u_2) \mid u_1 > 0\}$. In Figure 7 we show two options for W by drawing its level sets: in the first one W has a saddle at the origin and of course a minimum at a_1; in the second one, W has a maximum at the origin. We also draw a typical level set of a possible Q. The level sets of Q should be convex and should intersect the level sets of W in acute angles so that (21d) is satisfied. Certainly, this is not a proof for the existence of a Q but it gives the flavor of what is involved. One can see that the case of the maximum above cannot be handled with the simple choice $Q(u) = |u - a_1|$.

For $G = \mathcal{H}_2^3$ on the plane, F the $\frac{\pi}{3}$ sector, and $a_1 = (1, 0)$, it can be verified that the triple-well potential

$$W(u_1, u_2) = |u|^4 + 2u_1u_2^2 - \frac{2}{3}u_1^3 - |u|^2 + \frac{2}{3} \tag{25}$$

satisfies the Q-monotonicity condition in $D = \big\{(r, \theta) \mid r > 0, \theta \in (-\frac{\pi}{3}, \frac{\pi}{3})\big\}$, with $Q(u) = |u - a_1|$, where $u = (u_1, u_2)$. In Proposition 1 in Part II we verify that the potential in (25) satisfies Hypotheses 1–4.

For $n = 3$, $G = \mathcal{T}^*$, F the simplicial cone generated by

$$\big(\sqrt{\tfrac{2}{3}}, 0, \tfrac{1}{\sqrt{3}}\big),\ \big(0, \sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{3}}\big),\ \big(0, 0, \tfrac{1}{\sqrt{3}}\big),$$

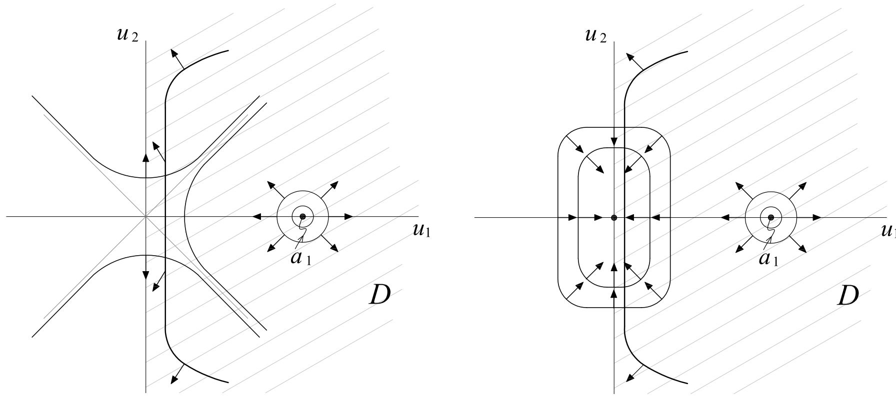

FIGURE 7.

and $a_1 = \left(\sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}}\right)$, we can take as an example the quadruple-well potential

$$W(u_1, u_2, u_3) = |u|^4 - \frac{4}{\sqrt{3}}(u_1^2 - u_2^2)u_3 - \frac{2}{3}|u|^2 + \frac{5}{9},$$

with $Q(u) = |u - a_1|$, where $u = (u_1, u_2, u_3)$, and D the simplicial cone generated by

$$\left(0, \sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{3}}\right), \ \left(0, -\sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{3}}\right), \ \left(\sqrt{\tfrac{2}{3}}, 0, -\tfrac{1}{\sqrt{3}}\right).$$

Finally, we give an example in $\mathbb{R}^n$, with G the reflection group generated by the coordinate planes, F the simplicial cone generated by the standard basis $e_1 = (1, \dots, 0)$, $\dots$, $e_n = (0, \dots, 1)$, and $a_1 = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$,

$$W(u) = \sum_{k=1}^{n} C_k \left(u_k^2(u_k^2 - 2\alpha_k^2) + \alpha_k^4\right), \tag{26}$$

for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, where C_k are given positive constants, with $D = F$ and $Q = |u - a_1|$. Note that in this last example a_1 is in the interior of $\bar{F}$. In Proposition 3 in Part II we establish that for each finite reflection group G acting on $\mathbb{R}^n$ there exist infinitely many smooth potentials W satisfying Hypotheses 1–4. Also, in the Remarks following Propositions 1 and 3 in Part II we give information on the way that a certain class of such Q's can be constructed.

We are now ready to state our result.

Theorem 1. *Under Hypotheses 1–4, there exists an equivariant (cf.* (4)*) classical solution u to system* (1) *such that:*

(i) $|u(x) - a_1| \leq K\mathrm{e}^{-kd(x,\partial D)}$, *for some positive constants k, K and for $x \in D$, where $d(x, \partial D)$ is the Euclidean distance between x and ∂D.*
(ii) $u(D) \subset D$ *(positivity).*

In particular, u connects the $N = |G|/|\mathrm{Stab}(a_1)|$ global minima of W:

$$\lim_{\lambda \to +\infty} u(\lambda g a_1) = g a_1, \ \textit{for all } g \in G.$$

The proof of Theorem 1 is based on a family of constrained minimization problems in $W^{1,2}_{\mathrm{E}}(B_R;\mathbb{R}^n)$ (the equivariant Sobolev maps)

$$\min J_{B_R},\ \text{where } J_{B_R}(u)=\int_{B_R}\Big\{\frac{1}{2}|\nabla u|^2+W(u)\Big\}\,\mathrm{d}x, \tag{27}$$

where $B_R=\{x\in\mathbb{R}^n \mid |x|<R\}$. We introduce two constraints. One enforces the desirable behavior at infinity,

$$|u(x)-a_1|\leq\bar{q}<r_0,\ \text{for } x\in C_R\subset D\cap B_R. \tag{28}$$

where C_R is another ball $B(x_R,2L)$ with $x_R=\frac{R}{2}x_0$ and $L>0$, fixed, independent of R, and sufficiently large. The other constraint is positivity. We minimize in the class of positive maps

$$\big\{u\in W^{1,2}_{\mathrm{E}}(B_R;\mathbb{R}^n)\mid u(\overline{D\cap B_R})\subset\bar{D}\big\}. \tag{29}$$

Before going further, let us point out that equivariance in *not* a constraint since by the results in Palais [30], a critical point of J_{B_R} in the equivariant class is aumatically a critical point in $W^{1,2}_{\mathrm{E}}(B_R;\mathbb{R}^n)$. At the beginning we thought that positivity should be a property of the minimizer u_R and thus be automatically satisfied. However, we were not able to prove this. Instead, by using the gradient flow

$$\begin{cases} u_t=\Delta u-W_u(u),\\ \dfrac{\partial u}{\partial \boldsymbol{n}}=0,\ \text{on } \partial B_R. \end{cases} \tag{30}$$

in the class $W^{1,2}_{\mathrm{E}}(B_R;\mathbb{R}^n)$, we were able to show that the set of positive maps remains invariant under (30) and, moreover, that the flow takes the positive maps into the set of *strongly positive* maps, that is, maps that map the interior of D into itself. This, together with the fact that the gradient flow reduces the free energy J_{B_R} allows us to remove the positivity constraint. The removal of the other constraint for all $R>R_0$, where R_0 is a certain size that can be estimated well and depends only on L and W, is achieved via comparison arguments through the Q-functions. However, the key role of the Q-monotonicity is to ensure, through an estimate, that the limit along subsequences

$$u(x)=\lim_{R\to\infty}u_R(x) \tag{31}$$

is not identically equal to zero, that is, trivial.

Part II

We begin by giving the details behind the construction of the potential in (25).

Let $e_j\in\mathbb{R}^2$, $j=1,2,3$, be the vertices of an equilateral triangle T inscribed in the unit circle. Then,

$$\langle e_j,e_j\rangle=1,\ \text{for } j=1,2,3,\quad\text{and}\quad \langle e_j,e_h\rangle=-\frac{1}{2},\ \text{for } j\neq h. \tag{32}$$

The group $\mathcal{H}^3_2$ of the symmetries of T is generated by the reflections $S_j:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$S_j(\xi e_j+\eta e_h)=\xi e_j+\eta e_k,\ \text{for } j\neq h,k,\ h\neq k,\ \text{with } \xi,\eta\in\mathbb{R}. \tag{33}$$

Proposition 1. *The potential* $W:\mathbb{R}^2\to\mathbb{R}$ *defined by*

$$W(u)=\langle u,u\rangle^2-\frac{1}{3}\prod_{j=1}^{3}\langle 2u-e_j,e_j\rangle+\frac{1}{3},\quad u\in\mathbb{R}^2 \tag{34}$$

satisfies Hypotheses 1–4 with $G=\mathcal{H}_2^3$, $a_j=e_j$, $j=1,2,3$, *and* $Q:\bar{D}\to\mathbb{R}$ *defined by*

$$Q(u)=|u-e_1|, \tag{35}$$

where $\bar{D}=\{u=\xi e_1+\eta(e_2-e_3)\mid \xi\geq 0,\ |\eta|\leq\xi\}$.

Proof. Hypothesis 2 follows trivially from the definition of W. We also have

$$W(e_j)=0,\ \text{for } j=1,2,3. \tag{36}$$

If we take

$$e_1=(1,0),\ e_2=\big(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\big),\ e_3=\big(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}\big), \tag{37}$$

then we have

$$W(u_1,u_2)=|u|^4-\frac{2}{3}u_1^3+2u_1u_2^2-|u|^2+\frac{2}{3} \tag{38}$$

with partial derivatives

$$\begin{cases}\dfrac{\partial W}{\partial u_1}(u_1,u_2)=4u_1|u|^2-2u_1^2+2u_2^2-2u_1,\\ \dfrac{\partial W}{\partial u_2}(u_1,u_2)=4u_2|u|^2+4u_1u_2-2u_2.\end{cases} \tag{39}$$

From (39) it follows that the critical points of W are the following: nondegenerate minima $u=e_j$, $j=1,2,3$, saddle points $u=-\frac{1}{2}e_j$, $j=1,2,3$, and a local minimum $u=0$. From this, (36), and the fact that $W(u)\to+\infty$ as $|u|\to+\infty$, we conclude that

$$W(u)>0,\ \text{for } u\neq e_j,\ j=1,2,3. \tag{40}$$

It remains to verify Hypothesis 4. To this end we study the sign of the function

$$V(u)=\frac{1}{2}\big\langle W_u(u),u-e_1\big\rangle=2|u|^4-3u_1^3+u_1u_2^2-2u_2^2+u_1. \tag{41}$$

We first analyze the sign of V on the boundary of the fundamental domain $F=\{(u_1,u_2)\mid 0<u_2<\sqrt{3}u_1,\ \text{for } u_1>0\}$. We have

$$\begin{cases}V(u_1,0)=2u_1\Big(u_1+\dfrac{1}{2}\Big)(u_1-1)^2\geq 0,\ \text{for } u_1\geq 0,\\ V(u_1,\sqrt{3}u_1)=32u_1\Big(u_1+\dfrac{1}{2}\Big)\Big(u_1-\dfrac{1}{4}\Big)^2\geq 0,\ \text{for } u_1\geq 0.\end{cases} \tag{42}$$

Next we look for the critical points of V in F. We have the equations

$$\begin{cases}\dfrac{\partial V}{\partial u_1}(u)=8u_1|u|^2-9u_1^2+u_2^2+1=0,\\ \dfrac{\partial V}{\partial u_2}(u)=8u_2|u|^2+2u_1u_2-4u_2=0.\end{cases} \tag{43}$$

For $u_2\neq 0$, (43b) implies $8|u|^2=4-2u_1$; $u_2^2=-u_1^2-\frac{1}{4}u_1+\frac{1}{2}$. Inserting these expressions into (43a) yields

$$u_1^2-\frac{5}{16}u_1-\frac{1}{8}=0, \tag{44}$$

which has a unique positive solution

$$\bar{u}_1 = \frac{5+\sqrt{153}}{32} > \frac{1}{2}\Big(1+\frac{1}{16}\Big) \tag{45}$$

It follows that V has a unique critical point $(\bar{u}_1, \bar{u}_2)$ in the positive quadrant and

$$\bar{u}_2^2 = \frac{1}{2} - \frac{1}{4}\bar{u}_1 - \bar{u}_1^2. \tag{46}$$

Using (46), (44), and

$$|u|^2 = \frac{1}{2} - \frac{1}{4}\bar{u}_1,$$

we get

$$\begin{aligned} V(\bar{u}_1, \bar{u}_2) &= 2\Big(\frac{1}{2} - \frac{1}{4}\bar{u}_1\Big)^2 - 3\bar{u}_1^3 + \frac{1}{2}\bar{u}_1 - \frac{1}{4}\bar{u}_1^2 - \bar{u}_1^3 - 1 + \frac{1}{2}\bar{u}_1 + 2\bar{u}_1^2 + \bar{u}_1 \\ &= -4\bar{u}_1^3 + \frac{15}{8}\bar{u}_1^2 + \frac{3}{2}\bar{u}_1 - \frac{1}{2} = \frac{5}{8}\bar{u}_1^2 + \bar{u}_1\frac{1}{2} > 0, \end{aligned} \tag{47}$$

where we have used (45). This implies that also Hypothesis 4 is satisfied. Indeed, from (42) we have $V(u) \geq 0$ on ∂F. On the other hand, $|u|$ sufficiently large implies $V(u) > 0$. Therefore, the existence of a point $\hat{u} \in F$ where $V(\hat{u}) \leq 0$ would imply the existence of a critical point $\bar{u}$ of V such that $V(\bar{u}) \leq 0$. This is not the case because we have shown that V has a unique critical point $\bar{u}$ in the positive quadrant and by (47), $V(\bar{u}) > 0$. This concludes the proof. □

Remark. The polynomial (38) is a special choice in the four-dimensional vector space $\mathcal{P}$ of polynomials of order ≤ 4 which are invariant under $\mathcal{H}_2^3$:

$$\mathcal{P} = \big\{P \mid P(u) = \alpha|u|^4 + \beta(u_1^3 - 3u_1u_2^2) + \gamma|u|^2 + \delta, \text{ for } \alpha,\beta,\gamma,\delta \in \mathbb{R}\big\}. \tag{48}$$

Hypothesis 1 requires $\alpha > 0$ and can be normalized to $\alpha = 1$. The coefficients β, γ, and δ are uniquely determined by the conditions

$$P_u(e_1) = 0,\ P(e_1) = 0, \text{ and } P_u\big(-\tfrac{1}{2}e_2\big) = 0. \tag{49}$$

The last condition is necessary in order that Hypothesis 4 holds with $Q(u) = |u - e_1|$.

In conclusion, the choice (38) is 'canonical' in the sense that it is the unique normalized polynomial of degree 4 that is invariant under $\mathcal{H}_2^3$, satisfies (49), and also satisfies Hypothesis 4 with $Q(u) = |u - e_1|$.

Let now $e_j \in \mathbb{R}^3$, $j = 1,2,3,4$ be the vertices of a tetrahedron $\mathcal{T}$ inscribed in the unit sphere. Then,

$$\langle e_j, e_j\rangle = 1, \text{ for } j = 1,2,3,4, \quad \text{and} \quad \langle e_j, e_h\rangle = -\frac{1}{3}, \text{ for } j \neq h. \tag{50}$$

If we take

$$e_{1,3} = \Big(\pm\sqrt{\tfrac{2}{3}}, 0, \tfrac{1}{\sqrt{3}},\ e_{2,4} = \Big(0, \pm\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{3}}\Big), \tag{51}$$

then one can check that the five homogeneous polynomials

$$1,\ |u|^2,\ (u_1^2 - u_2^2)u_3,\ (u_3^2 - 2u_1^2)(u_3^2 - 2u_2^2)),\ |u|^4$$

make up a basis for the five-dimensional vector space $\mathcal{P}$ of the polynomials of degree ≤ 4 which are invariant under the group $\mathcal{T}^*$ of the symmetries of $\mathcal{T}$. If one imposes to the generic $P \in \mathcal{P}$ the conditions

$$P(e_1) = 0,\ P_u(e_1) = 0, \tag{52}$$

and the necessary conditions

$$P_u\big(-\tfrac{1}{3}e_3\big)=0 \text{ and } P_u\big(\tfrac{1}{2}(e_2+e_4)\big)=0, \tag{53}$$

to ensure that Hypothesis 4 holds with $Q(u)=|u-e_1|$ and normalizes to 1 the coefficient of $|u|^4$, one gets the canonical polynomial

$$W(u)=|u|^4-\frac{4}{\sqrt{3}}(u_1^2-u_2^2)u_3-\frac{2}{3}|u|^2+\frac{5}{9}. \tag{54}$$

An analysis similar to the one in the proof of Proposition 1 shows that actually W satisfies Hypotheses 1–3 and Hypothesis 4 with $Q(u)=|u-e_1|$.

In the proposition that follows we give the proof of Lemma 3.1 in [5] in full generality.

Proposition 2. *Assume that*

(i) *$\Omega\subset\mathbb{R}^n$ is an open and connected set with a piecewise-smooth boundary and $0\in\Omega$.*

(ii) *$Q:\bar{\Omega}\to\mathbb{R}$ is a continuous function such that*

$$Q(u)>0 \quad and \quad Q_u(u)\neq 0, \quad for\ u\in\bar{\Omega}\setminus\{0\}, \tag{55}$$

with

$$Q(u)=|u|+H(u) \tag{56}$$

where $H:\bar{\Omega}\to\mathbb{R}$ a smooth function that satisfies

$$H(0)=H_u(0)=0. \tag{57}$$

(iii) *For each $u\in\partial\Omega$, $Q_u(u)$ points outside Ω.*

Then, for each $\nu\in\mathbb{S}^{n-1}$, the equation

$$\frac{\mathrm{d}u}{\mathrm{d}q}=\frac{Q_u(u)}{\big\langle Q_u(u),Q_u(u)\big\rangle}, \ for\ u\in\Omega\setminus\{0\} \tag{58}$$

has a unique solution $\tilde{u}(\cdot,\nu):(0,q_\nu)\to\mathbb{R}^n$, where $(0,q_\nu)$ is the maximal interval of existence, such that

$$\lim_{q\to0+}\tilde{u}(q,\nu)=0 \ and \ \lim_{q\to0+}\frac{\tilde{u}(q,\nu)}{|\tilde{u}(q,\nu)|}=\nu. \tag{59}$$

Moreover, the map h defined by

$$(q,\nu)\overset{h}{\mapsto}\tilde{u}(q,\nu), \tag{60}$$

is a diffeomorphism of $\Sigma=\big\{(q,\nu)\mid q\in(0,q_\nu),\ for\ \nu\in\mathbb{S}^{n-1}\big\}$ onto $\Omega\setminus\{0\}$.

Proof.
Step 1. Scalar multiplication of (58) by $Q_u(u)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}q}Q(u)=1. \tag{61}$$

This, (55), (56), and the fact that Ω is negatively invariant by (iii) imply

$$\lim_{q\to0+}u(q)=0 \text{ and } Q(u(q))=q \tag{62}$$

along any solution of (58). Using $q=|u|+H(u)$ we can rewrite (58) in the form

$$\frac{\mathrm{d}u}{\mathrm{d}q}=\frac{u}{q}+g(u), \tag{63}$$

where $g : \Omega \to \mathbb{R}^n$ is a smooth function that satisfies

$$g(0) = 0 \tag{64}$$

and is defined in terms of Q and H. From (63) we have

$$\frac{\mathrm{d}}{\mathrm{d}q}\frac{u(q)}{|u(q)|} = \frac{g(u(q))}{|u(q)|} - \Big\langle \frac{u(q)}{|u(q)|}, \frac{g(u(q))}{|u(q)|} \Big\rangle \frac{u(q)}{|u(q)|}. \tag{65}$$

This and (64) imply

$$\Big|\frac{\mathrm{d}}{\mathrm{d}q}\frac{u(q)}{|u(q)|}\Big| \leq \Big|\frac{g(u(q))}{u(q)}\Big| \leq C, \tag{66}$$

for some constant C. It follows that for each solution $u(q)$ of (58) there exists a unit vector $\nu \in \mathbb{S}^{n-1}$ such that

$$\lim_{q\to 0+}\frac{u(q)}{|u(q)|} = \nu. \tag{67}$$

The above argument shows that there is a map $\hat{h} : \Omega \setminus \{0\} \to \Sigma$ that associates to each $u \in \Omega \setminus \{0\}$ a pair $(q, \nu) \in \Sigma$ as follows: $q = Q(u)$ and ν is given by (67) with $u(\cdot)$ the unique solution of (63) through the point $(Q(u), u)$.

Step 2. From the implicit function theorem, the equation

$$p + H(p\nu) = q \tag{68}$$

has a unique smooth local solution $p = p(q, \nu)$ for which there holds

$$p(0, \nu) = 0 \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}q}(0, \nu) = 1. \tag{69}$$

Fix small positive numbers $0 < \varepsilon < \delta$ and let $u(q, \varepsilon, \nu)$, for $q \in (0, \delta)$, be the solution of (63) through the point $(\varepsilon, p(\varepsilon, \nu)\nu$. Using the fact that the fundamental matrix solution of the homogeneous equation

$$\frac{\mathrm{d}u}{\mathrm{d}q} = \frac{u}{q}$$

is

$$\frac{q}{s}\mathrm{Id},$$

we see that $u(\cdot, \varepsilon, \nu)$ is the unique solution of the integral equation

$$u(q, \varepsilon, \nu) = q\Big(\frac{p(\varepsilon, \nu)}{\varepsilon}\nu + \int_\varepsilon^q g(u(s, \varepsilon, \nu))\frac{\mathrm{d}s}{s}\Big), \text{ for } q \in (0, \delta). \tag{70}$$

Given $0 < \varepsilon_1 < \varepsilon_2 < \delta$, set

$$\phi(q) = \frac{1}{q}|u(q, \varepsilon_2, \nu) - u(q, \varepsilon_1, \nu)| \tag{71}$$

and observe that (69) and (64) imply

$$\Big|\frac{p(\varepsilon_2, \nu)}{\varepsilon_2} - \frac{p(\varepsilon_1, \nu)}{\varepsilon_2}\Big| \leq C|\varepsilon_2 - \varepsilon_1|, \tag{72}$$

$$\frac{1}{s}|g(u(s, \varepsilon_2, \nu)) - g(u(s, \varepsilon_1, \nu))| \leq C\phi(s). \tag{73}$$

Therefore, from (70) it follows that

$$\phi(q) \leq \begin{cases} C|\varepsilon_2-\varepsilon_1| + C\displaystyle\int_q^{\varepsilon_1}\phi(s)\,\mathrm{d}s + \int_{\varepsilon_1}^{\varepsilon_2}|g(u(s,\varepsilon_2,\nu))|\frac{\mathrm{d}s}{s} \\ \qquad\qquad \leq C|\varepsilon_2-\varepsilon_1| + \displaystyle\int_0^{\varepsilon_1}\phi(s)\,\mathrm{d}s, \quad \text{if } q<\varepsilon_1, \\ C|\varepsilon_2-\varepsilon_1| + \displaystyle\int_{\varepsilon_1}^{q}|g(u(s,\varepsilon_1,\nu))|\frac{\mathrm{d}s}{s} + \int_q^{\varepsilon_2}|g(u(s,\varepsilon_2,\nu))|\frac{\mathrm{d}s}{s} \\ \qquad\qquad \leq C|\varepsilon_2-\varepsilon_1|, \quad \text{if } \varepsilon_1<q<\varepsilon_2, \\ C|\varepsilon_2-\varepsilon_1| + \displaystyle\int_{\varepsilon_1}^{\varepsilon_2}|g(u(s,\varepsilon_1,\nu))|\frac{\mathrm{d}s}{s} + C\int_{\varepsilon_2}^{q}\phi(s)\,\mathrm{d}s \\ \qquad\qquad \leq C|\varepsilon_2-\varepsilon_1| + C\displaystyle\int_0^{q}\phi(s)\,\mathrm{d}s, \quad \text{if } \varepsilon_2<q. \end{cases} \tag{74}$$

This and Gronwall's lemma imply

$$\phi(q) \leq C|\varepsilon_2-\varepsilon_1|\mathrm{e}^{C\delta}, \tag{75}$$

that is,

$$|u(q,\varepsilon_2,\nu)-u(q,\varepsilon_1,\nu)| \leq Cq|\varepsilon_2-\varepsilon_1|\mathrm{e}^{C\delta}, \text{ for } \nu\in\mathbb{S}^{n-1}, q\in(0,\delta). \tag{76}$$

Therefore, $u(\cdot,\varepsilon,\cdot)$ converges uniformly on $(0,\delta)\times\mathbb{S}^{n-1}$ to a continuous function $\tilde{u}(\cdot,\cdot)$ which satisfies the limit equation

$$\tilde{u}(q,\nu) = q\Big(\nu + \int_0^q g(\tilde{u}(s,\nu))\frac{\mathrm{d}s}{s}\Big), \text{ for } q\in(0,\delta), \nu\in\mathbb{S}^{n-1}. \tag{77}$$

Thus, (77) shows that each $\nu\in\mathbb{S}^{n-1}$ uniquely determines a solution $\tilde{u}(\cdot,\nu)$ of (63), and $\nu_1\neq\nu_2$ implies $\tilde{u}(\cdot,\nu_1)\neq\tilde{u}(\cdot,\nu_2)$. From this and Step 1 it follows that (77) defines a map

$$h:(0,\delta)\times\mathbb{S}^{n-1}\to\Omega_\delta = \big\{u\in\Omega\setminus\{0\} \mid Q(u)<\delta\big\},$$

which is the inverse of the map $\hat{h}|_{\Omega_\delta}$, $\hat{h}$ the map defined in Step 1.

To conclude the proof, it suffices to show that $h:(0,\delta)\times\mathbb{S}^{n-1}\to\Omega_\delta$ is a diffeomorphism. The last statement of the Proposition then follows from the general theory of ordinary differential equations that allows to extend h to the whole Σ.

Step 3. Let $\alpha>0$ a small number and $\hat{\nu}:(-\alpha,\alpha)\to\mathbb{S}^{n-1}$ a smooth curve such that

$$\hat{\nu}(0)=\nu \text{ and } \hat{\nu}'(0)=\omega. \tag{78}$$

Set

$$\psi(s,\tau) = \frac{1}{s}\Big(\frac{\tilde{u}(s,\hat{\nu}(\tau))-\tilde{u}(s,\nu)}{\tau}\Big).$$

Then, (77) implies

$$|\psi(q,\tau)| \leq \Big|\frac{\hat{\nu}(\tau)-\nu}{\tau}\Big| + C\int_0^q|\psi(s,\tau)|\,\mathrm{d}s,$$

or,

$$|\psi(q,\tau)| \leq \Big|\frac{\hat{\nu}(\tau)-\nu}{\tau}\Big|\mathrm{e}^{C\delta}. \tag{79}$$

From (77) we also get

(80)
$$\begin{aligned}
\psi(q,\tau_2)-\psi(q,\tau_1) &= \frac{\hat\nu(\tau_2)-\nu}{\tau_2}-\frac{\hat\nu(\tau_1)-\nu}{\tau_1} \\
&\quad+\int_0^q\int_0^1 g_u\big(\tilde u(s,\nu)+\lambda(\tilde u(s,\hat\nu(\tau_2))-\tilde u(s,\nu))\big)\psi(s,\tau_2)\,\mathrm{d}\lambda\,\mathrm{d}s \\
&\quad-\int_0^q\int_0^1 g_u\big(\tilde u(s,\nu)+\lambda(\tilde u(s,\hat\nu(\tau_1))-\tilde u(s,\nu))\big)\psi(s,\tau_1)\,\mathrm{d}\lambda\,\mathrm{d}s \\
&= \frac{\hat\nu(\tau_2)-\nu}{\tau_2}-\frac{\hat\nu(\tau_1)-\nu}{\tau_1} \\
&\quad+\int_0^q\int_0^1 g_u(s,\lambda\tau_2)(\psi(s,\tau_2)-\psi(s,\tau_2))\,\mathrm{d}\lambda\,\mathrm{d}s \\
&\quad-\int_0^q\int_0^1 (g_u(s,\lambda,\tau_2)-g_u(s,\lambda,\tau_1))\psi(s,\tau_1)\,\mathrm{d}\lambda\,\mathrm{d}s
\end{aligned}$$

where we have set $g_u(s,\lambda,\tau)=g_u\big(\tilde u(s,\nu)+\lambda(\tilde u(s,\hat{nu}(\tau))-\tilde u(s,\nu))\big)$.

We note the estimates

$$\Big|\frac{\hat\nu(\tau_2)-\nu}{\tau_2}-\frac{\hat\nu(\tau_1)-\nu}{\tau_1}\Big|\le C(|\tau_1|+|\tau_2|), \tag{81}$$

$$|g_u(s,\lambda,\tau)|\le C, \tag{82}$$

$$\begin{aligned}
|g_u(s,\lambda,\tau_2)-g_u(s,\lambda,\tau_1)| &\le C\big|\tilde u(s,\hat\nu(\tau_2))-\tilde u(s,\hat\nu(\tau_1))\big| \\
&\le Cs(|\tau_2||\psi(s,\tau_2)+|\tau_1||\psi(s,\tau_1)|) \\
&\le C\delta(|\tau_1|+|\tau_2|),
\end{aligned} \tag{83}$$

where we have used (79) that implies $|\psi(s,\tau)|$ is bounded.

From this estimate, (79), and (80), it follows

$$|\psi(q,\tau_2)-\psi(q,\tau_2)|\le C(|\tau_1|+|\tau_2|)+C\int_0^q|\psi(s,\tau_2)-\psi(s,\tau_1)|\,\mathrm{d}s, \tag{84}$$

and therefore, we obtain for $(q,\nu)\in(0,\delta)\times\mathbb{S}^{n-1}$

$$\Big|\frac{\tilde u(q,\hat\nu(\tau_2))-\tilde u(q,\nu)}{\tau_2}-\frac{\tilde u(q,\hat\nu(\tau_1))-\tilde u(q,\nu)}{\tau_1}\Big|\le Cq(|\tau_1|+|\tau_2|) \tag{85}$$

From this inequality it follows that $\tilde u(q,\nu)$ is differentiable with respect to ν and that the derivative $D_\nu\tilde u(q,\nu):T_\nu\mathbb{S}^{n-1}\to\mathbb{R}^n$ depends continuously on (q,ν) and satisfies the integral equation

$$D_\nu\tilde u(q,\nu)\omega=q\Big(\omega+\int_0^q g_u(\tilde u(s,\nu))D_\nu\tilde u(s,\nu)\omega\frac{\mathrm{d}s}{s}\Big). \tag{86}$$

This and

$$D_q\tilde u(q,\nu)=\nu+\int_0^q g(\tilde u(s,\nu))\frac{\mathrm{d}s}{s}+g(\tilde u(q,\nu)), \tag{87}$$

show that $\tilde u(q,\nu)$ is differentiable at $(q,\nu)\in(0,\delta)\times\mathbb{S}^{n-1}$ and

$$D\tilde u(q,\nu)(\rho,\omega)=q\omega+\rho\nu+\text{higher-order terms} \tag{88}$$

and therefore that $D\tilde u(q,\nu)$ is nonsingular. This concludes the proof. $\square$

Proposition 3. *For each finite reflection group G acting on $\mathbb{R}^n$, there exist infinitely-many smooth potentials $W : \mathbb{R}^n \to \mathbb{R}$ that satisfy Hypotheses 1–4.*

Proof. Let

$$\mathrm{Int}\big(\textstyle\bigcup_{g\in \mathrm{Stab}(a)} g\bar{F}\big)$$

and let $Q : \bar{D} \to \mathbb{R}$ be a convex function such that the map $Q(\,\cdot + a) : \bar{D} \setminus a \to \mathbb{R}$ satisfies the assumptions in Proposition 1. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a G-invariant map such that

(i) $\rho(u) > 0$, for $u \in D \setminus \{a\}$, and $\rho(u) = 0$, for $u \in \{a\} \cup \partial D$,
(ii) $\rho(u + a) = |u| + \phi(u)$, for $u \in D$,

where ϕ is a smooth function such that $\phi(0) = \phi_u(0) = 0$.

Define

$$\begin{cases} W(u) = \displaystyle\int_0^{Q(u)} \rho\big(\tilde{u}(q, \nu(u))\big)\,\mathrm{d}q, \text{ for } u \in \bar{D} \setminus \{a\}, \\ W(a) = 0. \end{cases} \tag{89}$$

where $(q, \nu) \mapsto \tilde{u}(q, \nu) - a$ is the diffeomorphism defined in Proposition 2. We assume that W is extended G-equivariantly to the whole of $\mathbb{R}^n$. By definition, W satisfies $W(u) > 0$ for $u \neq ga$, $g \in G$. From (89) and the smoothness of the maps $u \mapsto Q(u)$ and $u \mapsto \nu(u)$, it follows that W is smooth on $D \setminus \{a\}$. From (ii) and assumption (56) in Proposition 2 we see that W extendes smoothly to a and a is nondegenerate zero of W. Finally, the smoothness of W in a neighborhood of ∂D follows from the assumption that $\rho(u)$ vanishes for $u \in \partial D$.

It remains to show that the function W defined by (89) and the convex function Q verify the condition

$$\big\langle Q_u(u), W_u(u)\big\rangle \geq 0, \text{ for } u \in D \setminus \{a\}. \tag{90}$$

To see this we note that differentiating with respect to q the identity $\nu(\tilde{u}(q, \nu)) = \nu$ we get

$$\nu_u(u) Q_u(u) = 0, \tag{91}$$

where we have also used (58). Differentiating (89) with respect to u yields

$$W_u(u) = \rho(u) Q_u(u) + \int_0^{Q(u)} \rho_u\big(\tilde{u}(q, \nu(u))\big) \tilde{u}_\nu(q, \nu(u)) \nu_u(u)\,\mathrm{d}q. \tag{92}$$

In view of (91), scalar multiplication of (92) by $Q_u(u)$ implies

$$\big\langle W_u(u), Q_u(u)\big\rangle = \rho(u)|Q_u(u)|^2 \geq 0. \tag{93}$$

This concludes the proof. □

Remark. Possible explicit choices for the functions Q and ρ considered in Propositions 1 and 2 are

$$Q(u) = |u| + \langle Au, u\rangle, \tag{94}$$

where A is a positive definite symmetric $n \times n$ matrix, and

$$\rho(u) = \prod_{\hat{a} \in \{ga \,|\, g \in G\}} d\big(u, \textstyle\bigcup_{g\in G} \partial(gD)\big)^2, \tag{95}$$

where $d(u, E)$ is the distance of u from the set E.

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Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece and Institute for Applied and Computational Mathematics, Foundation of Research and Technology – Hellas, 71110 Heraklion, Crete, Greece
E-mail address: `nalikako@math.uoa.gr`

Dipartimento di Matematica Pura ed Applicata, Università degli Studi dell'Aquila, Via Vetoio, 67010 Coppito, L'Aquila, Italy
E-mail address: `fusco@univaq.it`