

Lecture 5

Exercise 4

$M(A) = \text{smallest } \# a \text{ such that } \exists \text{ sequence } P_i \text{ of polyhedral chains, } P_i \rightarrow A \text{ in } W\text{-sense, and } M(P_i) \rightarrow a$

Flat chains of Finite Mass and Induced Measure

Let A be a flat chain with $M(A) < \infty$. With A we associate a measure μ_A , and with every μ_A -measurable set $X \subset \mathbb{R}^n$ a flat chain $A \llcorner X$, such that

$$\mu_A(X) = M(A \llcorner X),$$

by the following procedure:

Let $\{P_j\}$ a sequence of polyhedral chains tending to A , and so that $M(P_j) \rightarrow M(A)$. For $j=1,2,\dots$ let μ_j be the measure determined by the property $\mu_j(I) = M(P_j \llcorner I)$ if I is any n -dim. interval. Then $\mu_j(\mathbb{R}^n) = M(P_j)$ is bounded. By taking a subsequence we may assume that $\sum W(P_{j+1} - P_j) < \infty$ and also that μ_j tends weakly to a measure μ_A . It will be shown that μ_A is the same for all $\{P_j\}$ above.

1. In this section we will call an n -dim interval I exceptional if either $\sum W[(P_{j+1} - P_j) \llcorner I] = \infty$ or $\mu_A(\text{fr } I) > 0$. If I is non-exceptional then $\{\mu_j \llcorner I\}$ is Cauchy here the limit exists and is denoted by $A \llcorner I$, a flat chain. Since $\mu_j \xrightarrow{w} \mu_A$, and also $\mu_A(\text{fr } I) = 0 \implies \mu_j(I) \rightarrow \mu_A(I)$, $\mu_j(\mathbb{R}^n) \rightarrow \mu_A(\mathbb{R}^n)$

2. Lemma: If I is nonexceptional, then $M(A \cap I) = \mu_A(I)$,
 $M(A - A \cap I) = \mu_A(I^c)$.

Proof

From $\sum W[(P_{j+1} - P_j) \cap I] < \infty \Rightarrow \{P_j \cap I\}$ Cauchy,
 here say $P_j \cap I \rightarrow B_1$. Also $P_j - P_j \cap I \rightarrow A - B_1$.
 Notice also that $\sum W[(P_{j+1} - P_j) \cap I^c] < \infty$ as well.

Indeed

$$(P_{j+1} - P_j) \cap I^c + (P_{j+1} - P_j) \cap I = P_{j+1} - P_j$$

$$\Rightarrow (P_{j+1} - P_j) \cap I^c = (P_{j+1} - P_j) - [(P_{j+1} - P_j) \cap I]$$

$$\Rightarrow W[(P_{j+1} - P_j) \cap I^c] \leq W(P_{j+1} - P_j) + W[(P_{j+1} - P_j) \cap I]$$

$$\Rightarrow \sum \text{---} \leq \sum \text{---} + \sum \text{---}$$

$$\text{Hence } P_j \cap I \rightarrow B_1, \quad P_j \cap I^c \rightarrow B_2$$

By lower semicontinuity

$$M(B_1) \leq \underline{\lim} M(P_j \cap I), \quad M(B_2) \leq \underline{\lim} M(P_j \cap I^c)$$

$$M(P_j \cap I) = \mu_j(I) \rightarrow \mu_A(I), \quad M(P_j \cap I^c) = \mu_j(I^c) \rightarrow \mu_A(I^c)$$

\therefore

$$M(B_1) \leq \mu_A(I), \quad M(B_2) \leq \mu_A(I^c).$$

In conclusion

$$M(B_1) + M(B_2) \leq \mu_A(I) + \mu_A(I^c) = \mu_A(\mathbb{R}^n) \stackrel{(*)}{\leq} M(A) \leq M(B_1) + M(B_2).$$

$$\text{Thus } M(B_1) = \mu_A(I), \quad M(B_2) = \mu_A(I^c). \quad \square$$

$$(*) \mu_j(\mathbb{R}^n) = M(P_j) \rightarrow M(A), \quad \underline{\lim} \mu_j(\mathbb{R}^n) \geq \mu_A(\mathbb{R}^n)$$

$$(**) P_j = P_j - (P_j \cap I) + (P_j \cap I). \text{ Since } P_j \rightarrow A, P_j \cap I \rightarrow B_1 \Rightarrow A = B_1 + B_2$$

3. Next we extend the lemma to

$$M(A \cap X) = \chi_A(X), \quad M(A - A \cap X) = \chi_A(X),$$

$X = I_1 \cup I_2 \cup \dots \cup I_p$, nonoverlapping, nonempty and n -intends

Observation 1: $I_1 \cap I_2 = \emptyset \Rightarrow M(A \cap (I_1 \cup I_2)) = M(A \cap I_1) + M(A \cap I_2)$

Verification: Let $Q = A \cap (I_1 \cup I_2)$. We need to show that

$$\chi_Q(p) = \chi_{A \cap I_1}(p) + \chi_{A \cap I_2}(p).$$

Suppose $\chi_Q(p) = 1 \Rightarrow$ either $p \in A, p \in I_1$, or $p \in A, p \in I_2$ (not both). Then equality holds. Next suppose $\chi_Q(p) = 0 \Rightarrow p \notin A \cap I_1, p \notin A \cap I_2$. Again equality holds.

Hence

$$M(A \cap X) = M(A \cap I_1) + M(A \cap I_2) + \dots + M(A \cap I_p)$$

Observation 2: $I_1 \cap I_2 = \emptyset$. Let $A_1 = A \cap I_1, A_2 = A \cap I_2$.

$$\Rightarrow M(A_1 + A_2) = M(A_1) + M(A_2).$$

Verification: Take $P_i \rightarrow A_1 + A_2, M(P_i) \rightarrow M(A_1 + A_2)$

Set $P_i^j = P_i \cap I_j, j = 1, 2$.

Then

$$P_i^j \rightarrow A_j, \quad P_i = P_i^1 + P_i^2$$

$$M(P_i) = M(P_i^1 + P_i^2) = M(P_i^1) + M(P_i^2)$$

$$\downarrow \qquad \qquad \qquad \geq M(A_1) + M(A_2) \quad (\text{by f.s.c.}).$$

$$M(A_1 + A_2)$$

Since

$$M(A_1 + A_2) \leq M(A_1) + M(A_2) \text{ always,}$$

$$\therefore M(A_1 + A_2) = M(A_1) + M(A_2).$$

Hence utilizing both observations

$$M(A \cap X) = M(A \cap I_1) + \dots + M(A \cap I_p)$$

$$= \mu_A(I_1) + \dots + \mu_A(I_p) = \mu_A(I_1 \cup \dots \cup I_p) = \mu_A(X).$$

For the remaining half we proceed as follows:

Observation 3: $I_1 \cap I_2 = \emptyset \Rightarrow A - A \cap (I_1 \cup I_2) = A - [A \cap I_1 + A \cap I_2]$

Verification:

$$\chi_A(p) - \chi_{A \cap (I_1 \cup I_2)}(p) = \chi_A(p) - (\chi_{A \cap I_1}(p) + \chi_{A \cap I_2}(p))$$

holds true by observation 1.

Hence

$$A - A \cap (I_1 \cup I_2) = (A - A \cap I_1) - A \cap I_2$$

$$\bullet \text{IM}(A - A \cap (I_1 \cup I_2)) \geq \text{IM}(A - A \cap I_1) - \text{IM}(A \cap I_2)$$

Indeed

$$A - A \cap (I_1 \cup I_2) + A \cap I_2 = A - A \cap I_1$$

\Rightarrow

$$\text{IM}(A - A \cap (I_1 \cup I_2)) + \text{IM}(A \cap I_2) \geq \text{IM}(A - A \cap I_1)$$

\Rightarrow

$$\text{IM}(A - A \cap (I_1 \cup I_2)) \geq \text{IM}(A - A \cap I_1) - \text{IM}(A \cap I_2).$$

continuing

$$= \mu_A(I_1^c) - \mu_A(I_2) \quad (\text{by the lemma})$$

$$= \mu_A(I_1^c \cap I_2^c)$$

$$= \mu_A((I_1 \cup I_2)^c).$$

Hence

$$\text{IM}(A - A \cap X) \geq \mu_A(X^c).$$

On the other hand by utilizing

$$P_i \cap (I_1 \cup I_2)^c = P_i - P_i \cap (I_1 \cup I_2) \quad (*)$$

$$\text{IM} (P_i \cap (I_1 \cup I_2)^c) = \text{IM} (P_i - P_i \cap (I_1 \cup I_2))$$

$$\begin{matrix} \text{II} \\ \mu_j ((I_1 \cup I_2)^c) \\ \downarrow \end{matrix} \qquad \text{IV} \quad (\text{lower semicontinuity})$$

$$\mu_A ((I_1 \cup I_2)^c) \geq \mu (A - A \cap (I_1 \cup I_2))$$

$$\therefore \mu_A (X^c) \geq \mu (A - A \cap X)$$

Thus the extension is established.

Exercise 5

Verify (*), that is show that $\forall p \in \mathbb{R}^n$

$$\chi_{P_i \cap (I_1 \cup I_2)^c}(p) = \chi_{P_i}(p) - \chi_{P_i \cap (I_1 \cup I_2)}(p)$$