HETEROCLINIC TRAVELLING WAVES OF GRADIENT DIFFUSION SYSTEMS

NICHOLAS D. ALIKAKOS AND *NIKOLAOS I. KATZOURAKIS*

Abstract. We establish existence of travelling waves to the gradient system $u_t = u_{zz} - \nabla W(u)$ connecting two minima of W when $u : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{R}^N$, that is, we establish existence of a pair $(U, c) \in [C^2(\mathbb{R})]^N \times (0,\infty)$, satisfying

$$U_{xx} - \nabla W(U) = -c\, U_x$$
$$U(\pm\infty) = a^{\pm},$$

where $a^{\pm}$ are local minima of the potential $W \in C^2_{\text{loc}}(\mathbb{R}^N)$ with $W(a^-) < W(a^+) = 0$ and $N \geq 1$. Our method is variational and based on the minimization of the functional $E_c(U) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|U_x|^2 + W(U) \right\} e^{cx} dx$ in the appropriate space setup. Following Alikakos-Fusco [A-F], we introduce an artificial constraint to restore compactness and force the desired asymptotic behavior, which we later remove. We provide variational characterizations of the travelling wave and the speed. In particular, we show that $E_c(U) = 0$.

1. Introduction

Assume we are given a potential $W \in C^2_{\text{loc}}(\mathbb{R}^N)$ with several local minima, in general at *different levels*. Let a^+, a^- be local minima with $W(a^+) = 0$, $W(a^-) < 0$. We consider the problem of existence of a solution (U, c) to the system

$$\text{(1)} \qquad \begin{cases} U_{xx} - \nabla W(U) = -c\, U_x \\ U(\pm\infty) = a^{\pm} \end{cases}$$

where $c > 0$ and $U : \mathbb{R} \longrightarrow \mathbb{R}^N$ is in $[C^2(\mathbb{R})]^N$ connecting $a^{\pm}$, the dimension being any $N \geq 1$. A typical potential with two minima and $N = 2$ is shown in Fig. 1. Solutions of problem (1) are known as *heteroclinic travelling waves*. They are special solutions of the form $U(z-ct) = u(z,t)$ to the diffusion system with gradient structure:

$$\text{(2)} \qquad u_t = u_{zz} - \nabla W(u)\ , \quad u = u(z,t) : \ \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{R}^N,$$

and in addition heteroclinic connections of the dynamical system corresponding to the 2nd order ODE system $U_{xx} - \nabla W(U) = -c\, U_x$. Physically, problem (1) can be interpreted as the Newtonian Law of motion with force term $-\nabla(-W)$ due to the potential $-W$ and dissipation (friction) term $-cU_x$. In this context, $U(x)$ represents the trajectory of an ideal unit mass particle going from a global maximum to an

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other local maximum of $-W$, asymptotically in time.

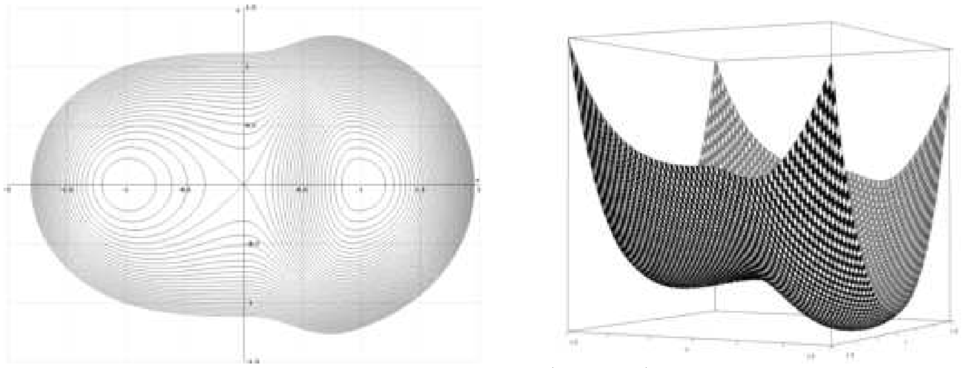


Fig. 1: Simulation of the standard 2-well W deformed (exmpl. 35), having minima at different levels

Problem (1) with $c = 0$ is a special case known as the "standing wave" heteroclinic connection problem. It reduces to a Hamiltonian system $U_{xx} = \nabla W(U)$ for a potential with minima at the *same level*. This case for general $N > 1$ has been studied by Sternberg in [St], Alikakos-Fusco in [A-F] and in great detail for $N = 2$ by Alikakos, Betelú, Chen in [A-Be-C].

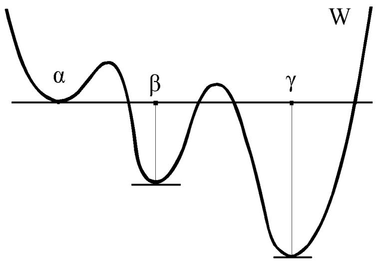


Fig. 2: In general, no α - γ connection exists

The scalar case $N = 1$ and $c > 0$ of (1) is textbook material from the viewpoint of existence (e.g [He] p. 128, [Ev], p. 175). The global stability of the connection for the scalar case of (2) has been studied in the classical papers of Fife and McLeod [F-McL], [F-McL2] and recently by Gallay and Risler in [G-R]. Already in the scalar case, existence for (1) of an heteroclinic between two minima is not always guaranteed in the presence of a third one, as it has been observed in [F-McL] (Fig. 2).

In the vector case $N > 1$ and $c \neq 0$ for (1) maximum and comparison principles are no longer available and as a result only special systems have been studied. We refer to the monograph of Volperts' [V] for monotone systems and numerous related references.

In the very recent paper E. Risler [R] has established existence of solutions to (1), as a byproduct of his study of the parabolic semiflow of (2). Among other results, Risler studies the case of a bistable potential and proves the existence of a travelling wave connecting the global minimum of W with a local minimum, as in the present paper. However, his hypotheses are more restrictive than our (h^{**}) (Sec. 8), which shows the advantage of the Direct Method we utilize.

Another very recent paper that establishes existence of travelling waves, actually for a generalization of (1) is Lucia-Muratov-Novaga [LMN]. Their method has similarities with ours, but their hypotheses are different and not directly addressing the potential W.

In the present paper we choose to work directly with the time independent problem (1). We prove existence of heteroclinic travelling waves for potentials with several minima under weak coercivity requirements which allow for potentials unbounded from below. We establish connections between possibly degenerate minima, imposing assumptions only on the geometry of the sublevel set $\big\{W \leq \alpha\big\} \subseteq \mathbb{R}^N$ for $\alpha > 0$ small, which encloses the minima (assumptions (h^*) in Sec. 6, (h^{**}) in Sec. 8).

Our approach is variational: we introduce a *weighted action* functional, an idea already introduced in Fife-McLeod ([F-McL], [F-McL2]), to obtain travelling wave solutions to (1) as (local) minimizers of the weighted action

$$E_c(U) = \int_{\mathbb{R}} \left\{ \frac{1}{2}\big|U_x\big|^2 + W(U) \right\} e^{cx} dx \tag{3}$$

in the *Fréchet space* of vector functions $[H^1_{\rm loc}(\mathbb{R})]^N$, utilizing certain devices to overcome the unboundedness and compactness problems of E_c. We show that action-minimizing travelling waves (U, c) are characterized by the property $E_c(U) = 0$ and they can be derived as solutions to

$$E_c(U) = \inf \left\{ E_c(V) :\ V \in [H^1_{\rm loc}(\mathbb{R})]^N,\ V(\pm\infty) = a^{\pm} \right\}, \qquad E_c(U) = 0. \tag{4}$$

We now give a brief description of our method. A formal computation shows that critical points of E_c correspond to weak solutions of (1). We wish to construct solutions of $U_{xx} - \nabla W(U) = -c\, U_x$, with the desired behavior at infinity $U(\pm\infty) = a^{\pm}$, by minimizing (3), in the appropriate setup. Minimization can not be done directly, because the unbounded domain $\mathbb{R}$ excludes strong compactness in all reasonable functional spaces, while the asymptotic behavior required in (1) can not be guaranteed.

In addition, (3) is *not* generally *bounded* from below for all $c > 0$, a difficulty not present when $c = 0$, and moreover it is sensitive to translations: $E_c\big(U(\cdot - \delta)\big) = e^{c\delta} E_c(U)$. Thus, a minimizing sequence may converge to the trivial minimizers $a^{\pm}$ with $E_c(a^+) = 0$, $E_c(a^-) = -\infty$.

To overcome these problems, we first solve a constrained minimization problem, utilizing the *unilateral constraint method* introduced by Alikakos and Fusco in [A-F]: we fix 2 arbitrary parameters $c, L > 0$ and we minimize E_c directly within the *admissible set* of functions in $[H^1_{\rm loc}(\mathbb{R})]^N$ whose graph lies in the cylinders $(-\infty, -L] \times \mathbb{B}(a^-, r_0)$ and $[L, +\infty) \times \mathbb{B}(a^+, r_0)$ enclosing the 2 minima $a^{\pm}$ to be connected. Minimization leads to a 2-parameter family of minimizers in $c, L > 0$. Then L is increased with the hope that the constraint is not realized for some minimizer, thus solving the Euler-Lagrange equation (1) for some specific value of the other parameter $c = c^* > 0$.

This device bounds from below (3), and allows us to "capture" an object which is close to a solution to (1). Constrained minimizers are *piecewise solutions* (except possibly at the rims $\{\pm L\} \times \partial\big(\mathbb{B}(a^{\pm}, r_0)\big)$) converging asymptotically to $a^{\pm}$, for all $c > 0$. The main effort in the proof is devoted to showing that the constraint is in fact *not* realized for a specific $c^* > 0$ and for sufficiently large L.

The role of "c" is as follows. We incorporate into E_c an *arbitrary parameter* $c > 0$ which, until Sec. 6, is always *arbitrary and fixed.* In particular, we do *not* view c as a functional $c(U)$ of U. The specific $c = c^*$ which guarantees existence is determined by the requirement that $E_{c^*}(U_L) = 0$ for sufficiently large $L \geq L^*$. This is necessary for existence of minimizers since translation sensitivity of (3) shows that the only possible *finite* infimum of (3) is zero. A more transparent characterization was pointed out by the referee and is as follows. First look for the smallest possible value $c > 0$ for which (3) is bounded form below over $\{U \in [H^1_{\rm loc}(\mathbb{R})]^N : U(\pm\infty) = a^\pm\}$. Then, for that c construct the travelling wave by minimizing (3). A nice consequence of this is a uniqueness property of the speed for minimizing travelling waves.

The paper is organized as follows. In Sec. 2 we solve the constraint problem for E_c in $[H^1_{\rm loc}(\mathbb{R})]^N$, resulting to a 2-parameter family of minimizers in $c > 0$ and $L > 0$. In Sec. 3, assuming a very mild local monotonicity (h) near the minima $a^\pm$, we show that constrained minimizers are piecewise solutions to $U_{xx} - \nabla W(U) = -c\ U_x$, solving it on $\mathbb{R} \setminus \{\pm L\}$ and converging to $a^\pm$ at $\pm\infty$.

In Sec. 4 we introduce the main tool for removing the constraint, two local replacement lemmas, modeled after Lemmas 3.3, 3.4 in [A-F]. The new ingredient is the introduction of a convex set in the place of a ball, which allows controlling the solution far from the minima. The presentation here is self-contained independent of the rest of the paper.

In Sec. 5 we establish certain energy identities. In particular, they imply an energy equipartition at $+\infty$ and that $E_c(U_L)$ measures the jumps $[[(U_L)_x]]\big|_{\pm L}$.

In Sec. 6 we introduce a global assumption (h^*) and determine the speed c^* of the travelling wave. c^* is defined by means of a variational formula (see (27)) which is similar to a formula of Heinze [Hei]. Utilizing tools from Sec. 4, 5, we prove that c^* satisfies the desired properties (Proposition 25). Hence, we distinguish the *suitable* E_{c^*} among all $\big\{E_c : c > 0\big\}$. The variational formulation (4) which implies existence for (1) is also given here.

In Sec. 7 we prove existence of solution by removing the constraint and derive explicit bounds on $c^* \in [c_{\min}, c_{\max}]$, by means of our variational formulation (4).

In Sec. 8 we show that the assumption (h^*) can be relaxed to include potentials that are unbounded from below or have other critical points besides $a^\pm$ (cf. [A-F]). Finally, in the Appendix we discuss the optimality of our assumptions.

Our proof includes the special case $W(a^-) = W(a^+) = 0$, $c = 0$ that was treated in [A-F].

2. The Constrained Minimization Problem

Here we solve a minimization problem for $E_c(U) = \int_{\mathbb{R}} \Big\{\frac{1}{2}\big|U_x\big|^2 + W(U)\Big\} e^{cx} dx$ in the local Sobolev space $[H^1_{\rm loc}(\mathbb{R})]^N$ of vector $U : \mathbb{R} \longrightarrow \mathbb{R}^N$. $[H^1_{\rm loc}(\mathbb{R})]^N$ admits a Fréchet topology, defined by the seminorms of $[H^1(-m, m)]^N$, $m \geq 1$. Technically, instead of $[H^1_{\rm loc}(\mathbb{R})]^N$ we use its isomorphic copy $[H^1_{\rm loc}(\mathbb{R}, e^{c\mathrm{Id}})]^N$ with weight $x \mapsto e^{cx}$, the standard Lebesgue measure dx being replaced by the absolutely continuous $e^{cx}dx$. It is only a matter of convenience, since minimization gives derivatives bounded in $[L^2(\mathbb{R}, e^{c\mathrm{Id}})]^N$. $C^k_{\rm loc}(\mathbb{R}^N)$ will denote the space of C^k functions equipped with the Fréchet topology of uniform convergence together with all the derivatives over compacts, while $C^k(\mathbb{R}^N)$ denotes the bounded C^k functions with its standard norm. We shall frequently decompose W as $W = W^+ - W^-$, where $W^+ = \max\{W, 0\}$ and $W^- = \max\{-W, 0\}$.

Lemma 1. ***(Characterization of the speed)*** *Assume that a solution (U,c) to (1) exists, satisfying $U_x(\pm\infty)=0$ up to sequences. Then:*

$$W^-(a^-) \;=\; c\int_{\mathbb{R}} \big|U_x\big|^2 dx \qquad \& \qquad c\big(a^+ - a^-\big) \;=\; \int_{\mathbb{R}} \nabla W\big(U\big)dx.$$

Proof of Lemma 1. The equation readily implies $-U_{xx}\cdot U_x \;+\; \nabla W(U)\cdot U_x = c\big|U_x\big|^2$. Hence,

$$\begin{aligned} c\int_{\mathbb{R}} \big|U_x\big|^2 dx \;&=\; -\int_{\mathbb{R}} \Big(\frac{1}{2}\big|U_x\big|^2\Big)_x dx \;+\; \int_{\mathbb{R}} \big(W(U)\big)_x dx \\ &=\; \pm\, 0 \;+\; W(U(+\infty)) \;-\; W(U(-\infty)) \\ &= -\, W(a^-). \end{aligned}$$

Moreover, again from the equation we have

$$\begin{aligned} \int_{\mathbb{R}} \nabla W\big(U\big)dx \;&=\; \int_{\mathbb{R}} \Big(U_{xx} \;+\; cU_x\Big)dx \\ &=\; 0 -\; 0 \;+\; c\big(a^+ - a^-\big). \quad \square \end{aligned}$$

As a consequence of Lemma 1, if $U(\pm\infty)=a^\pm$ and $W(a^+)=0>W(a^-)$, then c must be positive.
Take now $L>0$ and $r_0>0$ small, such that $W(u)\geq 0$ for $|a^+-u|\leq r_0$ and $W(u)<0$ for $|a^- - u|\leq r_0$. We introduce the constraint sets:

$$\begin{aligned} \mathcal{X}_L^+ \;&:= \Big\{U\in [H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N \;:\; |U(x)-a^+|\leq r_0,\; x\geq +L\Big\}, \\ \mathcal{X}_L^- \;&:= \Big\{U\in [H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N \;:\; |U(x)-a^-|\leq r_0,\; x\leq -L\Big\}, \end{aligned}$$

and set $\mathcal{X}_L := X_L^+ \bigcap \mathcal{X}_L^-$. Pointwise values make sense by means of the imbedding $[H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N \hookrightarrow [C^0_{\mathrm{loc}}(\mathbb{R})]^N$.

Theorem 2. ***(Existence of Constrained Minimizers)*** *Let W be a potential in $C^2_{loc}(\mathbb{R}^N)$ and $a^\pm$ two of its local minima, with $W(a^-)<0=W(a^+)$, and a^- its global minimum. We assume that $W^{-1}\big([W(a^-),0]\big)$ is compact in $\mathbb{R}^N$. If $L>0$, $c>0$ are fixed parameters, then the minimization problem*

$$E_c(U_L) \;=\; \inf_{\mathcal{X}_L}\, \big\{E_c\big\}$$

has a solution U_L in $\mathcal{X}_L \subseteq [H^1_{loc}(\mathbb{R}, e^{cId})]^N$.

The assumption on W implies $\liminf_{|u|\to\infty}\big[W(u)\big]\geq 0$. This will be relaxed in the sequel, allowing for potentials with several local minima and possibly unbounded negative values, by means of a localization. We denote the minimizers of E_c into $\mathcal{X}_L$ by U_L instead of the more accurate notation $U_{c,L}$, suppressing the dependence on the parameter $c>0$ which (until Sec. 6) is always fixed.

Proof of Theorem 2. We first show that $\mathcal{X}_L\neq\emptyset$ together with $-\infty<\inf_{\mathcal{X}_L}\big\{E_c\big\}<\infty$. Since we are interested only in increasing the parameter L later, we restrict, as we can, our attention to $L\geq 1$.
Claim. *There exists an affine function $U_{aff}\in\mathcal{X}_L\bigcap[W^{1,\infty}_{loc}(\mathbb{R})]^N$ such that*

$$-\infty \;<\; -\frac{e^{cL}W^-(a^-)}{c} \;\leq\; \inf_{\mathcal{X}_L}\,\big\{E_c\big\} \;\leq\; E_c(U_{aff}) \;<\; \infty.$$

Proof of Claim . Let χ_A denote the characteristic of $A \subseteq \mathbb{R}$. We set

$$U_{aff}(x) \;:=\; a^-\chi_{(-\infty,-1)} \;+\; \left(\frac{1-x}{2}a^- + \frac{1+x}{2}a^+\right)\,\chi_{[-1,1]} \;+\; a^+\chi_{(1,\infty)}.$$

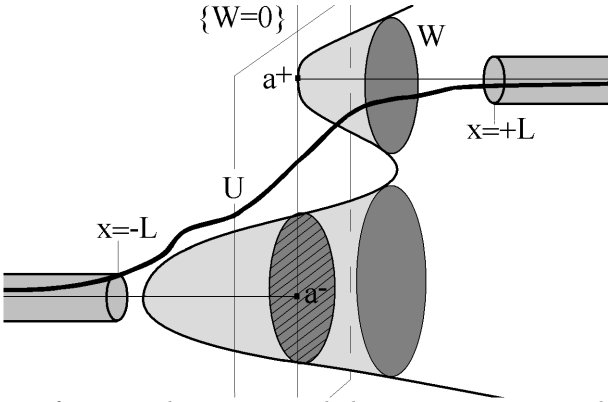


Fig.3 The device of constrained minimization which restores compactness and boundedness

Clearly, $(U_{aff})_x \in [L^\infty_{\mathrm{loc}}(\mathbb{R})]^N$ and exists a.e. on $\mathbb{R}$. Thus, $U_{aff} \in [H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$. We calculate

$$\begin{aligned} E_c(U_{aff}) \;&=\; \int_{-\infty}^{-1}(0+W(a^-))e^{cx}dx \;+\; \int_1^\infty (0+W(a^+))e^{cx}dx \\ &\quad+\; \int_{-1}^1 \left\{\frac{1}{2}\Big|\frac{a^+-a^-}{2}\Big|^2 + W\left(\frac{1-x}{2}a^- + \frac{1+x}{2}a^+\right)\right\}e^{cx}dx \\ &\le\; \int_{-1}^1 \left\{\frac{1}{2}\Big|\frac{a^+-a^-}{2}\Big|^2 \;+\; W^+\left(\frac{1-x}{2}a^- + \frac{1+x}{2}a^+\right)\right\}e^{cx}dx \\ &\quad+\; \frac{1}{c}e^{-c}W(a^-). \end{aligned}$$

Hence, if we set $E_c^+(U) := \int_{\mathbb{R}} \left\{\frac{1}{2}|U_x|^2 + W^+\big(U\big)\right\}e^{cx}dx$, we obtain

$$E_c(U_{aff}) \;\le\; -e^{-c}\frac{W^-(a^-)}{c} \;+\; e^c E_0^+(U_{aff}). \tag{5}$$

This implies the upper bound $\sup_{L\ge1}\inf_{\mathcal{X}_L}\big\{E_c\big\} \le \sup_{L\ge1}E_c(U_{aff}) < \infty$. If U lies in $\mathcal{X}_L$, we have $W^-(U(x)) = 0$ for $x \ge L$ and $W^+(U(x)) = 0$ for $x \le -L$. Hence, for any such U, utilizing that $W^-(U) \le W^-(a^-)$, we have

$$\begin{aligned} E_c(U) =\; &\int_{\mathbb{R}} \left\{\frac{1}{2}\big|U_x\big|^2 + W\big(U\big)\right\}e^{cx}dx \\ =\; &\frac{1}{2}\int_{\mathbb{R}}|U_x|^2e^{cx}dx \;+\; \int_{\mathbb{R}}W^+(U)e^{cx}dx \;-\; \int_{\mathbb{R}}W^-(U)e^{cx}dx \\ \ge\; &-\int_{\mathbb{R}}W^-(U)e^{cx}dx \\ \ge\; &-W^-(a^-)\int_{-\infty}^L e^{cx}dx \;=\; -\frac{W^-(a^-)}{c}e^{cL}. \quad \square \end{aligned}$$

By C^2 regularity of solutions to (1), we may assume that inf $\mathcal{X}_L\big[E_c\big] < E_c(U_{aff})$ strictly. We choose a minimizing sequence $\{U_L^n\}_{n\ge1}$ in $[H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$ such that

$E_c(U^n_L) \longrightarrow \inf_{\mathcal{X}_L}\big\{E_c\big\}$, as $n \to \infty$. The constraints immediately yield

$$\big|U^n_L(x)\big| \ \leq\ \max\big\{|a^+|, |a^-|\big\} \ +\ r_0\ , \qquad x \in\ (-\infty, -L] \cup [L, \infty).$$

Claim. (Uniform Bounds) *There exists a* $C = C(c, L, W) > 0$ *such that*

$$\sup_{n\geq 1}\ \big\|(U^n_L)_x\big\|_{[L^2(\mathbb{R}, e^{cId})]^N}\ \leq\ C\ ,\quad \sup_{n\geq 1}\ \big\|U^n_L\big\|_{[L^\infty(\mathbb{R})]^N}\ \leq\ C.$$

Proof of Claim . For any $x \in [-L, L]$, we have the estimates

$$\begin{aligned}|U^n_L(x)| \ &\leq\ |U^n_L(-L)| \ +\ \int_{-L}^{x}\big|(U^n_L)_t\big|e^{\frac{ct}{2}}e^{\frac{-ct}{2}}dt\\ &\leq \max\big\{|a^+|, |a^-|\big\} \ +\ r_0 \ +\ \left(\int_{-L}^{L} e^{-ct}dt\right)^{\frac{1}{2}}\left(\int_{-L}^{x}\big|(U^n_L)_t\big|^2 e^{ct}dt\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\big|(U^n_L)_x\big|^2 e^{cx}dx \ &\leq\ E_c(U_{aff}) \ -\ \int_{\mathbb{R}} W(U^n_L)e^{cx}dx\\ &\leq\ E_c(U_{aff}) - \int_{\mathbb{R}} W^+(U^n_L)e^{cx}dx \ +\ \int_{\mathbb{R}} W^-(U^n_L)e^{cx}dx\\ &\leq\ E_c(U_{aff}) \ +\ \int_{-\infty}^{L} W^-(U^n_L)e^{cx}dx\\ &\leq\ E_c(U_{aff}) \ +\ \frac{W^-(a^-)}{c}e^{cL}.\end{aligned}$$

We conclude:

$$\frac{1}{2}\big\|(U^n_L)_x\big\|^2_{[L^2(\mathbb{R}, e^{c\mathrm{Id}})]^N}\ \leq\ \frac{W^-(a^-)}{c}e^{cL} \ +\ E_c(U_{aff}).$$

Utilizing that $\big|U^n_L(x)\big| \leq \max\big\{|a^+|, |a^-|\big\} + r_0$ for $x \in (-\infty, -L] \bigcup [L, \infty)$, we get

$$\big\|U^n_L\big\|_{[L^\infty(\mathbb{R})]^N}\ \leq\ \max\big\{|a^+|, |a^-|\big\} + r_0 + \left(\frac{e^{cL} - e^{-cL}}{c}\right)^{\frac{1}{2}}\big\|(U^n_L)_x\big\|_{[L^2(\mathbb{R}, e^{c\mathrm{Id}})]^N}. \quad \square$$

We may now proceed to the existence of the minimizer. By the claim above, $(U^n_L)^\infty_1$ is *bounded in the locally convex sense* in $[H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$, with the derivatives bounded in $[L^2(\mathbb{R}, e^{cId})]^N$:

$$\begin{aligned}\sup\nolimits_{n\geq 1}\big\|U^n_L\big\|_{\left(H^1(I, e^{cId})\right]^N}\ &\leq\ C(c, L, W, I) \ \text{ for all } I \subset\subset \mathbb{R},\\ \sup\nolimits_{n\geq 1}\big\|(U^n_L)_x\big\|_{[L^2(\mathbb{R}, e^{cId})]^N}\ &\leq\ C(c, L, W).\end{aligned}$$

By standard compactness arguments, there is a $U_L \in [H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$ such that up to a certain subsequence $U^n_L \rightharpoonup U_L$ as $n \to \infty$ weakly in $[H^1_{\mathrm{loc}}(\mathbb{R})]^N$ and $U^n_L \longrightarrow U_L$ in $[L^2_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$ and a.e. on $\mathbb{R}$. By weak LSC of the weighted L^2 norm and the

Fatou Lemma for $W(U_L^n) + W^-(a^-)\chi_{(-\infty,L]} \geq 0$[1], we have

$$\int_{\mathbb{R}} \frac{1}{2}|(U_L)_x|^2 e^{cx} dx \ \leq \ \liminf_{n\to\infty} \int_{\mathbb{R}} \frac{1}{2}|(U_L^n)_x|^2 e^{cx} dx,$$

$$\int_{\mathbb{R}} \Big\{ W(U_L) + W^-(a^-)\chi_{(-\infty,L]} \Big\} e^{cx} dx \leq$$

$$\leq \ \liminf_{n\to\infty} \int_{\mathbb{R}} \Big\{ W(U_L^n) + W^-(a^-)\chi_{(-\infty,L]} \Big\} e^{cx} dx.$$

Hence, the theorem follows together with the bounds

$$-\frac{e^{cL}W^-(a^-)}{c} \ \leq \ E_c(U_L) \ \leq \ \lim_{n\to\infty} E_c(U_L^n)$$
$$\leq \ -e^{-c}\frac{W^-(a^-)}{c} \ + \ e^c E_0^+(U_{aff}). \quad \square$$

3. Constrained Minimizers are Piecewise Solutions

We will now prove that the constrained minimizers U_L of Theorem 2 are piecewise solutions in $C^2_{\mathrm{loc}}(\mathbb{R} \setminus \{-L, L\})^N$, while $U_L|_{(-\infty,-L]}$ and $U_L|_{[L,\infty)}$ are inside the cylinders and converge asymptotically to $a^\pm$. Following [A-F], we introduce the following local monotonicity assumption:

(h) There exists an $R_0 > 0$ such that the map $r \mapsto W(a^\pm + r\xi)$ has a strictly positive derivative for every $r \in (0, R_0)$ and every $\xi \in \mathbb{R}^N$, $|\xi| = 1$.

This is a rather weak non-degeneracy assumption, allowing for potentials with degenerate C^∞- *flat* minima. From now on we assume, as we can, that $r_0 < R_0$, hence $\mathbb{B}(a^\pm, r_0)$ are in the monotonicity region. We will need to express U_L in polar form: for any U in $[H^1_{\mathrm{loc}}(\mathbb{R}, e^{c\mathrm{Id}})]^N$, we set $U^\pm(x) := a^\pm + \rho^\pm(x) n^\pm(x)$. Then $|(U^\pm)_x|^2 = ((\rho^\pm)_x)^2 + (\rho^\pm)^2 |n^\pm_x|^2$. For any $I \subseteq \mathbb{R}$ measurable, we shall interpret integrals expressed in polar form as

$$\int_I \big|U_x\big|^2 e^{cx} dx = \int_{I \bigcap \{\rho^\pm > 0\}} \Big\{ (\rho^\pm_x)^2 + (\rho^\pm)^2 |n^\pm_x|^2 \Big\} e^{cx} dx,$$

since the imbedding $[H^1_{\mathrm{loc}}(\mathbb{R})]^N \hookrightarrow [C^0_{\mathrm{loc}}(\mathbb{R})]^N$ implies $|U_x| = 0$ a.e. on the closed sets $U^{-1}(\{a^\pm\})$, even when they have positive measure. For any $\mu < \nu$ in $\mathbb{R}$, we set

$$E_c\big(U, (\mu, \nu)\big) := \int_\mu^\nu \Big\{ \frac{1}{2}\big|U_x\big|^2 + W(U) \Big\} e^{cx} dx.$$

This is the action (3) restricted on $[\mu, \nu]$.

Lemma 3. *(cf.* [A-F]*) Assume W satisfies (h) and $c > 0$ is fixed. Let $a \in \{a^+, a^-\}$ and $U \in [H^1(\mu, \nu)]^N$ with $U = a + \rho n$, and suppose that*
(i) $0 < \rho(\mu) = \rho(\nu) = r \leq R_0$ (R_0 as in (h)*),*
(ii) $r \leq \rho(x) \leq R_0$, for all $x \in (\mu, \nu)$.
Then, there exists a $\widetilde{U} \in [H^1(\mu, \nu)]^N$, $\widetilde{U} = a + \widetilde{\rho} n$, such that $U(\mu) = \widetilde{U}(\mu)$, $U(\nu) = \widetilde{U}(\nu)$ and $\widetilde{\rho}(x) < r$, for all $x \in (\mu, \nu)$ while

$$E_c(\widetilde{U}, (\mu, \nu)) \ < \ E_c(U, (\mu, \nu)).$$

[1] We owe this argument to the referee.

In particular, locally minimizing solutions to $U_{xx} - \nabla W(U) = -c\, U_x$ on $[\mu,\nu]$ attain the maximum value r of their polar radius $\rho^{\pm} = \big|U - a^{\pm}\big|$ only at the endpoints $\{\mu\},\{\nu\}$.

Proof of Lemma 3. We note that the proof of Lemma 3.3 in [A-F] is based on a pointwise deformation and thus it holds generally for functionals of the form $\int(\frac{1}{2}|U_x|^2 + W(U))d\mu(x)$ with μ a positive Radon measure. See Lemma 10 for a similar argument. □

We now prove that in view of (h), the polar radii of U_L are weak subsolutions of the operator $L(\rho) := \rho_{xx} + c\rho_x$ in $[H^1(\mu,\nu)]^N$, for all $\mu < \nu < -L$ and $L < \mu < \nu$. We write $U^{\pm}(x) := a^{\pm} + \rho^{\pm}(x) n^{\pm}(x)$ (cf. Stefanopoulos [Stef]).

Proposition 4. ***(Constrained minimizers as radially weak*** H^1 ***subsolutions)*** *The minimizers* U_L *of Theorem 2 satisfy*

$$-\big(\rho_L^{\pm}\big)_{xx} - c\big(\rho_L^{\pm}\big)_x + \rho_L^{\pm}\big|(n_L^{\pm})_x\big|^2 + \nabla W\big(a^{\pm} + \rho_L^{\pm} n_L^{\pm}\big)\cdot n_L^{\pm} \le 0,$$

weakly in $H^1_{loc}\big((L,\infty)\bigcap\{\rho_L^{+} > 0\}\big)$ *and* $H^1_{loc}\big((-\infty,-L)\bigcap\{\rho_L^{-} > 0\}\big)$. *In particular, if* W *satisfies* (h), *we obtain*

$$\big(\rho_L^{\pm}\big)_{xx} + c\big(\rho_L^{\pm}\big)_x \ge 0.$$

Proof of Proposition 4. We construct local variations that do not violate the constraint $\rho_L^{\pm} \le r_0$. For definiteness we consider the case $a = a^-$, the other is similar. We take $\phi(x) := \theta(x) n_L^-(x)$, with θ in $C_c^{\infty}(-\infty,-L)$ and consider one-sided variations of the form

$$U_L^{\varepsilon}(x) := U_L(x) - \varepsilon\phi(x) = a^- + \big(\rho_L^-(x) - \varepsilon\theta(x)\big) n_L^-(x)$$

(radially inside the cylinder) which satisfy the constraint for small $\varepsilon \in [0,\varepsilon_\phi]$. Since U_L is a minimizer of E_c, $E_c(U_L^{\varepsilon}) \ge E_c(U_L)$, for all $\varepsilon \in [0,\varepsilon_\phi]$. Consequently,

$$\lim_{\varepsilon\to 0^+}\Big[\frac{1}{\varepsilon}\Big(E_c(U_L^{\varepsilon}) - E_c(U_L)\Big)\Big] \ge 0.$$

We calculate, using that $\mathrm{supp}(\theta) \subseteq (-\infty,-L)$,

$$\begin{aligned} E_c(U_L^{\varepsilon}) \;=\; & \int_{-\infty}^{-L}\Bigg\{\frac{1}{2}\big((\rho_L^-)_x - \varepsilon\theta_x\big)^2 \;+\; \frac{1}{2}\big(\rho_L^- - \varepsilon\theta\big)^2\big|(n_L^-)_x\big|^2 \\ & \;+\; W\big(a^- + (\rho_L^- - \varepsilon\theta) n_L^-\big)\Bigg\} e^{cx} dx \\ & \;+\; \int_{-L}^{\infty}\Big\{\frac{1}{2}\big|(U_L)_x\big|^2 + W(U_L)\Big\}\, e^{cx} dx. \end{aligned}$$

Taking one-sided $\frac{d}{d\varepsilon}\big|_{\varepsilon=0^+}$, we get

$$\int_{-\infty}^{-L}\Big\{-(\rho_L^-)_x(\theta_x e^{cx}) - \Big[\rho_L^-\big|(n_L^-)_x\big|^2 + \nabla W\big(a^- + \rho_L^- n_L^-\big)\cdot n_L^-\Big](\theta e^{cx})\Big\}\, dx \;\ge\; 0.$$

We write $\theta_x e^{cx} = (\theta e^{cx})_x - c\theta e^{cx}$ and substitute to get

$$\int_{-\infty}^{-L}\Bigg\{\big(\rho_L^-\big)_x\big(\theta e^{cx}\big)_x + \Big[\rho_L^-\big|(n_L^-)_x\big|^2 - \nabla W\big(a^- + \rho_L^- n_L^-\big)\cdot n_L^-\Big](\theta e^{cx})\Bigg\} dx \;\le\; 0.$$

We are done, since the multiplication operator $M_{e^{c\mathrm{Id}}}$ is a Fréchet automorphism on the dense subspace $C_c^{\infty}\big(-\infty,-L\big)$ of $H^1_{\mathrm{loc}}\big(-\infty,-L\big)$. □

It is now straightforward that all $(U_L)_{L\geq 1}$ realize the constraint at most at the rims of the cylinders.

Proposition 5. ***(Contact at most at the rims of the cylinders)*** *If* W *satisfies* (h), *then*
a) If $x_L^+ := \inf\{t\in\mathbb{R} : \rho_L^+ \leq r_0 \text{ on } [t,+\infty)\}$, *then we have* $\rho_L^+ < r_0$ *on* $(x_L^+,+\infty)$.
b) If $x_L^- := \sup\{t\in\mathbb{R} : \rho_L^- \leq r_0 \text{ on } (-\infty,t]\}$, *then we have* $\rho_L^- < r_0$ *on* $(-\infty,x_L^-)$.

Proof of Proposition 5. We drop sub/superscripts $L,\pm$ for ρ and prove only a), since b) is analogous. By definition, $x_L^+\in(-L,L]$ and it is the time at which U_L enters $\mathbb{B}(a^+,r_0)$ and remains inside it for all later times. Minimizers U_L are, by (h), radially weak H^1 subsolutions: $\rho_{xx}+c\rho_x\geq 0$. Let $x_0\in(x_L^+,\infty)$ be such that $\rho(x_0)=r_0$. Since the point x_0 lies in the interior of $[x_L^+,x_0+1]$, by the Strong Maximum Principle for weak C^0 subsolutions ([G-T]), we have that either $\rho(x_0)<r$, or $\rho\equiv r_0$ on $[x_L^+,x_0+1]$. Lemma 3 implies that ρ is not identically r_0, otherwise we obtain a contradiction to minimality of U_L. Hence, $\rho<r_0$ on $(x_L^+,+\infty)$. □

Proposition 6. ***(Constrained minimizers are piecewise solutions)*** *All* U_L *are solutions to* $U_{xx}-\nabla W(U)=-c\,U_x$ *in* $[C^2_{loc}(\mathbb{R}\setminus\{x_L^\pm\})]^N\bigcap[C^0(\mathbb{R})]^N$. *They are in* $[C^2_{loc}(\mathbb{R})]^N$ *except possibly when* $x_L^\pm=\pm L$.

Proof of Proposition 6. By Proposition 5, $|U_L(x)-a^\pm|<r_0$, for all $x\in\mathbb{R}\setminus[x_L^-,x_L^+]$. Take any point $x^*\in\mathbb{R}\setminus\{x_L^-,x_L^+\}$. By continuity, there exists an $\varepsilon_0>0$ and a compact tubular neighborhood $\{\mathbb{B}(U_L(x),\varepsilon_0)\ :\ x\in[x^*-\delta,x^*+\delta]\}$ of the graph of U_L not intersecting the boundary of the constraint cylinders, the assertion being trivial when $x^*\in(x_L^-,x_L^+)$. This holds for $x_L^\pm$ as well, when $x_L^+<L$ and $x_L^->-L$. We take variations of U_L the $U_L^\varepsilon:=U_L-\varepsilon\phi$, $|\varepsilon|\leq\varepsilon_0$ small, for all ϕ in $[C_c^\infty(x^*-\delta-\varepsilon_1,x^*+\delta+\varepsilon_1)]^N$, $\varepsilon_1>0$ small, whose restriction on $(x^*-\delta,x^*+\delta)$ is dense in $[H^1(x^*-\delta,x^*+\delta)]^N$. Using that $\phi_x e^{cx}=(\phi e^{cx})_x-c\phi e^{cx}$, we easily get that $U_{xx}-\nabla W(U)=-c\,U_x$ is solved weakly. Since $\nabla W\in[C^1_{\text{loc}}(\mathbb{R}^N)]^N$ and $(U_L)_x\in[L^2_{\text{loc}}(\mathbb{R})]^N$, there exists $(U_L)_{xx}\in[L^2_{\text{loc}}(\mathbb{R}\setminus\{x_L^\pm\})]^N$ and therefore $U_L\in[C^1_{\text{loc}}(\mathbb{R}\setminus\{x_L^\pm\})]^N$ which gives that $U_L\in[C^2_{\text{loc}}(\mathbb{R}\setminus\{x_L^\pm\})]^N$, since $\nabla W\in[C^1_{\text{loc}}(\mathbb{R}^N)]^N$. □

Remark 7. (i) **(Polar form of the equation)** Write the equation $U_{xx}-\nabla W(U)=-c\,U_x$ in polar coordinates $U_L=a^\pm+\rho_L^\pm n_L^\pm$ and multiply by $n_L^\pm$ to get that the polar radii $\rho_L^\pm$ of U_L satisfy the equation

$$(\rho)_{xx}\ +\ c(\rho)_x\ =\ \rho|n_x|^2\ +\ \nabla W(a^\pm+\rho n)\cdot n. \tag{6}$$

(ii) **(Energy formula)** Integrating once the equation as in the proof of Lemma 1, we get the formula

$$c\int_\mu^\nu \left|U_x\right|^2 dx=\left(W(U)-\frac{\left|U_x\right|^2}{2}\right)\Bigg|_\mu^\nu, \tag{7}$$

on any interval $[\mu,\nu]$, on which U solves $U_{xx}-\nabla W(U)=-c\,U_x$ classically.

Proposition 8. ***(Asymptotic behavior of constrained minimizers)*** *If* W *satisfies* (h), *then* $U_L(x)\longrightarrow a^\pm$ *as* $x\to\pm\infty$. *Moreover, the polar radii* $\rho_L^\pm$ *of* U_L *are eventually strictly monotone inside the cylinders and also* $(U_L)_x(\pm\infty)=0$ *at least up to sequences.*

Proof of Proposition 8. We treat both cases together, dropping indices $\pm, L$ of ρ.

Claim 1. *The polar radii are eventually strictly monotone in the cylinders.*
Indeed, by Lemma 3 and the action minimality of U_L, ρ can not be identically constant on any subinterval of $(-\infty, x_L^-)$ (x_L^+, ∞). Hence, by continuity of ρ the set of critical points $A := \{\rho_x = 0\}$ is discrete. Since ρ solves $\rho_{xx} + c\rho_x \geq 0$, the Maximum Principle implies that A does not contain maximum points. Moreover, A can not contain more than one minimum point; if a minimum point exists, then at all latter points (in the unbounded direction of time) ρ_x preserves its sign on both sides of the critical point. Hence, ρ is eventually strictly monotone.

Let now r^* denote the asymptotic limit of ρ. At $+\infty$ it readily follows that $r^* = 0$, since $e^{c\mathrm{Id}}W(U_L)$ is in $L^1(L, \infty)$. Indeed,

$$\begin{aligned}\int_L^\infty W^+(U_L)e^{cx}dx \;&\leq\; E_c(U_{aff}) \;+\; \int_{-\infty}^L W^-(U_L)e^{cx}dx\\ &\leq\; E_c(U_{aff}) \;+\; \frac{W^-(a^-)e^{cL}}{c} \;<\; \infty\end{aligned}$$

and a^+ is the only zero of W inside the ball $\mathbb{B}(a^+, r_0)$. Now we consider the limit at $-\infty$.

Claim 2. *For any $t \in \mathbb{R}$ such that $[t, t+1] \subseteq (-\infty, x_L^-)$, we have*

$$0 \;\leq\; \min_{\substack{t \leq s \leq t+1\\ |\xi|=1}} \Big[\nabla W\big(a^- + \rho(s)\xi\big)\cdot\xi\Big] \;\leq\; \rho_x(t+1)e^c \;-\; \rho_x(t). \tag{8}$$

Indeed, since $U_{xx} - \nabla W(U) = -c\, U_x$ is solved by U_L on $(-\infty, x_L^-)$, we integrate once the e^{cx} - multiple of equation (6) on $[t, t+1]$ to find

$$\begin{aligned}\int_t^{t+1}\big(\rho_x e^{cx}\big)_x dx \;&=\; \int_t^{t+1} e^{cx}\Big(\nabla W(a^- + \rho n)\cdot n + \rho\big|n_x\big|^2\Big)dx\\ &\geq\; e^{ct}\int_t^{t+1}\Big(\nabla W(a^- + \rho n)\cdot n + \rho\big|n_x\big|^2\Big)dx\\ &\geq\; e^{ct}\int_t^{t+1}\nabla W(a^- + \rho n)\cdot n\; dx\\ &\geq\; e^{ct}\min_{s\in[t,t+1]}\Big[\nabla W(a^- + \rho(s)n(s))\cdot n(s)\Big]\\ &\geq\; e^{ct}\min_{\substack{t \leq s \leq t+1\\ |\xi|=1}}\Big[\nabla W\big(a^- + \rho(s)\xi\big)\cdot\xi\Big].\end{aligned}$$

Utilizing assumption (h), we obtain (8).

Since the limit of ρ at $-\infty$ exists, there exists a sequence $x_n \longrightarrow -\infty$ such that $\rho_x(x_n) \longrightarrow 0$. Suppose first that eventually $\rho_x \geq 0$. By setting $t := x_n - 1$ in (8) and employing the monotonicity of ρ, we have

$$0 \;\leq\; \min_{|\xi|=1}\Big[\nabla W\big(a^- + \rho(x_n - 1)\xi\big)\cdot\xi\Big] \;\leq\; \rho_x(x_n)e^c.$$

By employing that $\rho(x_n - 1) \longrightarrow r^*$ and that $\rho_x(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, in the limit we obtain $\nabla W\big(a^- + r^*\xi\big)\cdot\xi = 0$ for some ξ. Since a^- is the only critical point in

$\mathbb{B}(a^-, r_0)$, it follows that $r^* = 0$. Similarly, if $\rho_x \leq 0$, we take $t := x_n$ to get

$$0 \ \leq \ \min_{|\xi|=1}\Big[\nabla W\big(a^- + \rho(x_n)\xi\big)\cdot\xi\Big] \ \leq \ \big|\rho_x(x_n)\big|$$

and again by passing to the limit as $n \longrightarrow \infty$ it follows that $r^* = 0$.
Now we consider the convergence of the derivative. By multiplying (6) by ρ and adding $(\rho_x)^2$, we obtain the identity

$$\big|U_x\big|^2 \ + \ \rho\nabla W(U)\cdot n \ = \ \frac{1}{2}\Big[(\rho^2)_{xx} \ + c\,(\rho^2)_x\Big]. \tag{9}$$

Since ρ^2 is also strictly increasing and has a limit at $-\infty$, we get $(\rho^2)_x \geq 0$ and that there exists a sequence $\xi_n \longrightarrow -\infty$ such that $(\rho^2)_x(\xi_n) \longrightarrow 0$. By (9), assumption (h) and integration on $[\xi_n - 1, \xi_n]$, we get

$$\begin{aligned}0 \ \leq \ \int_{\xi_n-1}^{\xi_n}\big|U_x\big|^2dx \ &\leq \ \frac{1}{2}\Big[(\rho^2)_x(\xi_n) - (\rho^2)_x(\xi_n-1)\Big] \ + \frac{c}{2}\Big[\rho^2(\xi_n) - \rho^2(\xi_n-1)\Big]\\ &\leq \ \frac{1}{2}\Big[(\rho^2)_x(\xi_n) \ + \ c\,\rho^2(\xi_n)\Big] \ \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow \infty$. The proof is complete. □

We conclude this section by proving that $(U_L)_x \in [L^2(\mathbb{R})]^N$, but not L-uniformly. In addition, U_L satisfies the first formula of Lemma 1 approximately, up to some additional terms which *relate c with the jump of $(U_L)_x$ at the rims.*

Proposition 9. ***(Approximate relation for** c**)*** *The 1-sided derivatives $(U_L)_x(\pm L^\pm)$ of U_L exist, and*

$$\begin{aligned}c\int_{\mathbb{R}}\big|(U_L)_x\big|^2dx \ = \ W^-(a^-) \ &+ \ \frac{1}{2}\Big(\big|(U_L)_x(-L^+)\big|^2 - \big|(U_L)_x(-L^-)\big|^2\Big)\\ &+ \ \frac{1}{2}\Big(\big|(U_L)_x(+L^+)\big|^2 - \big|(U_L)_x(+L^-)\big|^2\Big).\end{aligned}$$

In particular, $(U_L)_x \in [L^2(\mathbb{R})]^N$.

Proof of Proposition 9. Proposition 6 assures that we can apply formula (7) on $(-\infty, -L-\varepsilon)$, $(-L+\varepsilon, L-\delta)$ and $(L+\delta, \infty)$ for $\varepsilon, \delta > 0$ small utilizing by 8 the asymptotic behavior of U_L's and the continuity of W. We obtain three relations on these intervals. Utilizing Hölder's inequality, we easily find

$$\big|(U_L)_x(-L-\varepsilon)\big| \leq \sqrt{2}\Big(W(U_L(-L-\varepsilon)) \ + \ W^-(a^-)\Big)^{\frac{1}{2}},$$

$$\begin{aligned}\big|(U_L)_x(-L+\varepsilon)\big| \leq \sqrt{2}\Bigg(&ce^{c(L-\varepsilon)}\int_{-L+\varepsilon}^{L-\delta}\big|(U_L)_x\big|^2e^{cx}dx \ - \ W(U_L(+L-\delta))\\ &+ \ W(U_L(-L+\varepsilon)) \ + \ \frac{1}{2}\big|(U_L)_x(+L-\delta)\big|^2\Bigg)^{\frac{1}{2}},\end{aligned}$$

$$\big|(U_L)_x(+L-\delta)\big| \leq \sqrt{2}\Bigg(W(U_L(L-\delta)) - W(U_L(-L+\varepsilon)) + \frac{1}{2}\big|(U_L)_x(-L+\varepsilon)\big|^2\Bigg)^{\frac{1}{2}},$$

$$\big|(U_L)_x(+L+\delta)\big| \leq \sqrt{2}\Bigg(ce^{-c(L+\delta)}\int_{L+\delta}^{\infty}\big|(U_L)_x\big|^2e^{cx}dx \ + \ W(U_L(L+\delta))\Bigg)^{\frac{1}{2}}.$$

Letting $\varepsilon \longrightarrow 0^+$ and $\delta \longrightarrow 0^+$ separately, we obtain that the moduli of the one-sided limits exist, but may differ. Adding these relations and letting $\varepsilon,\ \delta \longrightarrow 0^+$ we obtain the formula for c. $\square$

4. The Local Replacement Lemmas.

We recall some basics from Differential Geometry. The canonical coordinates (p, d) on $\mathbb{R}^N$ with respect to a C^2 convex set $\mathcal{C} \subseteq \mathbb{R}^N$ are defined by

$$u \ =: \ p \ + \ dn \tag{10}$$

where p is the projection on the convex set $\mathcal{C}$, $0 \in \mathcal{C}$, d the signed distance from $\partial\mathcal{C}$ and n the outward unit normal of $\partial\mathcal{C}$. The latter is parameterized by the C^2 local coordinates

$$\mathbb{R}^{N-1} \ni s \ = \ (s_1, ..., s_{N-1}) \ \mapsto \ p(s_1, ..., s_{N-1}) \in \partial\mathcal{C}.$$

We may assume that the set of vectors

$$\frac{\partial p}{\partial s_i} \ = \ \vec{t_i} \ , \qquad i = 1, ..., N-1, \tag{11}$$

is an orthonormal frame in the tangent space at p, coinciding with the principal curvature directions ([DC], p. 144, p. 216). Thus,

$$\frac{\partial n}{\partial s_i} \ = \ \kappa_i \vec{t_i}, \quad \kappa_i = \kappa_i(s) \text{ the i-th principal curvature of } \partial\mathcal{C}. \tag{12}$$

The coordinate system (p, d) is defined for $-d_0 \ \leq \ d$, provided that $d_0\kappa_i \ \leq \ 1$, $i = 1, ..., N-1$ ([G-T]). The orientation is such that $\kappa_i \geq 0$ when $\mathcal{C}$ is convex. We write

$$U(x) \ = \ p(x) \ + \ d(x)n(x), \tag{13}$$

meaning $p(x) = p(s(x))$, $n(x) = n(s(x))$. By differentiating (13),

$$\begin{aligned} \dot{U}(x) \ &= \ \dot{p}(x) \ + \ \dot{d}(x)n(x) \ + \ d(x)\dot{n}(x) \\ &= \ \vec{t_i}\dot{s}_i \ + \ \dot{d}n \ + \ d\kappa_i\vec{t_i}\dot{s}_i. \end{aligned}$$

Hence,

$$|\dot{U}(x)|^2 \ = \ \sum_{i=1}^{N-1} \dot{s}_i^2(1 + \kappa_i d(x))^2 \ + \ (\dot{d}(x))^2. \tag{14}$$

Let now $\mathcal{C}' \subseteq \mathbb{R}^N$ be a convex set and assume that

$$W_u \cdot n \ \geq \ \frac{c_0}{2} > 0 \quad \text{on } \partial\mathcal{C}', \tag{15}$$

where $W \in C^1(\mathbb{R}^N)$ and (p, d) the canonical coordinates associated to $\partial\mathcal{C}'$. By the C^1 smoothness of W and (15), there is a $\bar{d} > 0$ such that

$$d \ \longmapsto \ W(p + dn) \quad \text{is increasing for } -\bar{d} \leq d \leq \bar{d}. \tag{16}$$

Lemma 10. *Let $x_1 < x_2$ in $\mathbb{R}$ and $U \in [H^1(x_1, x_2)]^N$ be such that*
(i) $d(x_1) = d(x_2) = 0$,
(ii) $0 \leq d(x) \leq \bar{d}$, for $x \in (x_1, x_2)$.
If (15) *and* (16) *are satisfied, then there exists $\tilde{U} \in [H^1(x_1, x_2)]^N$ with the following properties:*

$$\tilde{U}(x_1) \ = \ U(x_1), \quad \tilde{U}(x_2) \ = \ U(x_2),$$

$$-\bar{d} \ \leq \ \tilde{d}(x) \ < \ 0, \ \textit{for } x \in (x_1, x_2),$$

$$E_\mu(\tilde{U}, (x_1, x_2)) \ < \ E_\mu(U, (x_1, x_2)),$$

where $\tilde{U}(x) = \tilde{p}(x) + \tilde{d}(x)n(x)$ *and*

$$E_\mu(U, (x_1, x_2)) \ := \ \int_{x_1}^{x_2} \left(\frac{1}{2} |\dot{U}(x)|^2 + W(U(x)) \right) d\mu(x)$$

where μ *is a positive Radon measure on* $\mathbb{R}$*.*

Proof of Lemma 10 (cf. Lemma 3.3 in [A-F]). Let $\phi : [0,1] \longrightarrow \mathbb{R}$ be a smooth function such that $\phi(0) = \phi(1) = 0$, $\phi(\sigma) > 0$ for $\sigma \in (0,1)$. For small $\varepsilon \geq 0$ define

$$\widetilde{U}^\varepsilon(x) \ := \ p(x) - \varepsilon \phi \left(\frac{x - x_1}{x_2 - x_1} \right) n(x), \quad x \in [x_1, x_2],$$

where $U(x) = p(x) + d(x)n(x)$. By (14), we have

$$|\dot{U}(x)|^2 \ = \ \sum_{i=1}^{N-1} \dot{s}_i^2(x) \ + \ d^2 \sum_{i=1}^{N-1} \ \kappa_i^2 \dot{s}_i^2(x) \ + \ 2d \sum_{i=1}^{N-1} \kappa_i \dot{s}_i^2(x) \ + \ \dot{d}^2(x).$$

We note that

$$|\dot{\widetilde{U}}^\varepsilon|^2 \ = \ \sum_{i=1}^{N-1} \dot{s}_i^2 \ + \ \varepsilon^2 \phi^2 \sum_{i=1}^{N-1} \kappa_i^2 \dot{s}_i^2 \ - \ 2\varepsilon\phi \sum_{i=1}^{N-1} \kappa_i \dot{s}_i^2 \ + \ \varepsilon^2 \frac{\phi'^2}{(x_2 - x_1)^2}.$$

Thus, we have that

$$\begin{aligned} E_\mu(\tilde{U}^\varepsilon, (x_1, x_2)) \ &= E_\mu(\tilde{U}^0, (x_1, x_2)) \\ &- \ \varepsilon \int_{x_1}^{x_2} \phi \sum_{i=1}^{N-1} \kappa_i \dot{s}_i^2 d\mu \ + \ \frac{\varepsilon^2}{2} \int_{x_1}^{x_2} \phi^2 \sum_{i=1}^{N-1} \kappa_i^2 \dot{s}_i^2 d\mu \\ &- \int_{x_1}^{x_2} \Big(W(p) \ - \ W(p - \varepsilon\phi n) \Big) d\mu \\ &+ \ \frac{\varepsilon^2}{(x_2 - x_1)^2} \int_{x_1}^{x_2} \phi'^2 d\mu. \end{aligned} \tag{17}$$

By (16), (ii) above and convexity of $\mathcal{C}'$ we have

$$E_\mu(\tilde{U}^0, (x_1, x_2)) \ \leq \ E_\mu(U, (x_1, x_2)). \tag{18}$$

On the other hand, (16) also implies

$$
\begin{aligned}
&- \int_{x_1}^{x_2} \Big(W(p) - W(p-\varepsilon\phi n)\Big) d\mu \;+\; \frac{\varepsilon^2}{2(x_2-x_1)^2}\int_{x_1}^{x_2} \phi'^2 d\mu \\
&\qquad = -\int_{x_1}^{x_2}\left(\int_0^1 \frac{d}{d\tau}(W(p-\varepsilon\tau\phi n))d\tau\right) d\mu \\
&\qquad\quad + \frac{\varepsilon^2}{2(x_2-x_1)^2}\int_{x_1}^{x_2}\phi'^2 d\mu \\
&\qquad = -\varepsilon\int_{x_1}^{x_2}\left(\int_0^1 W_u(p-\varepsilon\tau\phi n)\cdot\phi n\right) d\tau d\mu \\
&\qquad\quad + \frac{\varepsilon^2}{2(x_2-x_1)^2}\int_{x_1}^{x_2}\phi'^2 d\mu \\
&\qquad \overset{(15)}{<} -C\varepsilon \;+\; \frac{\varepsilon^2}{2(x_2-x_1)^2}\int_{x_1}^{x_2}\phi'^2 d\mu \;<\; 0,
\end{aligned} \tag{19}
$$

for some $C > 0$ and small $\varepsilon > 0$. Finally, we observe that by the convexity of $\mathcal{C}'$,

$$
-\varepsilon\int_{x_1}^{x_2}\phi\sum_{i=1}^{N-1}\kappa_i \dot{s}_i^2 d\mu \;+\; \frac{\varepsilon^2}{2}\int_{x_1}^{x_2}\phi\sum_{i=1}^{N-1}\kappa_i^2\dot{s}_i^2 d\mu \le\; 0,
$$

for small $\varepsilon > 0$. From these inequalities and (17), the lemma follows with $\tilde{U} := \tilde{U}^{\varepsilon}$, $0 < \varepsilon << 1$. □

Hypotheses

(H1) $W : \mathbb{R}^N \longrightarrow \mathbb{R}$, C^2, with two minima $W(a^-) < W(a^+) = 0$.

(H2) $\{u|W(u) \le 0\} =: \mathcal{C}_0^- \cup \{a^+\}$, $\mathcal{C}_0^-$ compact, convex.

(H3) (i) $W_u \cdot n \ge c_0 > 0$ on $\partial\mathcal{C}_0^- =: \{W = 0\}_{(-)}$, n the outward unit normal on $\partial\mathcal{C}_0^-$.

(ii) $W_{uu} \ge c_0 I$ on $\{W = 0\}_{(-)}$.

Remark 11. a) By C^2 smoothness of W, there exists a $b > 0$ such that

$$
W_{uu} \;\le\; bI, \quad \text{on } \{u|W(u) \le 0\}. \tag{20}
$$

b) (H3) implies that the set $\{u|W(u) = \beta\}$ for $0 < \beta << 1$ is made up of two components, which we denote by

$$
\{W = \beta\}_{(-)} \quad \text{and} \quad \{W = \beta\}_{(+)},
$$

with $\{W = \beta\}_{(-)}$ convex and enclosing a^-. On the other hand, for $\beta < 0$ ($|\beta| << 1$), $\{u|W(u) = \beta\}$ is made up of one component which is convex. So more precisely there is an $\alpha_0 > 0$ such that $\{W = \beta\}_{(-)}$ is convex, $\alpha_0 \le \beta \le \alpha_0$. By the smoothness of W,

$$
W_u \cdot n \;\ge\; \frac{c_0}{2} \quad \text{on } \{W = \beta\}_{(-)},\ \alpha_0 \le \beta \le \alpha_0. \tag{21}
$$

Note that the sets $\{W = \beta\}_{(-)}$ are nested for $\alpha_0 \le \beta \le \alpha_0$.

Now we take $\alpha \in (0, \alpha_0)$ and furthermore restrict it as follows:

$$
0 \;<\; \alpha \;<\frac{c_0}{4}\lambda \;=:\; \bar{a}_0, \tag{22}
$$

where λ is a fixed number satisfying the conditions

$$0 \ \le \ \lambda \ \le \ \frac{c_0}{2b}\ , \quad 0 \ < \ \lambda \ \le \ d_0, \quad \lambda \ < \ \frac{1}{\max\{\kappa_1, ..., \kappa_{N-1}\}},$$

with b as in (20) above,

$$d_0 \ = \ \mathrm{dist}\Big(\{W = \alpha_0\}_{(-)}, \{W = -\alpha_0\}_{(-)}\Big),$$

and $\kappa_1, ..., \kappa_{N-1}$ the principal curvatures of $\{W = \beta\}_{(-)}$ (all positive by convexity). We note that

$$W(p \ - \ \lambda n(p)) \ < \ 0, \quad \text{for } p \in \{W = \alpha\}_{(-)}. \tag{23}$$

Indeed (dropping p in $n(p)$),

$$\begin{aligned} W(p) \ - \ W(p - \lambda n) \ &= \ -\int_0^\lambda \frac{d}{dt}[W(p - tn)]dt \\ &= \int_0^\lambda (W_u(p - tn) - W_u(p) + W_u(p)) \cdot n \ dt \\ &= \int_0^\lambda W_u(p) \cdot ndt \ - \ \int_0^\lambda \int_t^0 \frac{d}{ds}(W_u(p - sn)ds) \cdot n \ dt \\ &= \int_0^\lambda W_u(p) \cdot ndt \ - \ \int_0^\lambda \int_0^t W_{uu}(p - sn)n \cdot n \ dsdt \\ &\ge \ \frac{c_0}{2}\lambda \ - \ \frac{b}{2}\lambda^2 \qquad ((20), (21)) \\ &\ge \ \frac{c_0}{4}\lambda \qquad \Big(\lambda \le \frac{c_0}{2b}\Big). \end{aligned}$$

Therefore, we have

$$W(p) - \frac{c_0}{4}\lambda \ \ge \ W(p - \lambda n)$$

and so (by (22))

$$0 \ > \ \alpha \ - \ \frac{c_0}{4}\lambda \ \ge \ W(p - \lambda n).$$

Lemma 12. *Let $\mathcal{C}$ denote the component of $\{u|W(u) \ge \alpha\}$ with $\partial\mathcal{C} = \{W = \alpha\}_{(-)}$. Let (p, d) be the canonical coordinates with respect to $\mathcal{C}$. Assume that α is as in* (22)*, and assume that (H1), (H2), (H3) hold. Let also $x_1 < x_2 \in \mathbb{R}$ and $U \in [H^1(x_1, x_2)]^N$ be such that*
(i) $d(x_1) = d(x_2) = 0$,
(ii) $d(x_0) \ge 0$, for some $x_0 \in (x_1, x_2)$.

Then, there is a $\tilde{U} \in [H^1(x_1, x_2)]^N$ with the properties

$$\tilde{U}(x_1) \ = \ U(x_1)\ , \ \ \tilde{U}(x_2) \ = \ U(x_2),$$

where $-d_0 \le \tilde{d}(x) < 0$, for $x \in (x_1, x_2)$, and

$$E_\mu(\tilde{U}, (x_1, x_2)) \ < \ E_\mu(U, (x_1, x_2)),$$

where $\tilde{U}(x) = \tilde{p}(x) + \tilde{d}(x)n(x)$.

Proof of Lemma 12 (cf. Lemma 3.4 in [A-F]). Let

$$\rho_M \ := \ \max_{x \in [x_1, x_2]} d(x).$$

We can assume that $d(x_0) = \rho_M$. We first analyze the case $d(x_0) = \rho_M = 0$. In this case we can assume that $d(x) < 0$ for some $x \in (x_1, x_0)$ ($x \in (x_0, x_2)$), since otherwise, by Lemma 10 we can replace U with a function that satisfies this condition and has less action. From this and the continuity of U it follows the existence of $\hat{x}_1 \in (x_1, x_0)$, $\hat{x}_2 \in (x_0, x_2)$, $-\frac{d_0}{2} < \hat{d} < 0$, such that $d(\hat{x}_1) = d(\hat{x}_2) = \hat{d}$ and $\hat{d} < d(x) < 0$, for $x \in (\hat{x}_1, \hat{x}_2)$. We now consider the parallel hypersurface to $\partial\mathcal{C}$, parameterized by $p + \hat{d}n(p)$, $p \in \partial\mathcal{C}$. This is convex, and denote it by $\partial\mathcal{C}'$. It can be deduced by (21) that condition (15) holds on $\partial\mathcal{C}'$. Then we can apply lemma (10) on $\partial\mathcal{C}'$ and obtain a local replacement between $\hat{x}_1$ and $\hat{x}_2$ and conclude that the claim of the lemma is true if $\rho_M = 0$. Therefore we can assume $\rho_M > 0$. If $0 < \rho_M \leq d_0$, again we can conclude by Lemma 10 applied to the connected component I_0 of the set $\{x \in (x_1, x_2) | d(x) > 0\}$ that contains x_0. It remains to analyze the case $\rho_M > d_0$. We can identify (x_1, x_2) with I_0. Let $h : [0, d_0] \longrightarrow [-\lambda, 0]$, $h(\sigma) = -\lambda \frac{\sigma}{d_0}$, then $h(0) = 0$, $h(d_0) = -\lambda$. We define the deformation

$$\tilde{U}(x) := \begin{cases} p(x) + h(d(x))n(x), & \text{for } x \in [x_1, x_2],\ d(x) < d_0 \\ p(x) - \lambda n(x), & \text{for } x \in [x_1, x_2],\ d(x) \geq d_0, \end{cases}$$

$\tilde{U}(x_1) = U(x_1)$, $\tilde{U}(x_2) = U(x_2)$. For the kinetic energy we have the estimates

$$\begin{aligned} |\dot{U}(x)|^2 &= \sum_{i=1}^{N-1} \dot{s}_i^2 (1 + \kappa_i d(x))^2 + \dot{d}^2(x) \\ &\geq \sum_{i=1}^{N-1} \dot{s}_i^2 (1 + \kappa_i h)^2 + (h'(d))^2 \dot{d}^2(x) \\ &= |\dot{\tilde{U}}(x)|^2, \end{aligned}$$

when $d(x) < d_0$, while for $d(x) \geq d_0$ we have

$$\begin{aligned} |\dot{U}(x)|^2 &= \sum_{i=1}^{N-1} \dot{s}_i^2 (1 + \kappa_i d(x))^2 + \dot{d}^2(x) \\ &\geq \sum_{i=1}^{N-1} \dot{s}_i^2 (1 + \kappa_i d(x))^2 \\ &> \sum_{i=1}^{N-1} \dot{s}_i^2 (1 - \lambda\kappa_i)^2 \\ &= |\dot{\tilde{U}}(x)|^2. \end{aligned}$$

Hence,

$$\int_{x_1}^{x_2} |\dot{\tilde{U}}(x)|^2 d\mu(x) < \int_{x_1}^{x_2} |\dot{U}(x)|^2 d\mu(x).$$

For the potential energy we have the estimates

$$\begin{aligned} W(\tilde{U}(x)) &= W(p(x) \ + \ h(d(x))n(x)) \\ &= \ W\Big(p(x) \ - \ \frac{\lambda d(x)}{d_0}n(x)\Big) \\ &\leq \ W(p(x) \ + \ d(x)n(x)) \quad (\text{by}(21)) \\ &= \ W(U(x)), \end{aligned}$$

when $d(x) < d_0$, while for $d(x) \geq d_0$ we have by (23) and (H2) that

$$W(\tilde{U}(x)) \ \leq \ 0 \ \leq \ W(U(x)).$$

Putting it all together, we have

$$\int_{x_1}^{x_2} W(\tilde{U}(x))d\mu(x) \ < \ \int_{x_1}^{x_2} W(U(x))d\mu(x).$$

The argument so far establishes that

$$E_\mu(\tilde{U}, (x_1, x_2)) \ < \ E_\mu(U, (x_1, x_2)).$$

The proof of Lemma 12 is complete. □

5. Action Properties of Minimizers

We now show that $E_c(U_L)$ is a function of the jumps at the rims $\big|(U_L)_x(\pm L^+)\big|^2 - \big|(U_L)_x(\pm L^-)\big|^2$, while $E_c(U_L) = 0$ for minimizers in $[C^2(\mathbb{R})]^N$ which solve $U_{xx} - \nabla W(U) = -c\, U_x$ on $\mathbb{R}$. To prove this, we derive an *equipartition* relation *at* $+\infty$ (see [A-Be-C], [A-F] and our result Lemma 15). We first need a formula for the action of solutions:

Lemma 13. ***(1st integral)*** *Every solution to* $U_{xx} - \nabla W(U) = -c\, U_x$ *in* $[C^2(\mu,\nu)]^N$ *satisfies:*

$$E_c(U, (\mu,\nu)) = \int_\mu^\nu \Big\{\frac{1}{2}\big|U_x\big|^2 + W(U)\Big\}e^{cx}dx \ = \ \left\{\frac{e^{cx}}{c}\left(W(U) - \frac{\big|U_x\big|^2}{2}\right)\right\}\Bigg|_\mu^\nu.$$

Proof of Lemma 13. The equation $U_{xx} - \nabla W(U) = -c\, U_x$ implies $-U_{xx}\cdot U_x + \nabla W(U)\cdot U_x = c\big|U_x\big|^2$, hence we obtain

$$\left(\frac{1}{2}\big|U_x\big|^2 - W(U)\right)_x \ = \ -c\big|U_x\big|^2.$$

Integrating by parts the e^{cx} - multiple of this equation, we get

$$\begin{aligned} \left\{\frac{e^{cx}}{2}\big|U_x\big|^2\right\}\Big|_\mu^\nu \ - \ \frac{c}{2}\int_\mu^\nu \big|U_x\big|^2 e^{cx}dx \ - \ \Big(e^{cx}W(U)\Big)\Big|_\mu^\nu \ + \ c\int_\mu^\nu W(U)e^{cx}dx \\ = \ -c\int_\mu^\nu \big|U_x\big|^2 e^{cx}dx. \end{aligned}$$

which leads to the desired formula. □

Lemma 14. ***(The action in terms of the jumps)*** *The minimizers U_L satisfy*

$$\left.\begin{aligned} E_c(U_L) \;=\; \lim_{\omega\to\infty} \frac{e^{c\omega}}{c}\left(W(U_L(\omega)) - \frac{\big|(U_L)_x(\omega)\big|^2}{2}\right) \\ +\; \frac{e^{+cL}}{2c}\left(\big|(U_L)_x(+L^+)\big|^2 - \big|(U_L)_x(+L^-)\big|^2\right) \\ +\; \frac{e^{-cL}}{2c}\left(\big|(U_L)_x(-L^+)\big|^2 - \big|(U_L)_x(-L^-)\big|^2\right). \end{aligned}\right\} =: \; e_c(U_L)$$

The sum $e_c(U_L)$ comprises "error terms" which vanish if $U_L \in [C^2_{\mathrm{loc}}(\mathbb{R})]^N$.

Proof of Lemma 14. First note that $E_c(U_L) = \lim_{\omega\to\infty} E_c(U_L, (-\infty,\omega))$. Apply Lemma 13 to U_L which is a piecewise solution on $(-\infty,-L)$, $(-L,L)$, (L,ω) and add the three relations, utilizing the continuity of $W(U_L)$ at $\pm L$. Finally, let $\omega \longrightarrow \infty$. ☐

Solutions to $U_{xx} = \nabla W(U)$ in the well-studied case of $c = 0$ satisfy an equipartition property: $2W(U) = \big|U_x\big|^2$. Our dissipation term $-c\big|U_x\big|^2$ forces a similar behavior but *at* $+\infty$.

Lemma 15. ***(Equipartition limit of the energy at*** $+\infty$***)*** *The minimizers U_L satisfy*

$$\lim_{\omega\to\infty}\left[\frac{e^{c\omega}}{c}\left(W(U_L(\omega)) - \frac{\big|(U_L)_x(\omega)\big|^2}{2}\right)\right] = 0.$$

Proof of Lemma 15. By the formula (7) for $\mu = \omega$, $\nu = \infty$ and Proposition 8, we have

$$0 \;\le\; c\int_\omega^\infty \big|(U_L)_x\big|^2 dx \;=\; \frac{\big|(U_L)_x(\omega)\big|^2}{2} \;-\; W(U_L(\omega)).$$

This gives

$$\begin{aligned} 0 \;\le\; \frac{e^{c\omega}}{c}\left(\frac{\big|(U_L)_x(\omega)\big|^2}{2} \;-\; W(U_L(\omega))\right) \;&=\; e^{c\omega}\int_\omega^\infty \big|(U_L)_x\big|^2 dx \\ &\le\; \int_\omega^\infty \big|(U_L)_x\big|^2 e^{cx} dx. \end{aligned}$$

By Proposition 9, we have $(U_L)_x \in [L^2(\mathbb{R}, e^{c\mathrm{Id}})]^N$. Hence, letting $\omega \longrightarrow \infty$ we are done. ☐

Corollary 16. ***(The action measures the jump discontinuities)*** *We have that $E_c(U_L) = e_c(U_L)$, with $e_c(U_L)$ as in Lemma 14. In particular, $E_c(U_L) = 0$ if $U_L \in [C^2_{loc}(\mathbb{R})]^N$.*

6. Implications of the Local Replacement Lemmas. Determination of the Speed.

We first introduce our main hypothesis on the potential (cf. (H1)-(H3) in Sec. 4):

(h^*) W is in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, $a^\pm$ are minima, $W(a^-) < 0 = W(a^+)$ and $\min_{\mathbb{R}^N}\{W\} = W(a^-)$. Moreover:

(1) There is an $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$, we have $W^{-1}(\{\alpha\}) = \partial\mathcal{C}_\alpha^- \cup \partial\mathcal{C}_\alpha^+$, $\{u \in \mathbb{R}^N | W \leq \alpha\} = \mathcal{C}_\alpha^- \cup \mathcal{C}_\alpha^+$, where $\mathcal{C}_\alpha^-, \mathcal{C}_\alpha^+$ are disjoint compact, convex sets with C^2 boundaries, containing $a^\pm$ respectively. Moreover, $W_u \cdot n \geq c_0 > 0$ on $\partial\mathcal{C}_0^-$ and $W_{uu} \geq c_0 I$ on $\partial\mathcal{C}_0^-$, n the outward unit normal of $\partial\mathcal{C}_0^-$.
(2) The map $r \mapsto W(a^- + r\xi)$ has a strictly positive derivative as long as $a^- + r\xi \in \mathcal{C}_\alpha^-$, $|\xi| = 1$, $r > 0$.

Assumption (h^*) implies $\liminf_{|u|\to\infty}\big\{W(u)\big\} \geq \alpha_0$, thus W satisfies

$$W^{-1}\big([W(a^-), 0]\big) \ \subset\subset\ \mathbb{R}^N,$$

which was assumed in Theorem 2.

Definition 17. For $\alpha \in (0, \bar{\alpha}_0]$ and $L \geq 1$, we set

$$\begin{aligned} \lambda_L^- &:= \sup\big\{x \in \mathbb{R} \ : \ |U_L(x) - a^-| = r_0\big\}, \\ \lambda_L^+ &:= \inf\big\{x \in \mathbb{R} \ : \ |U_L(x) - a^+| = r_0\big\}, \\ \lambda_L^{\alpha-} &:= \sup\big\{x \in \mathbb{R} \ : \ U_L(x) \in \partial(\mathcal{C}_\alpha^-)\big\}. \end{aligned}$$

We will show that U_L intersects exactly once any of the sets $\partial\mathbb{B}(a^-, r_0)$, $\partial\mathbb{B}(a^+, r_0)$, $\partial\mathcal{C}_\alpha^-$. Decreasing $\alpha > 0$ if necessary, we may assume $\mathcal{C}_a^+ \subseteq \mathbb{B}(a^+, r_0)$ and that $\mathbb{B}(a^+, r_0)$ is disjoint from $\mathcal{C}_a^-$.

Proposition 18. ***(Global a priori control on action minimizers)*** *Assume W satisfies* (h) *and* (h^*)*, α is as in Definition 17 and let $(U_L)_{L\geq 1}$ be the family of minimizers of Theorem 2. For all $L \geq 1$, we have*
(I) U_L exits $\mathcal{C}_\alpha^-$ precisely once at $x = \lambda_L^{\alpha-}$, that is

$$x \in (-\infty, \lambda_L^{\alpha-}] \quad \Longrightarrow \quad W(U_L(x)) \leq \alpha.$$

(II) The image $U_L(\mathbb{R})$ restricted to $\mathbb{R}^N \setminus (\mathcal{C}_\alpha^- \bigcup \mathbb{B}(a^+, r_0))$ has only one connected component and

$$W(U_L(x)) \geq \alpha \text{ for } x \in [\lambda_L^{\alpha-}, \lambda_L^+].$$

(III) The image $U_L(\mathbb{R})$ restricted on $\mathcal{C}_\alpha^- \bigcup \mathbb{B}(a^+, r_0)$ has precisely two connected components and

$$W(U_L(x)) \leq \alpha \text{ for some } x \in \mathbb{R} \text{ then either } \ x \in (-\infty, \lambda_L^\alpha], \text{ or } \ x \in [\lambda_L^+, +\infty).$$

(IV) The numbers $\lambda_L^\pm$ are well defined as the unique times at which U_L crosses the spheres $\partial(\mathbb{B}(a^\pm, r_0))$.
(V) The polar radii $\rho_L^\pm = \big|U_L - a^\pm\big|$ are strictly monotone on $[\lambda_L^+, +\infty)$, $(-\infty, \lambda_L^{\alpha-}]$ respectively.

Proof of Proposition 18. 1. We first settle λ_L^-. We note that Lemma 3.4 of [A-F] applies because the local replacements in its proof are pointwise, and because $W(a^-) \leq W(a^+)$. Thus, λ_L^- is unique and half of (IV) is established.
2. Next we settle $\lambda_L^{\alpha-}$. By applying Lemma 12, we obtain the existence of a unique intersection of U_L with $\partial\mathcal{C}_a^-$, and so (I) is established.
3. We handle λ_L^+ as follows. Assume by contradiction that U_L intersects $\partial\mathbb{B}(a^+, r_0)$ more than once. Then, there are $x_1 < x_2$ such that $U_L(x_i) \in \partial\mathbb{B}(a^+, r_0)$, $i = 1, 2$ and $U_L(x_i) \notin \mathbb{B}(a^+, r_0)$, $x_1 < x < x_2$. Since by step 2. above, U_L can not intersect $\partial\mathcal{C}_\alpha^-$ for those x's, it follows that Lemma 3.4 in [A-F] applies and leads to a local

replacement with less action and thus to a contradiction. Thus, by step 1. above, (IV) has been established.
4. The previous arguments show that $U_L(x)$ can not exit $\mathcal{C}_\alpha^-$ before $x = \lambda_L^{\alpha-}$ and can not enter $\mathbb{B}(a^+, r_0)$ before $x = \lambda_L^+$. Thus we have control on the intervals for which U_L is in the monotonicity regions, which implies the L^∞ bounds

$$\|\rho_L^-\|_{L^\infty(-\infty,\lambda_L^{\alpha-})} \;\le\; \max_{u\in\mathcal{C}_\alpha^-}|u-a^-| \;\;,\;\; \|\rho_L^+\|_{L^\infty(\lambda_L^+,\infty)} \;\le\; r_0.$$

It follows that Lemma 3 can be applied to the minimizers inside $\mathbb{B}(a^\pm, r)$ with $r > r_0$ showing that they can not be identically constant on any subinterval. By (h^*), $\rho_L^\pm$ satisfy $(\rho_L^\pm)_{xx} + c(\rho_L^\pm)_x \ge 0$. By the Strong Maximum Principle, both $\rho_L^\pm$ can not have local maxima, thus they are strictly monotone. By Proposition 8 it follows that the same is true for $r < r_0$, thus (V) has been established. □

Remark 19. We have the ordering $-L \le \lambda_L^{\alpha-} \le \lambda_L^+$. We will prove existence by showing that for some $L < \infty$ large, the constraint is not realized: $-L < \lambda_L^-$ and $\lambda_L^+ < L$ strictly. We define

$$\lambda_L^{0-} \;:=\; \sup\{x \in \mathbb{R} \;:\; U_L(x) \in \partial\mathcal{C}_0^-\}.$$

Note that $-L \le \lambda_L^{0-} \le \lambda_L^{\alpha-} \le \lambda_L^+$.

In the sequel we will need the following estimate.

Lemma 20. *If $dist(\mathcal{C}_\alpha^-, \mathbb{B}(a^+, r_0)) =: d_\alpha$, then for all $\alpha \in [0, \bar{\alpha}_0]$ and $L \ge 1$, we have*

$$E_c(U_L) \;\ge\; -\frac{W^-(a^-)}{c}e^{c\lambda_L^{0-}} \;+\; \frac{\alpha}{c}\Big[e^{c\lambda_L^+} - e^{c\lambda_L^{\alpha-}}\Big] \;+\; \frac{c\, d_\alpha^2}{2\big(e^{-c\lambda_L^{\alpha-}} - e^{-c\lambda_L^+}\big)}.$$

Proof of Lemma 20. We have the identity

$$E_c(U_L) = \;-\int_{-\infty}^{\lambda_L^{0-}} W^-(U_L)e^{cx}dx \;+\; \int_{\lambda_L^{0-}}^{\infty} W^+(U_L)e^{cx}dx \;+\; \frac{1}{2}\int_{\mathbb{R}} \big|(U_L)_x\big|^2 e^{cx}dx.$$

We estimate each term separately, recalling that $W(U_L) \ge \alpha$ on $[\lambda_L^{\alpha-}, \lambda_L^+]$ and $W^-(U_L) \le W^-(a^-)$:

$$\begin{aligned}\int_{-\infty}^{\lambda_L^{0-}} W^-(U_L)e^{cx}dx \;&\le\; W^-(a^-)\int_{-\infty}^{\lambda_L^{0-}} e^{cx}dx \\ &=\; \frac{W^-(a^-)}{c}e^{c\lambda_L^{0-}},\end{aligned}$$

$$\begin{aligned}\int_{\lambda_L^{0-}}^{\infty} W^+(U_L)e^{cx}dx \;&\ge\; \int_{\lambda_L^{\alpha-}}^{\lambda_L^+} W^+(U_L)e^{cx}dx \\ &\ge\; \alpha\int_{\lambda_L^{\alpha-}}^{\lambda_L^+} e^{cx}dx \;=\; \frac{\alpha}{c}\Big[e^{c\lambda_L^+} - e^{c\lambda_L^{\alpha-}}\Big],\end{aligned}$$

$$\begin{aligned}d_\alpha \;\le\; \big|U_L(\lambda_L^{\alpha-}) - U_L(\lambda_L^+)\big| \;&\le\; \int_{\lambda_L^{\alpha-}}^{\lambda_L^+} \big|(U_L)_x\big|dx \\ &\le\; \left(\int_{\lambda_L^{\alpha-}}^{\lambda_L^+} e^{-cx}dx\right)^{\frac{1}{2}}\left(\int_{\lambda_L^{\alpha-}}^{\lambda_L^+} \big|(U_L)_x\big|^2 e^{cx}dx\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have

$$d_\alpha^2 \;\le\; \left(\frac{e^{-c\lambda_L^{\alpha-}} - e^{-c\lambda_L^{+}}}{c}\right)\int_{\mathbb{R}}\big|(U_L)_x\big|^2 e^{cx}dx.$$

Putting these bounds together, we obtain the desired estimate. ☐

The speed of the travelling wave. Thus far, all the results were valid for an arbitrary $c > 0$. It is easy to see that the specific $c = c^*$ that guarantees existence should be very special: by Proposition 9,

$$\begin{aligned}\Big(\big|(U_L)_x(+L^+)\big|^2 - \big|(U_L)_x(+L^-)\big|^2\Big) + \Big(\big|(U_L)_x(-L^+)\big|^2 - \big|(U_L)_x(-L^-)\big|^2\Big)&\\ + 2W^-(a^-) =&\; 2c\int_{\mathbb{R}}\big|(U_L)_x\big|^2dx\\ \ge&\; 2c\int_{-L}^{L}\big|(U_L)_x\big|^2dx\\ \ge&\; \frac{c\big|U_L(+L) - U_L(-L)\big|^2}{L}\\ \ge&\; \frac{c}{L}\Big(|a^+ - a^-| - 2r_0\Big)^2,\end{aligned}$$

which shows that if $c \longrightarrow +\infty$ we can not achieve the smooth matching of piecewise solutions at any $L < \infty$. On the other hand, by Corollary 16 and the a priori bound (5), we have

$$\begin{aligned}e^{+cL}\Big(\big|(U_L)_x(+L^+)\big|^2 - \big|(U_L)_x(+L^-)\big|^2\Big) + e^{-cL}\Big(\big|(U_L)_x(-L^+)\big|^2 - \big|(U_L)_x(-L^-)\big|^2\Big)&\\ =&\; 2cE_c(U_L)\\ \le&\; 2cE_c(U_{aff})\\ \le&\; -2e^{-c}W^-(a^-) + 2ce^{c}\big(E_0^+(U_{aff})\big),\end{aligned}$$

which shows that derivatives can not match if $c \longrightarrow 0^+$. The desired $c = c^*$ is the specific value, at which, for sufficiently large $L > L^* \ge 1$, $E_c(U_L) = 0$. This behavior of $E_{c>0}$ is not present in its $E_{c=0}$ counterpart ([A-F], [A-Be-C]) but it is *plausible*: $U_{xx} - \nabla W(U) = -c\;U_x$ *is* translation invariant while (3) *is not.* Translates $U(\cdot - \delta)$, $\delta \ne 0$ of solutions occur as minimizers to a rescaled $e^{c\delta}E_c$, but both waves have the *same action* only if $E_c(U(\cdot - \delta)) = E_c(U) = 0$.

Remark 21. Note that for fixed $c > 0$, the function $L \longmapsto E_c(U_L) : [1,\infty) \longrightarrow \big(-\infty, E_c(U_{aff})\big]$ is *non-increasing* in L: as L increases, $\mathcal{X}_L$ increases ($L < L'$ implies $\mathcal{X}_L \subset \mathcal{X}_{L'}$) and $E_c(U_L)$ decreases (see Sec. 2 for definitions).

The next two estimates are key ingredients and will allow determine of the speed and establish existence. The full strength of (h^*) is employed to show that U_L *can not get trapped for infinite time* inside $\mathcal{C}_\alpha^-$, after exiting the ball $\mathbb{B}(a^-, r_0)$. We set

$$R_{\max}^\alpha := \max_{u\in\partial\mathcal{C}_\alpha^-}\big|u - a^-\big|.$$

Lemma 22. *If W satisfies (h^*), there exists a $w^* > 0$ such that if $\alpha \in [0, \bar{\alpha}_0]$,*

$$\begin{aligned}\lambda_L^{\alpha-} - \lambda_L^- \le&\; \frac{1}{w^*}\left\{cR_{\max}^\alpha + \Big[\big(cR_{\max}^\alpha\big)^2 + 2w^*\big|R_{\max}^\alpha - r_0\big|\Big]^{\frac{1}{2}}\right\}\\ =:&\; \Lambda_{\alpha,-}.\end{aligned}\tag{24}$$

As w^* we may take

$$w^* := \min_{\substack{r_0 \le r \le R^\alpha_{\max} \\ |\xi|=1}} \left[\frac{d}{dt}\Big|_{t=r} W(a^- + t\xi) \right].$$

Proof of Lemma 22. Writing $U_{xx} - \nabla W(U) = -c\, U_x$ in polar form $U_L = a^- + \rho^-_L n^-_L$, we get (6). Employing (2) of (h^*) on $[\lambda^-_L, \lambda^{\alpha -}_L] \subseteq [-L, L]$, we estimate

$$\begin{aligned}(\rho^-_L)_{xx} + c(\rho^-_L)_x \ &\ge\ \nabla W(a^- + \rho^-_L n) \cdot n^-_L \\ &=\ \frac{d}{dt}\Big|_{t=\rho^-_L} W(a^- + t n^-_L) \\ &\ge\ \min_{\substack{r_0 \le r \le R^\alpha_{\max} \\ |\xi|=1}} \left[\frac{d}{dt}\Big|_{t=r} W(a^- + t\xi) \right] \ =:\ w^* \ >\ 0.\end{aligned}$$

Integrating once $(\rho^-_L)_{xx} + c(\rho^-_L)_x \ \ge\ w^*$ on $[\lambda^-_L, x]$, $x \le \lambda^{\alpha -}_L$ we get

$$(\rho^-_L)_x + c\rho^-_L \ \ge\ w^*(x - \lambda^-_L) \ +\ \Big\{ c\rho^-_L(\lambda^-_L) \ +\ (\rho^-_L)_x\big((\lambda^-_L)^+\big) \Big\}.$$

By Proposition 18, we have $\{\ .\ \} \ge 0$. By a further integration,

$$\int_{\lambda^-_L}^x (\rho^-_L)_z(z)dz \ +\ c\int_{\lambda^-_L}^x (\rho^-_L)(z)dz \ \ge\ w^* \int_{\lambda^-_L}^x (z - \lambda^-_L)dz.$$

Set $x := \lambda^{\alpha -}_L$. We utilize the a priori bound $\big\|\rho^-_L\big\|_{L^\infty[\lambda^-_L, \lambda^{\alpha -}_L]} \le R^\alpha_{\max}$ and that the right term equals $\frac{w^*}{2}\big[\lambda^{\alpha -}_L - \lambda^-_L\big]^2$ to obtain

$$\frac{\big|R^\alpha_{\max} - r_0\big|}{\lambda^{\alpha -}_L - \lambda^-_L} \ +\ \frac{c}{\lambda^{\alpha -}_L - \lambda^-_L}\left(\int_{\lambda^-_L}^x (\rho^-_L)(z)dz \right) \ \ge\ \frac{w^*}{2}\big[\lambda^{\alpha -}_L - \lambda^-_L\big],$$

which gives the desired inequality. Setting $\lambda^{\alpha -}_L - \lambda^-_L =: x$ and comparing with the solutions of the parabola $\frac{w^*}{2}x^2 - (cR^\alpha_{\max})x - \big|R^\alpha_{\max} - r_0\big| \le 0$ we obtain

$$\frac{w^*}{2}\big[\lambda^{\alpha -}_L - \lambda^-_L\big] \ \le\ \frac{\big|R^\alpha_{\max} - r_0\big|}{\lambda^{\alpha -}_L - \lambda^-_L} \ +\ cR^\alpha_{\max}$$

which clearly implies (24). ☐

Lemma 23. *For all $\alpha \in (0, \bar\alpha_0]$, we have the implication:*

$$E_c(U_L) \le 0 \quad \Longrightarrow \quad \lambda^+_L - \lambda^{\alpha -}_L \ \le\ \frac{1}{c}\ln\left(1 + \frac{W^-(a^-)}{\alpha}\right) \ =:\ \Lambda_{\alpha,+}. \tag{25}$$

Proof of Lemma 23. Follows directly from the estimate of Lemma 20:

$$\begin{aligned} 0 \ &\ge\ E_c(U_L) \\ &\ge\ e^{c\lambda^{\alpha -}_L}\left\{ -\frac{W^-(a^-)}{c} \ +\ \frac{\alpha}{c}\Big(e^{c(\lambda^+_L - \lambda^{\alpha -}_L)} - 1\Big) \ +\ \frac{c\, d^2_\alpha}{2\big(1 - e^{-c(\lambda^+_L - \lambda^{\alpha -}_L)}\big)} \right\} \\ &\ge\ \frac{e^{c\lambda^{\alpha -}_L}\alpha}{c}\Big\{ -\Big(\frac{W^-(a^-)}{\alpha} \ +\ 1\Big) \ +\ e^{c(\lambda^+_L - \lambda^{\alpha -}_L)}\Big\}. \quad \square \end{aligned}$$

Corollary 24. *The length of the time interval $\big[\lambda^-_L, \lambda^+_L\big]$ for which the graph of U_L remains between the constraint cylinders is L - uniformly bounded as long as $E_c(U_L) \le 0$.*

Proof of Corollary 24. By Lemmas 22 and 23, we have

$$
\begin{aligned}
\lambda_L^+ - \lambda_L^- \;&=\; \big(\lambda_L^+ - \lambda_L^{\alpha-}\big) \;+\; \big(\lambda_L^{\alpha-} - \lambda_L^-\big) \\
&\leq\; \Lambda_{\alpha,+} \;+\; \Lambda_{\alpha,-} \\
&=:\; \Lambda \;<\; \infty,
\end{aligned}
\tag{26}
$$

provided that $E_c(U_L) \leq 0$. This proves the bound. □

Proposition 25. ***(Determination of the speed of the travelling wave)*** *There exist $c^* > 0$ and $L^* \geq 1$ such that, for all $L \geq L^*$,*

$$
E_{c^*}(U_L) \;=\; \inf_{\mathcal{X}_L}\, [E_{c^*}] \;=\; 0.
$$

The proof consists of several lemmas.

Lemma 26. *For any $L \geq 1$ and any $V \in \mathcal{X}_L$, both fixed, the function $c \longmapsto E_c(V)$ is continuous on $F := \{c > 0 : |E_c(V)| < \infty\}$.*

Proof of Lemma 26. Let $c_m \longrightarrow c_\infty > 0$ as $m \longrightarrow \infty$. Since $V \in \mathcal{X}_L$, we have $W(V) = W^+(V) \geq 0$ on $[L, \infty)$ and as a result, for any $c \in F$,

$$
\begin{aligned}
0 \;&\leq\; \int_L^\infty \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{cx} dx \\
&=\; E_c(V) \;-\; \int_{-\infty}^L \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{cx} dx \\
&\leq\; E_c(V) \;+\; \sup_{(-\infty,L]} |W(V)| \int_{-\infty}^L e^{cx} dx \\
&<\; \infty.
\end{aligned}
$$

Hence, for m large we have on $(L, +\infty)$ that

$$
\left| \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{c_m Id} \right| \;\leq\; 2\left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{c_\infty Id} \;\in L^1(L, +\infty).
$$

Again for any $c \in F$, we have

$$
\begin{aligned}
\int_{-\infty}^L \left|\frac{1}{2}|V_x|^2 + W(V)\right| e^{cx} dx \;&\leq\; \int_{-\infty}^L \left(\left\{\frac{1}{2}|V_x|^2 + W(V)\right\} + \; 2|W(V)|\right) e^{cx} dx \\
&\leq\; E_c(V) \;+\; 2 \sup_{(-\infty,L]} |W(V)| \int_{-\infty}^L e^{cx} dx \\
&<\; \infty.
\end{aligned}
$$

Since $c_m \longrightarrow c_\infty$ as $m \longrightarrow \infty$, if we choose m large enough such that $c_m \leq \frac{3}{2} c_\infty$, we have $e^{c_m x} \leq e^{c_\infty L} e^{\frac{c_\infty}{2} x}$ for all $x \leq L$. Hence, for m large we have on $(-\infty, L)$ that

$$
\left| \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{c_m Id} \right| \;\leq\; e^{c_\infty L} \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{\frac{c_\infty}{2} Id} \;\in L^1(-\infty, L).
$$

By the pointwise convergence $\left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{c_m Id} \longrightarrow \left(\frac{1}{2}|V_x|^2 + W(V)\right) e^{c_\infty Id}$ as $m \longrightarrow \infty$, the lemma follows by application of the Dominated convergence theorem on $(-\infty, L)$ and $(L, +\infty)$ separately. □

Recall that U_L has so far always denoted the minimizer of E_c into $\mathcal{X}_L$ for fixed c. We will temporarily denote the dependence of U_L on c explicitly by $U_{L,c}$. Following an idea of Heinze [Hei], we introduce the following set

$$C \;:=\; \Big\{ c>0 \;\Big|\; \exists\; L\geq 1 \;:\; E_c(U_{L,c})<0 \Big\}. \tag{27}$$

Lemma 27. *The set* (27) *is open, non-empty and* $\sup\ C \leq \sqrt{2W^-(a^-)}{d_0}^{-1}$.

Proof of Lemma 27. By observing that C equals the set

$$\Big\{ c>0 \;\Big|\; \exists\; L\geq 1 \;\&\; \exists\; V\in\mathcal{X}_L : E_c(V)<0 \Big\},$$

Lemma 26 implies that C is open. By the bound (5) on $U_{\text{aff}}\in\bigcap_{L\geq1}\mathcal{X}_L$, we have $f(c)\;\geq\;E_c(U_{\text{aff}})$, where

$$f(c):=e^{-c}\left(-\frac{1}{c}W^-(a^-)+e^{2c}E_0^+(U_{\text{aff}})\right).$$

Moreover, the equation $f(c)=0$ has a unique solution $c_0>0$ since f changes sign and $f'>0$ on $(0,\infty)$. Hence, $(0,c_0)\subseteq C\neq\emptyset$. Moreover, by Lemma 20, for $c\in C$ fixed, we have

$$0\;>E_c(V)\;\geq\;E_c(U_L)\;\geq\;e^{c\lambda_L^{\alpha-}}\left[-\;\frac{W^-(a^-)}{c}\;+\;\frac{c\,d_\alpha^2}{2\big(1-e^{-c(\lambda_L^+-\lambda_L^{\alpha-})}\big)}\right].$$

which implies that $0\geq c^2d_\alpha^2-2W^-(a^-)$. Letting $\alpha\longrightarrow0^+$, we finally obtain

$$0<c_0\;\leq\;\sup C\;\leq\;\sqrt{2W^-(a^-)}{d_0}^{-1}.\quad\square$$

Lemma 28. *Suppose that* $L\geq1$ *is fixed and we have a sequence* $C\ni c_m\longrightarrow c_\infty$ *as* $m\longrightarrow\infty$, $c_\infty>0$. *Then, there exists a subsequence* $c_{m,k}\longrightarrow c_\infty$ *along which*

$$E_{c_{m,k}}(U_{L,c_{m,k}})\;\longrightarrow\;E_{c_\infty}(U_{L,c_\infty}),\quad \text{as}\quad k\longrightarrow\infty.$$

Proof of Lemma 28. Fix $\varepsilon>0$ and choose $V\in\mathcal{X}_L$ such that $E_{c_\infty}(V)-\varepsilon\leq E_{c_\infty}(U_{L,c_\infty})\leq E_{c_\infty}(V)$. Since $c_m\longrightarrow c_\infty$, by Lemma 26, we can choose $m(\varepsilon)\in N$ large such that $|E_{c_\infty}(V)-E_{c_m}(V)|\leq\varepsilon$, for all $m\geq m(\varepsilon)$. Thus,

$$\begin{aligned}E_{c_m}(U_{L,c_m})\;&\leq\;E_{c_m}(V)\\ &\leq\;E_{c_\infty}(V)\;+\varepsilon\\ &\leq\;E_{c_\infty}(U_{L,c_\infty})\;+2\varepsilon,\end{aligned}$$

which implies

$$\limsup_{m\to\infty}\;E_{c_m}(U_{L,c_m})\;\leq\;E_{c_\infty}(U_{L,c_\infty}). \tag{28}$$

By arguing as in the proof of Theorem 2, there exists a subsequence $c_{m,k}\longrightarrow c_\infty$ along which $U_{L,c_{m,k}}\longrightarrow\overline{U}$ in $[C^0_{\text{loc}}(\mathbb{R})]^N$ and $U_{L,c_{m,k}}\rightharpoonup\overline{U}$ weakly in $[H^1_{\text{loc}}(\mathbb{R})]^N$, as $k\longrightarrow\infty$. By weak LSC of the L^2 norm, we have

$$\liminf_{k\to\infty}\;\frac12\int_{\mathbb{R}}|(U_{L,c_{m,k}})_x|^2e^{c_{m,k}x}dx\;\;\geq\;\;\frac12\int_{\mathbb{R}}|(U_{L,c_\infty})_x|^2e^{c_\infty x}dx.$$

For k large, we have the lower bound

$$W(U_{L,c_{m,k}})e^{c_{m,k}Id}\geq-\big(e^{c_\infty L}W^-(a^-)\big)e^{\frac{c_\infty}{2}Id}\chi_{(-\infty,L]}$$

which is an $L^1(\mathbb{R})$ function. Hence, the Fatou lemma implies

$$\liminf_{k\to\infty} \int_{\mathbb{R}} W(U_{L,c_{m,k}})e^{c_{m,k}x}dx \ \geq\ \int_{\mathbb{R}} W(U_{L,c_\infty})e^{c_\infty x}dx.$$

We conclude that

$$\liminf_{k\to\infty}\ E_{c_{m,k}}(U_{L,c_{m,k}}) \ \geq\ E_{c_\infty}(\overline{U}) \ \geq\ E_{c_\infty}(U_{L,c_\infty}). \tag{29}$$

Putting (28) and (29) together, the proof follows. □

Lemma 29. *If $c^* := \sup\ C$, then $E_{c^*}(U_{L,c^*}) = 0$ for all $L \geq \Lambda$.*

Proof of Lemma 29. By (27), there exists a sequence $C \ni c_m \longrightarrow c^*$ as $m \longrightarrow \infty$ such that $E_{c_m}(U_{L_m,c_m}) < 0$. By the negativity of the action we may employ the bound (26) to obtain

$$\lambda^+_{L_m} - \lambda^-_{L_m} \ \leq\ \Lambda$$

which is uniform in $m \in \mathbb{N}$. Moreover, since $E_{c_m}(U_{L_m,c_m}) < 0$, we necessarily have $\lambda^+_{L_m} = L_m$, since otherwise a translation to the right would contradict minimality of U_{L_m,c_m}. By observing that the translate $U_{L_m,c_m}(\cdot + L_m)$ is in $\mathcal{X}_\Lambda$, we have

$$\begin{aligned} E_{c_m}(U_{\Lambda,c_m}) \ &\leq\ E_{c_m}(U_{L_m,c_m}(\cdot + L_m)) \\ &=\ e^{-c_m L_m}E_{c_m}(U_{L_m,c_m}) \\ &<\ 0. \end{aligned}$$

By Lemma 28, the passage to the limit as $m \longrightarrow \infty$ (along a subsequence if necessary) implies

$$\begin{aligned} E_{c^*}(U_{\Lambda,c^*}) \ &=\ \lim_{m\to\infty}\ E_{c_m}(U_{\Lambda,c_m}) \\ &\leq\ 0. \end{aligned}$$

Since $c^* = \sup\ C$ and C is open, $c^* \notin C$ and as a result $E_{c^*}(U_{\Lambda,c^*}) \geq 0$. By Remark 21 and (27), we conclude that $E_{c^*}(U_{L,c^*}) = 0$ for all $L \geq \Lambda$. □

Proof of Proposition 25. By putting Lemmas 26, 27, 28 and 29 together, the proof of Proposition 25 follows with $c^* = \sup C$, $L^* = \Lambda$. □

Proposition 25 provides a c^* for which $E_{c^*}(U_L) = 0$ for large L and this is sufficient for existence. However, c^* is the unique possible speed of minimizing travelling waves[2]:

Proposition 30. ***(Uniqueness of the speed)*** *Assume that a minimizing solution (U, c) to (1) exists. Then, there exists precisely one constant c_* such that (U, c_*) solves (1).*

Corollary 31. *Since minimizers of (3) have vanishing action, we have $c_* = c^*$. Hence, Proposition 25 provides the unique constant for which $E_{c^*}(U) = 0$.*

Proof of Proposition 30. Let (U_1, c_1^*), (U_2, c_2^*) be two solutions of (1) with $0 < c_1^* < c_2^*$ and possibly $U_1 = U_2$. The differential form of the formula in Lemma 13 is

$$\frac{|U_x|^2}{2} + W(U) \ =\ e^{-cx}\left(\frac{e^{cx}}{c}\left[W(U) - \frac{|U_x|^2}{2}\right]\right)_x.$$

[2]this fact together with a sketch of its proof has been kindly pointed out by the referee.

We set $c := c_2^*$, $U := U_2^*$, multiply by $e^{c_1^* x}$ and integrate by parts the right hand side to obtain

$$\int_{-t}^{t} e^{c_1^* x}\left(\frac{|(U_2)_x|^2}{2} + W(U_2)\right) dx \;=\; \left(\frac{e^{c_1^* x}}{c_2^*}\left[W(U_2) - \frac{|(U_2^*)_x|^2}{2}\right]\right)\Big|_{-t}^{t} - (c_1^* - c_2^*)\int_{-t}^{t} \frac{e^{c_1^* x}}{c_2^*}\left[W(U_2) - \frac{|(U_2)_x|^2}{2}\right] dx.$$

We rewrite this identity as

$$\begin{aligned}
\left(e^{c_1^* x}\left[W(U_2) - \frac{|(U_2^*)_x|^2}{2}\right]\right)\Big|_{-t}^{t} \;&=\; c_2^*\int_{-t}^{t} e^{c_1^* x}\left(\frac{|(U_2)_x|^2}{2} + W(U_2)\right) dx \\
&\quad + \; c_1^*\int_{-t}^{t} e^{c_1^* x}\left[W(U_2) - \frac{|(U_2)_x|^2}{2}\right] dx \\
&\quad - \; c_2^*\int_{-t}^{t} e^{c_1^* x}\left[W(U_2) - \frac{|(U_2)_x|^2}{2}\right] dx \\
&= \; c_2^*\int_{-t}^{t} e^{c_1^* x}|(U_2)_x|^2 dx \\
&\quad - \; c_1^*\int_{-t}^{t} e^{c_1^* x}\frac{|(U_2)_x|^2}{2} dx \; + \; c_1^*\int_{-t}^{t} e^{c_1^* x} W(U_2) dx \\
&= \; (c_2^* - c_1^*)\int_{-t}^{t} e^{c_1^* x}|(U_2)_x|^2 dx \; + \; c_1^* E_{c_1^*}\big(U_2, (-t,t)\big).
\end{aligned}$$

Hence, we have the identity

$$c_1^* E_{c_1^*}\big(U_2, (-t,t)\big) \;=\; (c_1^* - c_2^*)\int_{-t}^{t} |(U_2)_x|^2 e^{c_1^* x} dx + \left(e^{c_1^* x}\left[W(U_2) - \frac{|(U_2)_x|^2}{2}\right]\right)\Big|_{-t}^{t}.$$

By Proposition (8), $(U_2)_x \longrightarrow 0$ as $t \to \pm\infty$ up to sequences. Since $E_{c_2^*}(U_2) = 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}}\left\{\frac{|(U_2)_x|^2}{2} + W^+(U_2)\right\} e^{c_2^* x} dx \;&=\; \int_{\mathbb{R}} W^-(U_2) e^{c_2^* x} dx \\
&\leq\; W^-(a^-)\frac{e^{c_2^* L_2^*}}{c_2^*} \\
&<\; \infty.
\end{aligned}$$

where L_2^* is a large constant as in Proposition 25. Hence, since $c_1^* < c_2^*$ we may let $t \to \infty$ to obtain

$$c_1^* E_{c_1^*}(U_2) \;=\; (c_1^* - c_2^*)\int_{\mathbb{R}} |(U_2)_x|^2 e^{c_1^* x} dx \; < 0.$$

But this contradicts that $c_1^* E_{c_1^*}(U_2) \geq 0$. □

We therefore in the remaining assume that $c = c^*$, the unique speed provided by Proposition 25.

A Variational characterization of minimizing travelling waves. Summarizing, solutions (U, c) to the system of equations

$$E_c(U) = \inf\left\{E_c(V) : \; V \in [H^1_{\rm loc}(\mathbb{R})]^N, \; V(\pm\infty) = a^\pm\right\}, \qquad E_c(U) = 0,$$

are heteroclinic travelling waves and solve the differential equations

$$\begin{cases} U_{xx} - \nabla W(U) = -c\, U_x \\ U(\pm\infty) = a^{\pm}. \end{cases}$$

Both the weight $e^{c\mathrm{Id}}$ of (3) and its minimizer are unknown. The first equation of the system involves the minimization problem for E_c in the class $\{E_c \mid c > 0\}$ and, the second one selects $c = c^*$ so that the minimum zero.

7. Removing the Constraints.

In this section we prove existence of solution to problem (1).

Theorem 32. (Existence) *Assume the potential W satisfies* (h), (h^*). *Then, there exists a travelling wave solution $(U, c) \in [C^2(\mathbb{R})]^N \times (0, +\infty)$ to*

$$\begin{cases} U_{xx} - \nabla W(U) = -c\, U_x \\ U(\pm\infty) = a^{\pm}. \end{cases}$$

The speed c equals the constant c^* in Proposition 25 which is unique. In particular, $E_{c^*}(U) = 0$.

Proof of Theorem 32. By Proposition 25, we have $E_{c^*}(U_{L,c^*}) = 0$, for all $L \geq L^*$. By Corollary 24, if we choose $L > \Lambda$ we obtain a minimizer $U := U_L$ of E_c with $c = c^*$ for which $E_c(U) = 0$. Thus either U or a translate $U(\cdot - \delta)$ (with necessarily the same action) does not realize the constraint, solving (1) on $\mathbb{R}$. The proof is complete. ☐

Corollary 33. *The speed c^* has the variational characterization*[3]

$$c^* \;=\; \sup_{c>0} \left\{ c \;\Big|\; \inf_{V \in \mathcal{X}} \; E_c(V) < 0 \right\},$$

where $\mathcal{X} := \big\{ V \in [H^1_{loc}(\mathbb{R})]^N : V(\pm\infty) = a^{\pm} \big\}$.

We now derive a priori bounds on c^*. We take $t > 0$ and consider the affine $[W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})]^N$ function

$$U^t_{aff}(x) \;:=\; a^- \chi_{(-\infty,-t)} \;+\; \left(\frac{t-x}{2t} a^- + \frac{t+x}{2t} a^+ \right) \chi_{[-t,t]} \;+\; a^+ \chi_{(t,\infty)}. \tag{30}$$

Proposition 34. ***(A priori bounds on*** c^****)*** *There exist $0 < c_{\min} < c_{\max} < \infty$ depending only on W, such that*

$$c_{\min} \;\leq\; c^* \;\leq\; c_{\max}.$$

Moreover, if $d_0 := \lim_{\alpha \to 0^+} d_\alpha$, then

$$c_{\max} \;=\; \frac{\sqrt{2W^-(a^-)}}{d_0},$$

$$c_{\min} \;=\; \sup_{t>0} \left[\frac{W^-(a^-)}{e^{2tc_{\max}}} \left(\frac{1}{2} \left\{ \frac{|a^+ - a^-|}{2t} \right\}^2 + \int_{-t}^{t} W^+ \Big(\frac{t-x}{2t} a^- + \frac{t+x}{2t} a^+ \Big) dx \right)^{-1} \right].$$

---

[3]Analogous characterizations have been obtained in [H-P-S] and [He] for other travelling wave problems.

Proof of Proposition 34. The upper bound follows by Lemmas 27 and 29. For the lower bound, we utilize (30) and take as we can $t = L$. This gives as in (5) that the inequality $0 = E_c(U_t) \leq E_c(U^t_{aff})$ implies

$$0 \ \leq \ -e^{-ct}\frac{W^-(a^-)}{c} \ + \ e^{ct}\int_{-t}^{t}\Big\{\frac{1}{2}\Big|\frac{a^+-a^-}{2t}\Big|^2 + W^+\Big(\frac{t-x}{2t}a^- + \frac{t+x}{2t}a^+\Big)\Big\}dx.$$

Hence, for all $t > 0$,

$$c \ \geq \ \frac{W^-(a^-)}{e^{2ct}}\left(\int_{-t}^{t}\Big\{\frac{1}{2}\Big|\frac{a^+-a^-}{2t}\Big|^2 + W^+\Big(\frac{t-x}{2t}a^- + \frac{t+x}{2t}a^+\Big)\Big\}dx\right)^{-1}.$$

Utilizing the upper bound and maximizing with respect to $t > 0$, we are done. □

8. Extensions.

Utilizing ideas related to those in [A-F], we relax (h^*) to a localized version. The new (h^{**}) requires the existence of two convex components $\mathcal{C}^{\pm}_{\alpha}$ of the sublevel set $\big\{W \leq \alpha\big\}$, but only when W is restricted in a large convex $\Omega \subseteq \mathbb{R}^N$ without any restriction on $W|_{\mathrm{ext}(\Omega)}$. As a consequence, (h^{**}) allows for potentials with several other minima and/or unbounded values to $-\infty$.

(h^{**}) There exists a C^2 convex closed set $\Omega \subseteq \mathbb{R}^N$ which encloses the minima $a^{\pm}$ and satisfies (H3), such that (h^*) holds for W within Ω. Moreover, The values of W on $\partial\Omega$ exceed those in the interior : if $u \in int(\Omega)$, then $W(u) < \min_{\partial\Omega} W$.

Example 35. ***(N-d potentials satisfying*** (h^*), (h^{**})***)*** *(i) We construct a deformation of the 2-well potential* $W(u) := |u-a^+|^p|u-a^-|^p$, $p \geq 2$, $u \in \mathbb{R}^N$. *We take* $\varepsilon > 0$ *and set*

$$F_\varepsilon(u) := \big\{\varepsilon \exp\big((|u-a^-|^2-\delta^2)^{-1}\big)+1\big\}\chi_{\mathbb{B}(a^-,\delta)} \ + \ \chi_{\mathbb{R}^N\setminus\mathbb{B}(a^-,\delta)},$$

where $C := \max_{|u-a^-|=\delta}\{W(u)\}$ *and define* $W_\varepsilon(u) := F_\varepsilon(u)W(u) - C(F_\varepsilon(u)-1)$. *The potentials* W_ε *satisfy our assumptions and* $W_\varepsilon - W \longrightarrow 0$ *in* $C^2(\mathbb{R}^N)$, *as* $\varepsilon \to 0^+$.

(ii) (G. Paschalides) The following deformation of the 2-well planar potential

$$W_C(u_1,u_2) \ := \ \begin{cases} W(u_1,u_2)\ , & u_1 < 0\ ,\ u_2 \in \mathbb{R}, \\ W(u_1,u_2) - C\big[6u_1^5 - 15u_1^4 + 10u_1^3\big]\ , & 0 \leq u_1 \leq 1\ ,\ u_2 \in \mathbb{R}, \\ W(u_1,u_2) - C\ , & u_1 > 1\ ,\ u_2 \in \mathbb{R}, \end{cases}$$

(with $a^{\pm} = (\pm 1, 0)$*) satisfies the assumptions* (h^*), (h^{**}) *for any* $C > 0$.

Remark 36. Can monotonicity of (h^*), (h^{**}) be relaxed? In the Appendix we construct a class of W's which are monotone except for merely one critical point a^0 in $W^{-1}([W(a^-),0])$. This implies existence of a connection $a^+ - a^0$, different from $a^+ - a^-$, which generally obstructs existence. Critical points at lower level attract, for $c > 0$, the flow of $U_{xx} - \nabla W(U) = -c\, U_x$ (see also Risler [R]).

Extension of Theorem 32 under the assumption (h^{**})**.** In this case we solve a related problem for a modified "better" $\overline{W}$ and then show that the solution we construct is also a solution of the original problem as well. We modify W to a new $\overline{W}$ by setting:

$$\overline{W} \ := \ W\chi_{\{W \geq \min_{\partial\Omega} W\}} \ + \ \big(2\min_{\partial\Omega} W - W\big)\chi_{\{W < \min_{\partial\Omega} W\}}.$$

This is the reflection the graph of W with respect to the hyperplane $\{w = \min_{\partial\Omega} W\}$ which maps any parts of $Gr(W)$ lying into $\{W \leq \min_{\partial\Omega} W\}$, to the opposite halfspace. $\overline{W}$ is sufficiently coercive and Lemma 12 applied to Ω and to E_c provides an $[L^\infty(\mathbb{R})]^N$ - bound for the minimizers, showing that they are localized inside Ω. Since $\overline{W}$ satisfies (h^*) inside Ω, problem (1) for $\overline{W}$ has a solution U in $[C^2(\mathbb{R})]^N$. By construction $W\big|_\Omega \equiv \overline{W}\big|_\Omega$, so U solves (1) for W as well.

9. Appendix

On the optimality of the assumptions. We construct a class of W's for which there is a heteroclinic between a local minimum a^+ with $W(a^+) = 0$ and a critical point a^0 with $0 > W(a^0) > W(a^-)$, a^- the global minimum. Hence, the existence of additional solutions which may obstruct the existence of $a^+ - a^-$ connections can not be excluded without monotonicity as in (h^*).

(h1) We assume that $W \in C^2_{\text{loc}}(\mathbb{R}^N)$ and

1. W has at least 3 critical points, $a^\pm$, a^0 with $a^\pm$ local minima, a^0 critical point and $W(a^+) = 0 > W(a^0) > W(a^-)$.
2. For $2 \leq j \leq N$, $W_{u_j}(u_1, 0, ..., 0) = 0$ and $[a^-, a^0]$, $[a^0, a^+]$ are on the u_1-axis.

If $N = 1$ and a^0 is a local minimum, then generally no $a^+ - a^-$ connection exists ([F-McL]), depending on the speeds $c_{-,0}$ and $c_{0,+}$ of the solutions $a^- - a^0$ and $a^0 - a^+$. For $N > 1$, (b) implies the existence of solutions $U = (u, 0, ..., 0)$ to $U_{xx} - \nabla W(U) = -c\, U_x$ for the slice $\overline{W}(u) := W(u, 0, ..., 0)$. Thus, we may only impose assumptions on $\overline{W}$:

(h2) We assume that (h^{**}) holds, with the exception that $\overline{W}$ is monotone on (a^-, a^0) (a^0, a^+) separately, instead of (a^-, a^+).

Proposition. *If W satisfies* (h1), (h2), *there exists a solution* $(U, c) \in [C^2(\mathbb{R})]^N \times (0, \infty)$ *to*

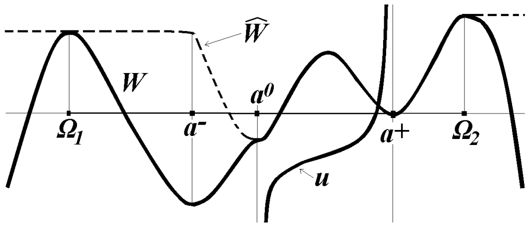
$$\begin{cases} U_{xx} - \nabla W(U) = -c\, U_x \\ U(+\infty) = a^+ \ , \ \ U(-\infty) = a^0. \end{cases}$$

Proof of Proposition .We deform smoothly the slice $\overline{W}$ to a new $\widehat{W}$ for which the *nature of the critical point* a^0 *is changed*, being a global *minimum* of $\widehat{W}$. Then, the problem for $\widehat{W}$ can be tackled by the foregoing theory, and, by a localization argument, the solution we construct solves also the original problem. Let $F : (a^-, a^0) \longrightarrow (0, \infty)$ be the "half" of the standard bell function $F(u) := K \exp\big((u - a^0)^{-1}(u - a^0 + 2a^-)^{-1}\big)$, $K > 0$ to be chosen, and consider the following transformation

$$\widehat{W}(u) \ := \ \begin{cases} \overline{W}(\Omega_2) \ , & u \geq \Omega_2 \\ \overline{W}(u) \ , & u \in [a^0, \Omega_2) \\ -\big(F(u)\overline{W}(u) - 2\overline{W}(a^-)\big) \ , & u \in (a^-, a^0) \\ -\big(F(a^-)\overline{W}(u) - 2\overline{W}(a^-)\big) \ , & u \leq a^-. \end{cases}$$

We choose $K > 0$, such that $\widehat{W}(a^-) \geq \overline{W}(\Omega_1)$. Assumptions (h1), (h2) imply that $\widehat{W}$ satisfies (h^*), giving an $a^+ - a^0$ heteroclinic which solves $u_{xx} - \widehat{W}'(u) = -cu_x$ (Theorem 32). Lemma 12 provides the $L^\infty(\mathbb{R})$ - bound

$$a^- \ \leq \ u(x) \ \leq \ \Omega_2 \ , \qquad \text{for all } x \in \mathbb{R}.$$



The function u solves $u_{xx} - \overline{W}'(u) = -cu_x$ as well. Indeed, it suffices to improve the bound on u to $a^0 \leq u(x) \leq a^+$, for all $x \in \mathbb{R}$. Since by construction $\overline{W}\big|_{[a^0,a^+]} \equiv \widehat{W}\big|_{[a^0,a^+]}$. Lemma 12 applied to (3) for $\widehat{W}$ gives the desired localization. $\square$

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