# Lectures on GMT

**References** (in increasing importance for these lectures)


1. Lectures by Brian White, Spring 2009, Stanford.
2. W. Fleming "Flat chains over a finite Coefficient Group" TAMS, V 121, No 1, 1966 pp 160–186
3. H. Whitney, Geometric Integration Theory, Dover 2005
4. H. Federer – W. Fleming, Normal and Integral Currents, Ann. of Math. 72 (1960), 458–520.

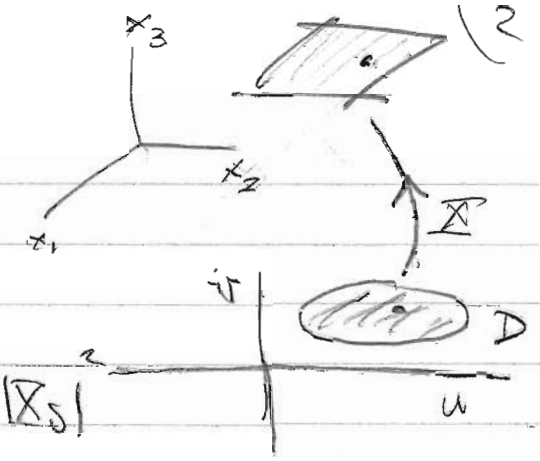

**Objective**: The Plateau Problem:

Closed curve (or closed $m-1$ submanifold) $C$ in Euclidean space. Find $m$-surface $M$ of least area with $\partial M = C$

**Existence?**

- Direct Method: Let $\alpha = \inf(\text{area}(M) : \partial M = C)$
  $\exists$ minimizing sequence $M_i$ that converges to a solution.
  Does not work here (at least not directly).

- For 2-d surfaces: classical theorem of Douglas and Rado ('30s) uses the fact that surfaces are locally conformally parametrized by $D \subseteq \mathbb{R}^2$




$\left( \underline{X}(u,v) \right.$ , $|\underline{X}_u|^2 = |\underline{X}_v|^2 = 1$ ,

$\underline{X}_u \cdot \underline{X}_v = 0$

$$\Delta \underline{X} = 2\lambda^2 H \quad , \quad \lambda = |\underline{X}_u|^2 = |\underline{X}_v|^2$$

Minimal $\Rightarrow$ $H = 0$ , $\Delta \underline{X} = 0$

Complex function theory applies (for example)

See Do Carmo, Diff. Geom. of Closed Surfaces [D. C], pp 201 – .
Struwe pp 19 – ... ).

## Structure of Notes

Part I : Functional Analytic Framework based on Whitney's flat Norm for making the Direct Method applicable – Existence.

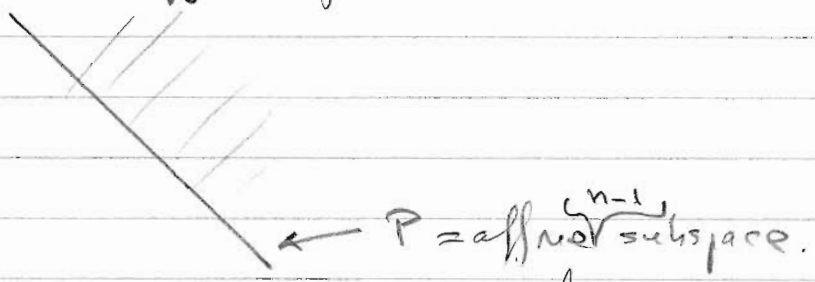
Part II : The Monotonicity Formula and consequences – Regularity.

# PART I

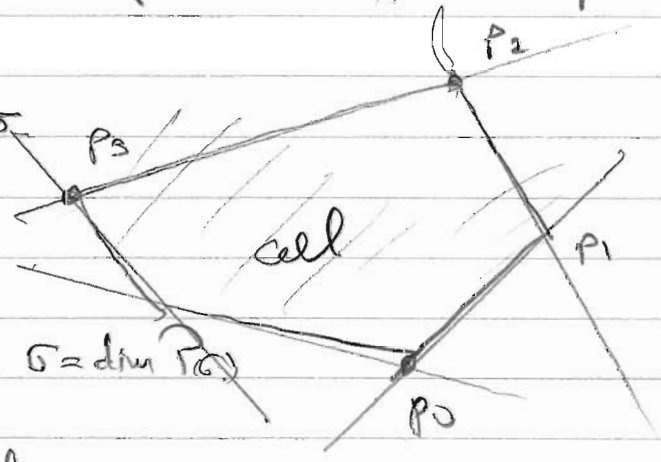
## Flat chains

- Cells, simplexes in an affine space $E$.

Closed half space




convex polyhedral cell $\sigma$




intersection of finite half spaces. $P(\sigma)$ = smallest affine subspace $\supset \sigma$, $\dim \sigma = \dim P(\sigma)$

$r = \dim(\sigma)$ $\quad$ $r$-cell

$\sigma_1 \cap \sigma_2 \neq \emptyset \Rightarrow$ cell

frontier, $\mathrm{fro}(\sigma)$, of $\sigma$, considered as a subset of $P(\sigma)$, is the boundary $\partial\sigma$ of $\sigma$. If $\sigma$ oriented then $\partial\sigma$ is set lying in the chain (see below) called $\partial\sigma$.
$\mathrm{Int}(\sigma) := \sigma - \partial\sigma$. If $\dim(\sigma_1) = \dim(\sigma_2)$, $\sigma_1, \sigma_2$ nonoverlapping if $\mathrm{int}(\sigma_1) \cap \mathrm{int}(\sigma_2) = \emptyset$.

$\partial\sigma$ is expressible as a finite union of $(n-1)$ cells the $(n-1)$-faces of $\sigma$. Each has faces, etc...

vertices are faces of dimension 0, points interior to no segment lying in $\sigma$. $p_0, \dots, p_r$ vertices then $\sigma$ is the smallest convex set containing these points:

(1) $\quad p = \mu_0 p_0 + \dots + \mu_r p_r, \quad \mu_i \geq 0, \quad \sum \mu_i = 1$

Allowing $\mu_i$'s negative get $\mathcal{T}(\sigma)$, Edges are 1-faces of $\sigma$. The simplex is (1) (with) $\sum \mu_i$'s <u>independent</u>*. Then $\dim(\sigma) = r$, expression (1) unique, $\mu_i$ the barycentric co-ordinates. Write $\sigma = p_0 \cdots p_r$. $\sigma$ cell then

(2) $p_\sigma = \alpha p_0 + \dots + \alpha p_k$, $\alpha = \frac{1}{k+1}$, center.

Let $f : \underset{\underset{E}{\cap}}{Q} \to E'$ affine, $f : Q^{k+1} \to E'$

$\left( f(\sum \alpha_i p_i) = \sum \alpha_i f(p_i) \quad (\sum \alpha_i = 1,\ p_i, \sum \alpha_i p_i \in Q) \right)$.

Let $f$ an affine map of the $r$-simplex $\sigma = p_0 \dots p_r$ into $E$; then $f(\sigma)$ is an $r$-simplex iff $q_i = f(p_i)$ independent iff $J_p \neq 0$. Note that diameter$(\sigma)$ = length of longest edge of $\sigma$.

- Polyedra, Complexes.

Polyedron $Q$ is finite union of cells. Its dimension is the largest dim. of the cell involved. If all cells of same dimension then $Q$ is polyhedral region

A complex $K$ is a finite set $S$ of cells such that: If $\sigma \in K \Rightarrow$ each face of $\sigma$ is the union of cells of $S$. No $\mathrm{int}(\sigma)$ intersects cells of lower dim. of $S$. Each $\sigma \cap \sigma'$ is the union of cells of $S$. Also each face of a cell $\sigma$ of $S$ is itself a cell of $S$ (no further subdivisions). The union of cells of $K$ form a polyedron, which we call $K$. Note that each point of $K$ is exactly in one of the sets $\mathrm{int}(\sigma)$, $\sigma \in S$.

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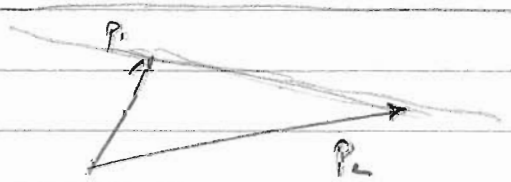
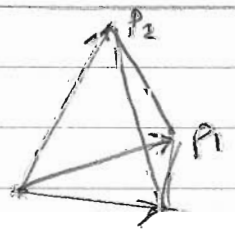



2-simplex




3-simplex

$p_1, p_2, \dots$ <u>independent</u>